Faltings's Proof of the Mordell Conjecture

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Abstract

Our plan is to try to understand Faltings's proof of the Mordell conjecture. The focus will be on his first proof, which is more algebraic in nature, proves the Shafarevich and Tate conjectures, and also gives us a chance to learn about some nearby topics, such as the moduli space of abelian varieties or p-adic Hodge theory. The seminar will meet 4:10–5:30 every Thursday in 1360. Some relevant references are [CS86; Fal86; Tat66; Tat67], as well as notes from a seminar on this topic at Stanford recently:

`http://math.stanford.edu/~akshay/ntslearn.html`

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1 September 15: Overview (Andrew Snowden)

Today we will list the results of Faltings that lead to the proof of the Mordell conjecture, and then give an overview of the proof. Afterward, we will distribute the talks that do not have speakers yet.

1.1 Statements of Faltings's results

Let K be a number field, and let S be a finite set of places. Fix $g \geq 2$. Faltings proved a number of statements, which we will label **A**–**G**. The main goal of this seminar is the following:

A. The Mordell conjecture. *If X is an algebraic curve over K of genus $g \geq 2$, then there are only finitely many rational points, i.e., the set $X(K)$ of K-rational points is finite.*

This was conjectured by Mordell around 1910, and was proved by Faltings around 1983.

Example 1.1. **A** implies that the curve $x^n + y^n - 1$ has finitely many K-points if $n \geq 4$.

A is in fact the last statement we will prove; we will continue by listing results in reverse logical order. The following statement is about objects, not points:

B. The Shafarevich conjecture (for curves). *There exist only finitely many (smooth, projective) curves (up to isomorphism) defined over K of genus g and with good reduction outside of S.*

B sort of looks like **A**, but instead concerns the K-points in the moduli space of genus g curves. The following statement is almost like **B** (if you replace a curve with its Jacobian), but has the added information of a polarization:

C. The Shafarevich conjecture (for abelian varieties). *Fix $d \geq 1$. Then, there exist only finitely many abelian varieties over K (up to isomorphism) of dimension g with good reduction outside S, and with a polarization of degree d.*

Here we recall that the degree of a polarization is the order of the kernel of the polarization, which we recall is an isogeny. Also note that **C** probably holds with $g = 1$ as well.

To state the next results, we need to recall the notion of a Tate module.

Recall 1.2. If A is an abelian variety of dimension g, and ℓ is a prime, then the *ℓ-adic Tate module* of A is defined to be

$$T_\ell(A) = \varprojlim_n A[\ell^n](\overline{K}),$$

where the transition maps are multiplication by ℓ. As long as ℓ is coprime to the characteristic, then $A[\ell^n](\overline{K}) \cong (\mathbf{Z}/\ell^n\mathbf{Z})^{2g}$, and so $T_\ell(A) \cong \mathbf{Z}_\ell^{2g}$. The Tate module $T_\ell(A)$ has an action of $G_K = \operatorname{Gal}(\overline{K}/K)$. The *rational Tate module* is

$$V_\ell(A) = T_\ell(A)\left[\tfrac{1}{\ell}\right] \cong \mathbf{Q}_\ell^{2g}.$$

The Tate module contains a whole ton of arithmetic information about the abelian variety it is constructed from; in fact, it is *almost* a complete invariant. This is the content of **E**.

D. Semi-simplicity of the Tate module. *If A is an abelian variety over the number field K, then $V_\ell(A)$ is a semi-simple representation of G_K.*

D is true over a finite field as well.

E. Faltings's isogeny theorem. *If A and B are two abelian varieties, then the natural map*

$$\operatorname{Hom}_K(A, B) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \longrightarrow \operatorname{Hom}_{G_K}(T_\ell(A), T_\ell(B))$$

is an isomorphism.

This is what we mean when we say that the Tate module is almost a complete invariant: if two Tate modules are isomorphic, then there is an isogeny between the abelian varieties they are defined from. This therefore reduces a geometric problem to a problem concerning Galois representations, which is almost linear algebra.

We will start by proving **D** and **E**, and then work up. The first eight talks will focus on **D** and **E**, and the last two talks will do **A**–**C**.

1.2 Overview of the proof

Before Faltings proved his results, Tate in the 1960s showed the analogues of **D** and **E** in the case where the number field K is replaced by a finite field. Tate's proof is therefore the "model argument" for Faltings's proof, and so we start by reviewing Tate's proof. While we won't show Tate's theorem in its entirety, we will show one key piece of the argument: it shows how to get geometric information from representation-theoretic information about Tate modules. This key piece is the following:

Key Result 1.3. *If A is an abelian variety over a finite field K, and $W \subset V_\ell(A)$ is a subrepresentation of the rational Tate module, then there exists $u \in \operatorname{End}(A) \otimes \mathbf{Q}_\ell$ such that $W = u(V_\ell(A))$.*

You can reduce an analogue of **E** to this result by taking W to be the graph of the map on Tate modules.

Idea of Proof. Let X_n be the integral elements in the W-subspace, that is,

$$X_n = (T_\ell(A) \cap W) + \ell^n T_\ell(A).$$

Note that X_n is of full rank/has finite index in T_ℓ. A finite index subrepresentation in the Tate module corresponds to an isogeny to another abelian variety, and so we can let $f_n \colon B_n \to A$ be this isogeny corresponding to X_n, so that $f_n(T_\ell(B_n)) = X_n$.

We then get an infinite sequence of isogenies, and we can use the following:

Fact (∗). Up to isomorphism, there are only finitely many abelian varieties of fixed dimension g over K (a finite field).

The idea is to embed the moduli space of principally polarized abelian varieties into a projective space, on which there are only finitely many points, and then use Zarhin's trick (which says an eightfold product of an abelian variety is principally polarized) to reduce to this case.

Now the B_n's fall into finitely many isomorphism classes, and so you can assume infinitely many are isomorphic; for simplicity, suppose that infinitely many B_n are isomorphic to A, and label them $B_{n_1}, B_{n_2}, B_{n_3}, \ldots$. Now pick an isomorphism $g_{n_i} \colon A \to B_{n_i}$ for all i, and consider the composition

$$h_i \colon A \xrightarrow{g_{n_i}} B_{n_i} \xrightarrow{f_{n_i}} A \in \operatorname{End}(A).$$

Now $h_i \in \operatorname{End}(A) \otimes \mathbf{Z}_\ell$, and the h_i accumulate in the space $\operatorname{End}(A) \otimes \mathbf{Z}_\ell$ to give a map $u \in \operatorname{End}(A) \otimes \mathbf{Z}_\ell$. □

So **E** reduces to Key Result 1.3.

To prove **D** and **E** for number fields, Fact (∗) is no longer true! But we are only using it in a very restricted way: we only used that the B_n's land in finitely many isomorphism classes. Since by construction, they were all isogenous to A, the following analogue of (∗) would be enough to carry out this argument over a number field K to get Key Result 1.3:

F. *If A is an abelian variety, then there are only finitely many abelian varieties over K, up to isomorphism, that are isogenous to A.*

We will actually show a more restrictive version version of **F**, since in our particular setting all of the B_n's arise from the same construction using W.

It is not apparent how to prove **F**; we need some sort of finiteness to get this argument going. The idea comes from elliptic curves, for example over $\mathbf{Q}$. Recall that elliptic curves are (more or less) parametrized by their j-invariant, and so over $\mathbf{Q}$, elliptic curves correspond to rational numbers. The standard way to cut $\mathbf{Q}$ into a finite set, at least in arithmetic, is to use heights. The basic idea is that $\operatorname{height}(\frac{a}{b}) = \max(|a|, |b|)$. You can define a similar function for $\mathbf{Q}^n$ and projective spaces, and then projective varieties by looking at $\mathbf{Q}$-points when embedded in a projective space.

Now we want to apply a similar idea to the moduli space of abelian varieties. There exists a moduli space $\mathscr{A}_g$ of abelian varieties of dimension g, which gives an embedding $\mathscr{A}_g \to \mathbf{P}^N$ by using (a power of) the Hodge bundle. Then, you can consider heights of points in $\mathscr{A}_g$ by using the height function on $\mathbf{P}^N$ to get the *moduli-theoretic height*

$$H \colon \mathscr{A}_g(K) \longrightarrow \mathbf{R}.$$

So for elliptic curves, the map is a map to $\mathbf{P}^1$, and you take heights of points.

Now the issue is that while there are only finitely many points of bounded height, there is really no way to be able to tell what the height of a given abelian variety is. Faltings got around this issue by defining another height function intrinsically from the datum of an abelian variety, which we now call the *Faltings height*, denoted $h(A)$.

Example 1.4. Let $E/\mathbf{Q}$ be an elliptic curve. We can define the Faltings height $h(E)$ of E as follows. Let $\mathscr{E}/\mathbf{Z}$ be the Néron model of E, and consider the embedding

$$\mathbf{Z} \cong H^0(\mathscr{E}, \Omega^1) \in H^0(E, \Omega^1) \cong \mathbf{Q}.$$

Choose a generator ω for $\mathbf{Z} \cong H^0(\mathscr{E}, \Omega^1)$. We then define

$$h(E) = \int_{E(\mathbf{C})} \omega \wedge \overline{\omega}.$$

For higher dimensional abelian varieties, you basically do the same procedure, by taking more generators for $H^0(\mathscr{A}, \Omega^1)$.

This is nice since you don't need to know anything about the moduli space. The downside is that there is no way to be able to tell there are only finitely many abelian varieties of bounded height. Faltings got around this by showing

G. *h and H are not very different.*

The precise statement implies the following:

Corollary 1.5. *Up to isomorphism, there are only finitely many abelian varieties over K with bounded Faltings height h.*

This is the last of the statements we wanted to state, and is the first one we will try to prove. We now give some indication to how to prove each statement from the previous one.

G $\Rightarrow$ **F**. The key is to understand how Faltings height changes under isogenies that show up in our situation. There is a simple formula that tells you exactly how it changes in terms of heights and arithmetic information about the kernel. A long computation with group schemes and some p-adic Hodge theory (the Tate decomposition) gives the conclusion F. □

F $\Rightarrow$ **D** + **E**. This mirrors Tate's original argument, as we have already said. □

D + **E** $\Rightarrow$ **C**. This is done in two parts:

1. We first show finiteness up to isogeny. By **D** and **E**, it suffices to show that there are only finitely many possibilities for the Tate module, as a Galois representation. This is reasonably straight-forward, by using Chebotarev's density theorem, and investigating Weil numbers.
2. We can then show finiteness up to isomorphism. To do this, we again need to study the behavior of heights under isogenies. This is similar to the argument for **G** $\Rightarrow$ **F**: you use group schemes and p-adic Hodge theory, although the key input this time is Raynaud's theorem about group schemes. □

C $\Rightarrow$ **B**. Take Jacobians. □

B $\Rightarrow$ **A**. This is an argument due to Paršin. Roughly, the idea is as follows: let X/K be a curve, and let P be a rational point. Then, there is a geometric construction that produces a finite cover $X_P \to X$ of X that ramifies only at P, and given such a map you can recover P. The genus and field of definition of P increase, but this increase is bounded independently of P, and you get a map

$$X(K) \longrightarrow \{\text{possible } X_P\text{'s}\}.$$

The latter set is finite by **B**, and this map is in fact finite-to-one, and so **A** follows. □

2 September 22: Endomorphisms of abelian varieties over finite fields (following Tate) (Emanuel Reinecke)

The main reference for this lecture is [Lic10]. See also [Tat66; MilAV; Mum08, App. 1].

We first start by restating Tate's theorem on abelian varieties over finite fields:

E. Tate's isogeny theorem. *Let A and B be two abelian varieties over $k = \mathbf{F}_q$. Let ℓ be a prime that is coprime to $\operatorname{char} k$, and let $G = G_k$ be the absolute Galois group. We have the following commutative diagram:*

$$\begin{tikzcd} \mathrm{Hom}_k(A,B)\otimes \mathbf{Z}_\ell \arrow[r, hook] \arrow[dr, "(\star)"'] & \mathrm{Hom}_{\mathbf{Z}_\ell}(T_\ell(A),T_\ell(B)) \circlearrowleft G \\ & \mathrm{Hom}_G(T_\ell(A),T_\ell(B)) = \mathrm{Hom}_{\mathbf{Z}_\ell}(T_\ell(A),T_\ell(B))^G \arrow[u, phantom, "\cup"] \end{tikzcd}$$

where the G-action on $\mathrm{Hom}_{\mathbf{Z}_\ell}(T_\ell(A),T_\ell(B))$ is given by $(gf)(x) = gf(g^{-1}(x))$. Then, the morphism $(\star)$ is an isomorphism.

The proof of the theorem will also give

D. Semisimplicity of Tate modules. *Let A be an abelian variety over $k = \mathbf{F}_q$. Then, the rational Tate module $V_\ell(A) = T_\ell(A) \otimes \mathbf{Q}_\ell$ is a semisimple G-representation.*

2.1 Motivation

We begin with some motivation. In this subsection we are working exclusively over a finite field $k = \mathbf{F}_q$. Then, the Frobenius endomorphisms π_A and π_B relative to k act on A and B, respectively. Tensoring the map $(\star)$ up by $\mathbf{Q}_\ell$, we get a map

$$\mathrm{Hom}_k(A,B) \otimes \mathbf{Q}_\ell \longrightarrow \mathrm{Hom}_G(V_\ell(A), V_\ell(B)) \tag{$\star\star$}$$

Via this map, π_A and π_B induce endomorphisms $V_\ell(\pi_A)$ and $V_\ell(\pi_B)$ on $V_\ell(A)$ and $V_\ell(B)$, respectively, and we can consider the characteristic polynomials P_A and P_B for $V_\ell(\pi_A)$ and $V_\ell(\pi_B)$. We can then use

A toy case of the Weil conjectures 2.1. *P_A and P_B have $\mathbf{Z}$-coefficients, and they are independent of ℓ.*

Next, provided that the action of Frobenius on $V_\ell(\pi_A)$ and $V_\ell(\pi_B)$ is semisimple (we will talk about this later), a bit of linear algebra shows that you can find $r = r(P_A, P_B) \in \mathbf{Z}$ such that

$$\dim_{\mathbf{Q}_\ell} \mathrm{Hom}_G(V_\ell(A), V_\ell(B)) = r(P_A, P_B).$$

In particular, since the right-hand side does not depend on ℓ, neither does the left-hand side; this will be used in the proof.

Now we combine this with what Tate's theorem would tell us to obtain

$$\operatorname{rk} \mathrm{Hom}(A,B) = \dim_{\mathbf{Q}_\ell} \mathrm{Hom}_G(V_\ell(A), V_\ell(B)) = r(P_A, P_B).$$

We state a first corollary of Tate's theorem.

Corollary 2.2. *Let A and B be two abelian varieties over $k = \mathbf{F}_q$. Let P_A and P_B be the characteristic polynomials of the (relative) Frobenii π_A and π_B acting on $V_\ell(A)$ and $V_\ell(B)$, respectively. Then,*

- (a) $\operatorname{rank} \mathrm{Hom}(A,B) = r(P_A, P_B)$;
- (b) *The following are equivalent:*
  - $(b1)$ *B is k-isogeneous to an abelian subvariety of A;*
  - $(b2)$ *$V_\ell(B))$ is G-isomorphic to a G-subrepresentation of $V_\ell(A)$ for some $\ell \neq \operatorname{char} k$;*
  - $(b3)$ *$P_B \mid P_A$ in $\mathbf{Q}[t]$.*
- (c) *The following are equivalent:*
  - $(c1)$ *B is k-isogeneous to A;*
  - $(c2)$ *$P_B = P_A$;*

(*c*3) *$\zeta_B = \zeta_A$, that is, A and B have the same number of rational points over any finite extension of k.*

This continues the theme of expressing the geometry of A solely through algebra. The last part (c) is particularly interesting: it says that if two abelian varieties are such that their rational points over any field extension are the same, then the abelian varieties are the same.

Let us see how Tate's theorem implies the Corollary.

Proof. $(b1) \Rightarrow (b2)$. Let $u\colon B \to A$ be a k-homomorphism. Then,

$$2\dim(\ker u) = \dim(V_\ell(\ker u)) = \dim(\ker V_\ell(u)). \tag{†}$$

So u is an isogeny if and only if $V_\ell(u)$ is injective.

$(b2) \Rightarrow (b1)$. Let $\alpha\colon V_\ell(B) \hookrightarrow V_\ell(A)$. Then, we can find by Tate's theorem some $u \in \mathrm{Hom}(B, A) \otimes \mathbf{Q}_\ell$ such that $V_\ell(u) = \alpha$, and moreover we can choose $u \in \mathrm{Hom}(B, A) \otimes \mathbf{Q}$ such that $V_\ell(u)$ is arbitrarily close to α. Since matrix rank is a lower semi-continuous function, if $V_\ell(u)$ is close enough to α, then we can ensure $V_\ell(u)$ is injective. Now taking multiples, we can assume that $u \in \mathrm{Hom}(B, A)$. Using (†), you get that u is an isogeny to an abelian subvariety of A.

The rest is left as an exercise: some are easy, some require knowing that π_A and π_B act semisimply on $V_\ell(A)$ and $V_\ell(B)$, and some use the Weil conjectures. ☐

The other big application of Tate's theorem comes from understanding the endomorphism algebra of an abelian variety.

Corollary 2.3. *Let A be an abelian variety of dimension g over $k = \mathbf{F}_q$. Let π_A be the relative Frobenius endomorphism of A, and P_A its characteristic polynomial. Then,*

(*a*) *$F := \mathbf{Q}[\pi_A]$ is the center of the endomorphism algebra $E := \mathrm{End}_k(A) \otimes \mathbf{Q}$.*

(*b*) $2g \le \dim_{\mathbf{Q}} E = r(P_A, P_A) \le (2g)^2$.

2.2 The isogeny category

In this subsection and the next, k can be any field.

We want to make the notion of considering abelian varieties "up to isogeny" precise.

Recall 2.4. If $f\colon A \to B$ is an isogeny of degree n, then there exists a map $g\colon B \to A$ such that the diagram

$$\begin{tikzcd} A \arrow[rr, "n"] \arrow[dr, "f"'] & & A \\ & B \arrow[ur, "g"'] & \end{tikzcd}$$

commutes. We can then define the category AV_k°, which has abelian varieties over k as objects, and morphisms are given by $\mathrm{Hom}^\circ(A, B) := \mathrm{Hom}(A, B) \otimes \mathbf{Q}$. In particular, if $f \in \mathrm{Hom}(A, B)$ is an isogeny, then $f^{-1} \in \mathrm{Hom}^\circ(B, A)$.

This category satisfies some very nice properties:

Theorem 2.5 (Poincaré complete reducibility)**.** *If $B \subset A$ is an abelian subvariety, then there exists another abelian subvariety $C \subset A$ such that the canonical map $B \times C \to A$ is an isogeny.*

The idea of the proof is very simple: if $B \hookrightarrow A$, then you can look at the dual map $A^\vee \twoheadrightarrow B^\vee$. But the dual of an abelian variety is isomorphic to the abelian variety itself in the isogeny category, and so you get a splitting $B \to A$ of this surjection. See [MilAV, Ch. I, Prop. 10.1].

Corollary 2.6. *AV_k° is semisimple.*

Proof. If A_1 splits off from A, then $A_1 \times A' \to A$, and so you get an isogeny $A_1 \times A_2 \times \cdots \times A_n \to A$ where A_i are simple by induction. ☐

Remark 2.7. Normally being abelian is part of being a semisimple category, and in fact, AV_k° is abelian.

Corollary 2.8. $(\star)$ *is injective.*

Proof. We in fact show that

$$\mathrm{Hom}_k(A,B) \to \mathrm{Hom}_{\mathbf{Z}_\ell}(T_\ell(A),T_\ell(B))$$

is injective; see [MilAV, Thm. 10.15] for the proof that tensoring up to $\mathbf{Z}_\ell$ does not hurt injectivity.

Suppose $u\colon A \to B$ is such that $T_\ell(u)=0$. Then, $u(x)=0$ for all $x \in A[\ell^n](k^s)$. This is bad: if $A' \subset A$ is a simple abelian subvariety, then $\ker u|_{A'}$ is either A' or zero, but since it contains $A'[\ell^n](k^s)$, we must have $\ker u|_{A'} = A'$. By semisimplicity, this implies that $u=0$. □

2.3 Initial reductions

Our first reduction reduces studying the morphism $(\star)$ involving $\mathbf{Z}_\ell$-modules to studying the morphism $(\star\star)$, which involves $\mathbf{Q}_\ell$-vector spaces, which are nicer to study.

Lemma 2.9. $(\star)$ *is an isomorphism if and only if* $(\star\star)$ *is an isomorphism.*

Proof. First, since $(\star\star)$ is just $(\star)$ tensored up by $\mathbf{Q}_\ell$, and we already know that $(\star)$ is injective (Corollary 2.8), we know that $(\star\star)$ is injective. So it suffices to show that $(\star)$ is surjective if and only if $(\star\star)$ is surjective. Moreover, the forward implication follows already from right-exactness of the tensor product, and to show the converse implication, it suffices to show that

$$\mathrm{cok}(\star) = \mathrm{Hom}_G(T_\ell(A),T_\ell(B))/\operatorname{im}(\star)$$

is torsion-free, for if $\mathrm{cok}(\star) \neq 0$, then it must be torsion. In fact, $\mathbf{Z}_\ell$-modules can only have ℓ^n-torsion, so it is enough to show that $\mathrm{cok}(\star)$ does not have ℓ-torsion.

So let $\alpha\colon T_\ell(A) \to T_\ell(B)$ be ℓ-torsion, that is, $\ell\alpha = T_\ell(u)$ for some $u \in \mathrm{Hom}(A,B)\otimes \mathbf{Z}_\ell$. We want to show that $\alpha = T_\ell(v)$ for some $v \in \mathrm{Hom}(A,B)\otimes\mathbf{Z}_\ell$. Assume for simplicity that $u \in \mathrm{Hom}(A,B)$ (if not, we can approximate everything by morphisms that look like this). Then, we have the following commutative diagram:

$$\begin{array}{ccc} T_\ell(A) & \xrightarrow{\ell\alpha = T_\ell(u)} & T_\ell(B) \\ \downarrow & & \downarrow \\ A[\ell](k^s) & \xrightarrow{u=0} & A[\ell](k^s) \end{array}$$

where the vertical maps are the projection map from $T_\ell(A) = \varprojlim_n A[\ell^n](k^s)$. This implies $A[\ell](k^s) \subset \ker(u)$, but then u factors as

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "\ell"'] & & B \\ & A \arrow[ur, "v"'] & \end{tikzcd}$$

so that $T_\ell(v) = \alpha$, that is, $\alpha \in \operatorname{im}(\star)$, and so $\alpha = 0$ in $\mathrm{cok}(\star)$. □

Lemma 2.10. *It suffices to show that* $(\star\star)$ *is an isomorphism for one* ℓ.

Proof. Since $(\star\star)$ is always injective, this follows by our remark about the Weil conjectures 2.1 that $\dim_{\mathbf{Q}_\ell} \mathrm{Hom}_G(V_\ell(A),V_\ell(B))$ is independent of ℓ. □

The next reduction is trickier: we have already reduced the question to one of $\mathbf{Q}_\ell$-vector space maps, but we want to further reduce to the case where everything is a $\mathbf{Q}_\ell$-algebra. We do this by showing it suffices to consider $(\star\star)$ in the case where $B=A$, since the endomorphism ring of the abelian variety A has a natural algebra structure.

Lemma 2.11. *It suffices to show that*

$$\mathrm{End}_k(A)\otimes\mathbf{Q}_\ell \longrightarrow \mathrm{End}_G(V_\ell(A)) \qquad (\star\star\star)$$

is an isomorphism for all abelian varieties A *over* k.

Proof. Consider the following diagram, which follows from functoriality:

$$\begin{array}{ccccccccc}
\mathrm{End}_k(A\times B)\otimes\mathbf{Q}_\ell & \xrightarrow{\sim} & \mathrm{End}_k(A)\otimes\mathbf{Q}_\ell & \oplus & \mathrm{Hom}_k(A,B)\otimes\mathbf{Q}_\ell & \oplus & \mathrm{Hom}_k(B,A)\otimes\mathbf{Q}_\ell & \oplus & \mathrm{End}_k(B)\otimes\mathbf{Q}_\ell \\
{\scriptstyle(\star\star\star)}\big\downarrow & & \cap & & \cap & & \cap & & \cap \\
\mathrm{End}_G(V_\ell(A\times B)) & \xrightarrow{\sim} & \mathrm{End}(V_\ell(A)) & \oplus & \mathrm{Hom}_G(V_\ell(A),V_\ell(B)) & \oplus & \mathrm{Hom}_G(V_\ell(B),V_\ell(A)) & \oplus & \mathrm{End}_G(V_\ell(B))
\end{array}$$

We already know the vertical maps are inclusions by Corollary 2.8, and so equality on the very left implies all the inclusions on the right are in fact equalities. ☐

Note that in particular, even if you were only interested in showing Tate's theorem for elliptic curves, this method of proof requires proving statements about higher dimensional abelian varieties.

Now let

$$\begin{aligned}
E_\ell &\coloneqq \mathrm{End}_k(A)\otimes\mathbf{Q}_\ell \subset \mathrm{End}_{\mathbf{Q}_\ell}(V_\ell(A))\\
F_\ell &\coloneqq \mathbf{Q}_\ell[G] \qquad\qquad\quad\;\; \subset \mathrm{End}_{\mathbf{Q}_\ell}(V_\ell(A))
\end{aligned}$$

that is, F_ℓ is the $\mathbf{Q}_\ell$-subalgebra of $\mathrm{End}_{\mathbf{Q}_\ell}(V_\ell(A))$ generated by automorphisms of $V_\ell(A)$ coming from G. Since the endomorphisms E_ℓ of V_ℓ that come from k-rational endomorphisms of A commute with the Galois action,

$$F_\ell \subset C_{\mathrm{End}_{\mathbf{Q}_\ell}(V_\ell(A))}(E_\ell), \tag{1}$$

where $C_{\mathrm{End}_{\mathbf{Q}_\ell}(V_\ell(A))}(E_\ell)$ denotes the centralizer of E_ℓ in $\mathrm{End}_{\mathbf{Q}_\ell}(V_\ell(A))$.

Lemma 2.12. *To show* $\mathbf{E}$*, it is enough to show* (1) *is an equality. More precisely,*

(*a*) $(\star\star\star)$ *is an isomorphism if and only if* $C(C(E_\ell)) = \mathrm{End}_G(V_\ell(A))$*;*

(*b*) *If* F_ℓ *is semisimple, then* $(\star\star\star)$ *is an isomorphism if and only if* $C(E_\ell) = F_\ell$*.*

Note that the assumption in (*b*) that F_ℓ be semisimple immediately shows $\mathbf{D}$.

Proof. For (*a*), if we show that E_ℓ is semisimple, then by the double centralizer theorem, we have that $C(C(E_\ell)) = E_\ell$. To show semisimplicity, we use Poincaré complete reducibility (Theorem 2.5), which implies A is isogenous to $\prod A_i^{n_i}$, where A_i, A_j are simple and not isogenous if $i \neq j$. Thus,

$$\mathrm{End}^\circ(A) = \mathrm{End}^\circ(\textstyle\prod A_i^{n_i}) = \prod \mathrm{End}^\circ(A_i^{n_i}) = \prod \mathrm{Mat}_{n_i}(\mathrm{End}^\circ(A_i)).$$

But $\mathrm{End}^\circ(A_i)$ is a division algebra, and matrix algebras over finite-dimensional division algebras are semisimple. In particular, $E_\ell = \mathrm{End}^\circ(A)\otimes\mathbf{Q}_\ell$ is semisimple.

For (*b*), if F_ℓ is semisimple, then $C(E_\ell) = F_\ell$ if and only if $E_\ell = C(C(E_\ell)) = C(F_\ell) = \mathrm{End}_G(V_\ell(A))$. ☐

Remark 2.13. Using that $\mathbf{Q}[\pi_A]$ is contained in the center of the semisimple algebra $\mathrm{End}^\circ(A)$, one can in fact show that $\pi_A \circlearrowright A$ acts semisimply on $V_\ell(A)$ (Exercise).

2.4 Proof of Tate's theorem

So far, we haven't used anything about the field. We start with stating the hypothesis on our field:

$$\begin{aligned}
\mathrm{Hyp}(k,A,\ell)\colon\ &\text{there exist only finitely many (up to } k\text{-isomorphism) abelian varieties } B\\
&\text{such that there is a } k\text{-isogeny } B\to A \text{ of } \ell\text{-power degree}
\end{aligned}$$

To simplify notation, we set $D \coloneqq C(E_\ell)$, the right-hand side of (1). Recall that by Lemma 2.12, we want to show that $C(D) = \mathrm{End}_G(V_\ell(A))$.

We already know the inclusion $\subset$ holds since $E_\ell \subset \mathrm{End}_G(V_\ell(A))$. It remains to show $C(D) \supset \mathrm{End}_G(V_\ell(A))$. So let $\alpha \in \mathrm{End}_G(V_\ell(A))$. We want to show that α commutes with everything in D. We restate this in terms of G-stable subspaces. Set W to be the graph of α:

$$W \coloneqq \Gamma_\alpha = \{(x, \alpha x) \in V_\ell(A\times A)\} \subset V_\ell(A\times A) = V_\ell(A)\times V_\ell(A).$$

Note it is G-stable. Then, $\alpha \in C(D)$ if and only if for all $x \in V_\ell(A)$ and $d \in D$, the equality $\alpha d x = d\alpha x$ holds. Applying d to the graph W above, we see that this is in turn equivalent to having $(d\oplus d)W \subset W$ for all $d \in D$. We then use the key result which Andrew mentioned last time:

Key Proposition 2.14. *If $W \subset V_\ell(A)$ is a G-stable subspace, then there exists $u \in E_\ell$ such that $uV_\ell(A) = W$.*

Applying this to the abelian variety $A \times A$ and letting $W = \Gamma_\alpha$ as before, we would have

$$(d \oplus d)W = (d \oplus d)uV_\ell(A \times A) = u(d \oplus d)V_\ell(A \times A) \subset uV_\ell(A \times A) = W,$$

which as we remarked, suffices to show that $C(D) \supset \mathrm{End}_G(V_\ell(A))$.

Recall the proof that Andrew gave in the last talk:

Proof of Key Proposition. Let $X_n = (T_\ell(A) \cap W) + \ell^n T_\ell(A)$. Then, this is G-stable and of finite index in $T_\ell(A)$, so it corresponds to an isogeny $f_n \colon B_n \to A$ such that $f_n(T_\ell(B_n)) = X_n$. Now we use $\mathrm{Hyp}(k, A, \ell)$: there exists an increasing sequence $0 \le n_1 < n_2 < \cdots$ such that the B_{n_i}'s all fall into one k-isomorphism class. Now pick isomorphisms $g_{n_i} \colon B_{n_1} \to B_{n_i}$, and consider the diagram

$$\begin{tikzcd} B_{n_1} \arrow[r, "g_{n_i}"] \arrow[d, "f_{n_1}"'] & B_{n_i} \arrow[d, "f_{n_i}"] \\ A \arrow[r, dashed, "u_i"] & A \end{tikzcd}$$

We can define $u_i \in \mathrm{End}^\circ(A)$ as $f_{n_i} \circ g_{n_i} \circ f_{n_1}^{-1}$. On Tate modules, this corresponds to a diagram

$$\begin{tikzcd} T_\ell(B_{n_1}) \arrow[r] \arrow[d] & T_\ell(B_{n_i}) \arrow[d] \\ X_{n_1} \arrow[r, "u_i"] & X_{n_i} \end{tikzcd}$$

where we also denote u_i as the morphism induced by the morphism $u_i \in \mathrm{End}^\circ(A)$ on the Tate module $T_\ell(A)$. Since also $X_{n_i} \subset X_{n_1}$ by definition of X_n, the morphism u_i is an element of $\mathrm{End}_{\mathbf{Z}_\ell}(X_{n_1}) \cap E_\ell$. This is a compact space, and so the sequence u_i has an accumulation point $u \in \mathrm{End}_{\mathbf{Z}_\ell}(X_{n_1}) \cap E_\ell$. You can check that u satisfies the properties we wanted. ☐

Remark 2.15. Using this proposition, Tate proved **D**.

We still have to check the hypothesis on the field.

2.5 Why does $\mathrm{Hyp}(k, A, \ell)$ hold if $k = \mathbf{F}_q$?

Fact 2.16. There exists a moduli space of d-polarized dimension g abelian varieties $A_{g,d}$, which is a stack of finite type over k. Its k-valued points can be described as follows:

$$A_{g,d}(k) = \left\{ (A, \lambda) \;\middle|\; \begin{array}{l} A \text{ is an abelian variety over } k \\ \lambda \colon A \to A^\vee \text{ is a degree } d \text{ polarization} \end{array} \right\}$$

Since this moduli space is of finite type, for any fixed g, d there are only finitely many abelian varieties of that type. This is not strong enough for $\mathrm{Hyp}(k, A, \ell)$ to hold, however, since d can be an arbitrary natural number.

To get around this, we need two statements about abelian varieties:

- Zarhin's trick: for any abelian variety A, the abelian variety $(A \times A^\vee)^4$ is principally polarized;
- Finiteness of direct factors: up to isomorphism, an abelian variety has only finitely many direct factors.

Recall that a direct factor $B \subset A$ of an abelian variety A is an abelian subvariety B such that there exists another abelian subvariety $C \subset A$ and an isomorphism $A \simeq B \times C$. Note that an abelian variety can still have infinitely many (isomorphic) direct factors.

Assuming these statements, we have the following:

Corollary 2.17. *If $k = \mathbf{F}_q$ is a finite field, then there are only finitely many non-isomorphic abelian varieties of dimension g.*

Proof. A is a direct factor of $(A \times A^\vee)^4 \in A_{8g,1}$. ☐

This resembles the strategy used by Zarhin for k a function field of positive characteristic and by Faltings for k a number field. Faltings still manages to bound the number of rational points in $A_{g,d}(k)$ that belong to the same isogeny class by using the Faltings height.

2.6 The weaker finiteness statement $\mathrm{Hyp}(k, A, d, \ell)$

Tate did not actually show $\mathrm{Hyp}(k, A, \ell)$; he instead used a weaker finiteness statement:

$\mathrm{Hyp}(k, A, d, \ell)$: there exist only finitely many (up to k-isomorphism) abelian varieties B such that
(a) there exists a polarization $\lambda\colon B \to B^\vee$ where $\deg \lambda = d$; and
(b) there exists a k-isogeny $B \to A$ of ℓ-power degree.

Assuming this, he proves the theorem for one ℓ for which $F_\ell \simeq \prod \mathbf{Q}_\ell$ (Exercise: Such an ℓ always exists if k is finite) and concludes by Lemma 2.10.

We now want to explain how to adapt the proof assuming $\mathrm{Hyp}(k, A, \ell)$ to work with the weaker finiteness statement $\mathrm{Hyp}(k, A, d, \ell)$. First, pick a polarization $\varphi_{\mathscr{L}}\colon A \to A^\vee$. By assumption, F_ℓ is semisimple, so by Lemma 2.12, it suffices to show $F_\ell = C(E_\ell) =: D$. One direction is clear: $F_\ell \subset D$, since the endomorphisms E_ℓ of $V_\ell(A)$ coming from (k-rational) isogenies of A commute with the Galois action.

So we want to show $D \subset F_\ell$. If $F_\ell = \prod \mathbf{Q}_\ell$, then we have a decomposition

$$V_\ell(A) = \bigoplus V_i$$

of $V_\ell(A)$ into V_i, which correspond to simple factors of F_ℓ. But then, F_ℓ is precisely the subalgebra of $\mathrm{End}(V_\ell(A))$ consisting of endomorphisms which act as scalars on the individual V_i. So it suffices to show that D acts via scalars on all of the V_i. But there is a simple way to check this: it suffices to show that every G-stable line $L \subset V_i$ is D-stable.

Now, observe that a G-stable line $L \subset V_i$ in fact has more structure: it is isotropic for the Weil pairing

$$\begin{aligned} e^{\mathscr{L}} = e_\ell^{\mathscr{L}} \colon V_\ell(A) \times V_\ell(A) &\longrightarrow \mathbf{Q}_\ell \\ (a, a') &\longmapsto e_\ell(a, \varphi_{\mathscr{L}}(a')) \end{aligned}$$

This is symplectic and thus alternating, so it must be zero on the line L. We will use this isotropicity in the proof.

Proposition 2.18. *Suppose $F_\ell = \prod \mathbf{Q}_\ell$, and $W \subset V_\ell(A)$ is a G-subrepresentation isotropic with respect to the Weil pairing $e^{\mathscr{L}}$. Then, W is D-stable.*

Proof Sketch. The proof is by descending induction on the dimension of W. We will only show the base case, where $\dim W = g$. We go through the same proof as before, just a bit more carefully.

Let $X_n = (T_\ell(A) \cap W) + \ell^n T_\ell(A)$. As before, it corresponds to an isogeny $f_n\colon B_n \to A$ such that $f_n(T_\ell(B_n)) = X_n$, and this isogeny has degree ℓ^{ng}. The induced polarization on B_n is given by

$$f_n^*\varphi_{\mathscr{L}} = \varphi_{f_n^*\mathscr{L}} \colon B_n \xrightarrow{f_n} A \xrightarrow{\varphi_{\mathscr{L}}} A^\vee \xrightarrow{f_n^\vee} B^\vee$$

and $\deg \varphi_{f_n^*\mathscr{L}} = (\ell^{ng})^2 \cdot \deg \varphi_{\mathscr{L}}$. But for all $x, y \in T_\ell(B_n)$, we have

$$e_{B_n}^{f_n^*\mathscr{L}}(x, y) = e_{B_n}(x, f_n^\vee \varphi_{\mathscr{L}} f_n y) = e_A(f_n x, \varphi_{\mathscr{L}} f_n y) = e_A^{\mathscr{L}}(f_n x, f_n y) \in e^{\mathscr{L}}(X_n, X_n).$$

Now $e^{\mathscr{L}}(X_n, X_n) = e^{\mathscr{L}}(\ell^n T_\ell(A), X_n) \subset \ell^n \mathbf{Z}_\ell(1)$ since W is $e^{\mathscr{L}}$-isotropic, and so there exists a polarization $\varphi_{\mathscr{M}}\colon B_n \to B_n^\vee$ such that $\varphi_{f_n^*\mathscr{L}} = \varphi_{\mathscr{M}} \circ \ell^n$, and

$$\deg \varphi_{\mathscr{M}} = \frac{\deg \varphi_{f_n^*\mathscr{L}}}{\deg \ell^n} = d.$$

Thus, each B_n is represented as a point in $A_{g,d}$, so the finiteness hypothesis $\mathrm{Hyp}(k, A, d, \ell)$ is enough to guarantee that the B_n's fall into finitely many k-isomorphism classes. The rest of the proof is identical to what we had before. ☐

3 September 29: Semiabelian varieties (Brandon Carter)

The references for this lecture are [Fal86] in [CS86], [FC90], and Conrad's notes from the Stanford seminar [Con11a; Con11b].

3.1 Motivation

We start by recalling our long-term goal, which we will not be able to accomplish today:

F. *Let A be an abelian variety over a number field K. Then, there are only finitely many abelian varieties over K, up to isomorphism, that are isogenous to A.*

The idea of the proof, as given by Andrew, is to define a suitable height function on the moduli space $\mathcal{A}_g$. One way to do so is to define the *modular height* H, which is induced from the height on $\mathbf{P}^N$ for suitable N, into which $\mathcal{A}_g$ embeds. However, we run into a couple of issues when we try to use this function:

Problems 3.1.
- Given an abelian variety, how can we determine its modular height?
- How does the modular height change under isogeny?

We resolve this by doing the following:

Fix 3.2. Use another height function h, called the *(stable) Faltings height*, which changes according to a nice formula under isogeny, and isn't too different from H.

Now the hope is that there are only finitely many abelian varieties over K of fixed dimension and bounded Faltings height h, since this will suffice to show **F**. However,

Problem 3.3. There can be infinitely many abelian varieties over K of fixed dimension and bounded Faltings height h.

Example 3.4. Let $E/\mathbf{Q}$ be an elliptic curve. Then, all quadratic twists of E have the same Faltings height, but are not isomorphic.

The solution will be to only consider *semistable* abelian varieties, which in this example corresponds to throwing out all but finitely many of these quadratic twists.

3.2 Néron models

We need these to define semistable abelian varieties.

Let R be a Dedekind ring, and let $K = \mathrm{Frac}(R)$; for example, we could have $R = \mathbf{Z}_p$ and $K = \mathbf{Q}_p$.

Definition 3.5. If X is a smooth, separated scheme of finite type over K, then the *Néron model* of X is a scheme $\mathfrak{X}$ over R, such that

(*i*) $\mathfrak{X}_K \cong X$; and

(*ii*) (Néron mapping property) For every smooth scheme $\mathfrak{Y}$ over R, and for every map $\mathfrak{Y}_K \to \mathfrak{X}_K$, there is a unique extension $\mathfrak{Y} \to \mathfrak{X}$, i.e., we can fill in the diagram below:

$$\begin{tikzcd} \mathfrak{X}_K \arrow[r, hook] \arrow[d] & \mathfrak{X} \arrow[d, dashed] \\ \mathfrak{Y}_K \arrow[r, hook] & \mathfrak{Y} \end{tikzcd}$$

Néron models do not exist in general, but they do for abelian varieties:

Fact 3.6. Néron models exist if X is an abelian variety, and are automatically unique by the Néron mapping property (*ii*).

We fix some "non-standard" notation:

Notation 3.7. $N(A)$ denotes the Néron model of A, and $N(A)^\circ$ denotes the open subscheme that restricts to the connected component of the identity in each fiber.

If you've never seen Néron models before, you can look at Andrew's notes [Sno13, Lec. 9], where he computed explicit examples for some curves.

Note 3.8. $N(A)^\circ$ is a group scheme over R.

3.3 Semiabelian varieties and semistable reduction

To define semiabelian varieties, we use the following old theorem of Chevalley, which says that any group scheme over a perfect field can be "split apart" into a linear algebraic group and an abelian variety.

Theorem 3.9 (Chevalley)**.** *If G is a smooth, connected k-group scheme, and k is perfect, then there exists a short exact sequence*

$$1 \longrightarrow H \longrightarrow G \longrightarrow B \longrightarrow 1,$$

where H is a linear algebraic group, and B is an abelian variety.

With this, we can make the following definition:

Definition 3.10. Let S be any base scheme (or stack). Then, a *semiabelian variety* of relative dimension g over S is a smooth group scheme $A \to S$, whose (geometric) fibers are connected of dimension g, and are extensions of abelian varieties by (possibly zero-dimensional) tori, that is, each fiber has no unipontent components in the Chevalley decomposition from Theorem 3.9.

Remark 3.11. In Definition 3.10, $A \to S$ is a scheme in the relative sense: the total space A may not actually be a scheme.

Remark 3.12. Proper semiabelian varieties are abelian.

Recall that we are interested in abelian varieties over fraction fields $K = \mathrm{Frac}(R)$ of a Dedekind ring R.

Definition 3.13. Let A be an abelian variety over K. Then, we say A has *semistable reduction* if $N(A)^\circ$ is a semiabelian variety.

We should think of semistable reduction as the analogue of non-additive reduction for elliptic curves.

Example 3.14. An elliptic curve E over $\mathbf{Q}$ or a number field has three types of reduction at each place:

- *Good reduction*, where the special fiber of $N(E)$ is an elliptic curve;
- *Multiplicative reduction*, where the special fiber of $N(E)$ is $\mathbf{G}_m$; and
- *Additive reduction*, where the special fiber of $N(E)$ is $\mathbf{G}_a$.

In the first case, the special fiber is an abelian variety, and in the second, we have a torus. Thus, places of additive reduction are where the elliptic curve does not have semistable reduction.

The way we get around this issue of non-semistable reduction is to use the following old theorem of Grothendieck:

Theorem 3.15 (Semistable reduction)**.** *Every abelian variety has semistable reduction after finite base change.*

The point is that you have non-semistable reduction at only finitely many places, which is fixed by passing to a finite base field extension.

The reason why taking more finite base changes does not cause any issues is the following:

Fact 3.16. Taking Néron models commutes with base change if A has semistable reduction.

We will use these facts with Jacobians of curves: Theorem 3.15 implies that they will have semistable reduction after finite base change, and Fact 3.16 implies that the property of having semistable reduction does not change when passing to futher base changes.

Example 3.17 (Deligne–Mumford)**.** Let $C \to S$ be a stable curve of genus g, that is, a fibration of curves such that for all $s \in S$, the fiber C_s satisfies the following properties:

- C_s is (geometrically) connected;
- C_s has (arithmetic) genus $g \geq 2$;
- C_s has at worst ordinary double points (nodes); and
- each rational component of C_s contains at least three points which also lie in other components.

Then, $\mathrm{Pic}^\circ(C/S) \to S$ is semi-abelian of relative dimension g.

Note that the last defining property of a stable curve is equivalent to saying that the automorphism groups of each fiber C_s are finite.

3.4 The Hodge line bundle $\omega_{A/S}$

Definition 3.18. Let $p\colon A \to S$ be semiabelian, and let $s\colon S \to A$ be the zero section. Then, we define a sheaf $\omega_{A/S}$ on S by

$$\omega_{A/S} \coloneqq s^*\left(\bigwedge^g \Omega^1_{A/S}\right),$$

which is a line bundle. $\omega_{A/S}$ is sometimes called the *Hodge bundle*.

The Faltings height will come from defining a metric on this line bundle $\omega_{A/S}$.

Remarks 3.19.

(*a*) If p is proper, then $\omega_{A/S} = p_*(\Omega^g_{A/S})$.

(*b*) $\omega_{A/S}$ commutes with base change.

(*c*) If $A = \operatorname{Pic}^\circ(C/S)$ where $q\colon C \to S$ is a stable curve, then $\omega_{A/S} \cong \bigwedge^g q_*(\omega_{C/S})$, where $\omega_{C/S}$ is the relative dualizing sheaf—this is just saying that one-forms on a curve are the same as one-forms on the Jacobian.

(*d*) If $S = \operatorname{Spec} \mathbf{C}$ and p is proper, i.e., A is an abelian variety over $\mathbf{C}$, then $\omega_{A/\mathbf{C}} = \Gamma(A, \Omega^g_{A/\mathbf{C}})$, and it has a canonical Hermitian pairing, defined by

$$\langle \omega, \omega' \rangle = \frac{1}{2^g}\int_A |\omega \wedge \overline{\omega}'| = \left(\frac{i}{2}\right)^g \int_A \omega \wedge \overline{\omega}'$$

for $\omega, \omega' \in \Gamma(A, \Omega^g_{A/C})$.

3.5 A Néron mapping property for semiabelian varieties

The following is a very useful property of semiabelian varieties, which Bhargav will use next time. It is called "Faltings' lemma with proof in 1000 ways" in [Con11a].

Proposition 3.20 (Néron mapping property for semiabelian varieties)**.** *Let S be a noetherian, normal scheme, and let $U \subset S$ be some open dense subscheme. Then, the restriction functor $A/S \mapsto A_U/U$, which sends a semiabelian variety A over S to its pullback over U, is fully faithful.*

Note that in this Proposition, the morphisms in each category are morphisms of semiabelian varieties over the base scheme; in particular, they are group homomorphisms. If S is a DVR, this statement is similar to the Néron mapping property from before.

We give a sketch of a proof; see [FC90, Ch. I, Prop. 2.7] for details.

Proof Sketch. Let $\phi\colon (A_1)_U/U \to (A_2)_U/U$ be a map defined on the pullback of A_1 to the pullback of A_2. Let Γ_ϕ denote the graph of ϕ, and let $\widetilde{\Gamma}_\phi$ be its normalization:

$$\begin{tikzcd}
\widetilde{\Gamma}_\phi \arrow[d, "\text{normalization}"'] \arrow[dr] \arrow[drr, bend left, "p_1"] & & \\
\Gamma_\phi \arrow[r, hook] & A_1 \times A_2 \arrow[r] & A_1
\end{tikzcd}$$

Then, to show that ϕ extends to a map $A_1/S \to A_2/S$, it suffices to show that the map $p_1\colon \widetilde{\Gamma}_\phi \to A$ is proper with one-point fibers, since then, the normality of S would imply that p_1 is an isomorphism by Zariski's Main Theorem. (Note that we will use this "graph trick" often.)

By the valuative criterion (plus minor considerations, in particular that of fiber size), you can reduce to the case where $S = \operatorname{Spec} R$ for a DVR R, and so the map ϕ is defined at the generic point $\eta \in \operatorname{Spec} R$. There are then three possibilities for what the generic fiber $(A_2)_\eta$ looks like:

Case 1. $(A_2)_\eta$ is an abelian variety.

In this case, $A_2 = N((A_2)_\eta)^\circ$, and so the map extends by the usual Néron mapping property.

Case 2. $(A_2)_\eta$ is a torus.

After base change, you can assume that $(A_2)_\eta = \mathbf{G}_m$ by passing to a base extension until the torus splits, and treating each factor separately; a descent argument shows you can descend from this base extension to the original field of definition. Then, ϕ_η is defined by a regular function f on $(A_1)_\eta$, with $\operatorname{div}(f)$ supported only on the special fiber. With a little bit of work (using that ϕ_η is a group homomorphism), this implies that $\operatorname{div}(f) = 0$.

Case 3. The general case.

$(A_2)_\eta$ is an extension of some abelian variety B_η by a torus T_η. Let $B = N((A_2)_\eta)^\circ$, and so by Grothendieck's Theorem 3.15, we may assume B is semiabelian after a finite base change. Then, $(A_2)_\eta \to B_\eta$ extends to $A_2 \to B$ by the Néron mapping property, and you can also extend $(A_1)_\eta \to (A_2)_\eta \to B_\eta$, after a little bit of work. Finally, you have to patch together the extensions on T_η and on B_η somehow. $\square$

Remark 3.21. Case 1 is the main case we will care about, and is possibly the only case we really need, since we think we only need Proposition 3.20 when A_2 is the universal family of abelian varieties.

3.6 Gabber's Lemma

Let $\overline{\mathcal{M}}_g$ be the moduli stack of stable curves of genus g, which we recall is proper over $\mathbf{Z}$. Let $\mathcal{A}_g$ be the moduli stack of principally polarized abelian varieties (PPAV's) of dimension g, with A_g the corresponding coarse moduli space. Recall that $\mathcal{A}_g$ is *not* proper, but some power of $\omega_{A/\mathcal{A}_g}$ gives a very ample line bundle on $A_g/\mathbf{Q}$, where

$$\begin{tikzcd} A \arrow[d] \\ \mathcal{A}_g \end{tikzcd}$$

is the universal abelian variety, and $\omega_{A/\mathcal{A}_g}$ is the Hodge bundle that we defined in Definition 3.18. Note that this implies $A_g/\mathbf{Q}$ is quasi-projective, with the embedding into $\mathbf{P}^N_{\mathbf{Q}}$ given by $(\omega_{A/\mathcal{A}_g})^{\otimes r}$.

Now let $\overline{A_g}/\mathbf{Q} \subset \mathbf{P}^N_{\mathbf{Q}}$ be the closure of $A_g/\mathbf{Q}$ in $\mathbf{P}^N_{\mathbf{Q}}$, and let $\overline{A_g}/\mathbf{Z}$ denote the closure of $\overline{A_g}/\mathbf{Q}$ in $\mathbf{P}^N_{\mathbf{Z}}$. Then, letting $\mathscr{M} = \mathcal{O}(1)$ on $\overline{A_g}/\mathbf{Z}$, we see that $\mathscr{M}$ extends $(\omega_{A/\mathcal{A}_g})^{\otimes r}$. The payoff of this eventually will be that the modular height comes from a metric on $\mathcal{O}(1)$, which we can compare with the Faltings height.

The issue with this naïve compactification of $A_g/\mathbf{Z}$ is that the universal family of abelian varieties does not necessary extend to a universal family of semiabelian varieties on the compactification, and in particular the naïve compactification does not have a natural moduli-theoretic interpretation. The content of Gabber's Lemma below is that there does exist a compactification that satisfies this property. Note that [Fal86] instead compactifies the stack $\mathcal{A}_g/\mathbf{Z}$.

Theorem 3.22 (Gabber's Lemma)**.** *Let S be a noetherian scheme, and let $f\colon X \to S$ be a separated S-scheme of finite type. Let $u\colon A \to X$ be an abelian scheme, i.e., a proper semiabelian scheme. Then, there exists a proper surjective map $\pi\colon X' \to X$ and an open immersion of X' into a proper S-scheme $\overline{X'}$, such that the pullback $u_{X'}\colon A_{X'} \to X'$ of the family u extends to a semiabelian scheme $\overline{u}\colon A' \to \overline{X'}$.*

We can organize this data into the following commutative diagram:

$$\begin{tikzcd} A \arrow[d, "u"'] & A_{X'} \arrow[l] \arrow[r, hook] \arrow[d, "u_{X'}"] \arrow[dl, phantom, very near start, "\llcorner"] \arrow[dr, phantom, very near start, "\lrcorner"] & A' \arrow[d, "\bar{u}"] \\ X \arrow[dr, "f"'] & X' \arrow[l, two heads, "\pi"'] \arrow[r, hook] \arrow[d] & \overline{X'} \arrow[dl] \\ & S & \end{tikzcd}$$

where both squares are cartesian.

Remarks 3.23. If S is a Dedekind scheme, then Gabber's Lemma is basically the semistable reduction theorem 3.15, but in higher dimension. Also, we will mostly only be interested when S is of finite type over $\mathbf{Z}$, e.g., when $S = \operatorname{Spec} R$ for a number ring R.

Since every abelian scheme is a quotient of a Picard scheme, with some work, you can reduce to the case of curves (we will return to this step later):

Theorem 3.24 (Gabber's Lemma for curves)**.** *Let S and X be as in the statement of Theorem 3.22. Let $u\colon C \to X$ be a smooth, proper curve, whose geometric fibers are connected curves of genus $g \geq 2$. Then, there exist $\pi\colon X' \to X$ and $\overline{X'}$ as in Theorem 3.22, such that the pullback $u_{X'}\colon C_{X'} \to X'$ of the family u extends to a family of semistable curves $\overline{u}\colon C' \to \overline{X'}$.*

Since the family $\overline{u}$ will come from a pullback of the universal family on $\overline{\mathcal{M}}_g$, it will in fact be a family of stable curves.

Proof. By Chow's lemma, we can reduce to the case where X is quasi-projective, i.e., X is an open subset of a projective scheme $\overline{X}$. Let $\mathcal{M}_S = \overline{\mathcal{M}}_g \times_{\mathbf{Z}} S$. Then, by the universality of the universal family, the morphism $u\colon C \to X$ corresponds to a morphism $X \to \mathcal{M}_S$.

We now use the "graph trick" again. Let $\overline{X'}$ denote the closure of the graph of $X \to \mathcal{M}_S$ in $\overline{X} \times \mathcal{M}_S$. Then, the morphism $\overline{X'} \to \mathcal{M}_S$ extends the morphism $X \to \mathcal{M}_S$, and by pulling back the universal family on $\mathcal{M}_S$, this extension gives a family $C' \to \overline{X'}$ of stable curves that extends the family $C \to X$. □

There is one problem with this proof:

Problem 3.25. C' is a proper Deligne–Mumford stack, not a scheme.

To fix this problem, we use a version of Chow's lemma for Deligne–Mumford stacks (this is true for Artin stacks as well):

Chow's Lemma for stacks 3.26. *Let $\mathcal{M} \to S$ be a separated Deligne–Mumford stack of finite type over a noetherian scheme S. Then, there exists a proper surjective map $X \to \mathcal{M}$, where X is a quasi-projective S-scheme, such that X is projective if and only if $\mathcal{M}$ is proper.*

So in the proof of Gabber's Lemma for curves (Theorem 3.24), pulling back to the scheme X provided by this version of Chow's Lemma gives the family of curves that we want.

We now describe how to get the full version of Gabber's Lemma (Theorem 3.22) from the version for curves (Theorem 3.24).

Full proof. Let η denote the generic point of X, and let A_η be the abelian variety over η. Let $\operatorname{Pic}^\circ(C_\eta) \to A_\eta$ be a realization of A_η as a quotient of a Picard scheme. Then, there exists another abelian variety B_η such that $A_\eta \times B_\eta$ is isogenous to $\operatorname{Pic}^\circ(C_\eta)$, by looking at duals and using Poincaré complete reducibility (Theorem 2.5). Let $\operatorname{Pic}^\circ(C_\eta) \to A_\eta \times B_\eta$ be this isogeny; spreading out over $U \subset X$ gives an isogeny

$$\operatorname{Pic}^\circ_{C_U/U} \longrightarrow A_U \times B_U \tag{2}$$

of abelian schemes.

Using Gabber's lemma for curves (Theorem 3.24), $\operatorname{Pic}^\circ_{C_U/U} \to U$ extends to a family $C' \to \overline{X'}$, that is, we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Pic}^\circ_{C_U/U} \arrow[d] & (\operatorname{Pic}^\circ_{C_U/U})_{X'} \arrow[l] \arrow[r, hook] \arrow[d] \arrow[dl, phantom, "\llcorner"] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Pic}^\circ_{C'/\overline{X'}} \arrow[d] \\
U & X' \arrow[l, two heads] \arrow[r, hook] & \overline{X'}
\end{tikzcd}$$

The Néron mapping property for semiabelian varieties (Proposition 3.20) now implies that the pullback of the isogeny (2) to X' extends to an isogeny

$$\operatorname{Pic}^\circ_{C'/\overline{X'}} \longrightarrow A_{\overline{X'}} \times B_{\overline{X'}},$$

where you have to check that A_U and B_U extend to some abelian varieties $A_{\overline{X'}}$ and $B_{\overline{X'}}$ over $\overline{X'}$. This allows you to treat the univeral family of abelian varieties as if it were the universal family of Jacobians of curves; see [Con11a, §§3–5] for details. □

There is an interesting Corollary, which Bhargav will explain next week. The main idea is that we have constructed two compactifications of $A_g/\mathbf{Q}$, and you can in fact construct a map from the Gabber compactification to $\overline{A_g}/\mathbf{Z}$. Then, the pullback of the line bundle $\mathscr{M}$ via this map will be close enough to the power $(\omega_{A/\mathcal{A}_g})^{\otimes r}$ of the Hodge bundle used to define $\overline{A_g}/\mathbf{Z}$, such that the modular height and the Faltings height won't be too different.

4 October 6: The Faltings height and basic finiteness theorems (Bhargav Bhatt)

Today's goal is to explain the basic finiteness theorem of Faltings [Fal86, Thm. 1]. References for this talk are Deligne's Seminaire Bourbaki article [Del85] and Silverman's article [Sil86] in Cornell–Silverman [CS86] for background material on heights.

We start by recalling what we did last time. Gabber's Lemma 3.22 gives an analogue over $\mathbf{Z}$ of the following picture which exists over the complex numbers. Let A_g denote the moduli space of g-dimensional principally polarized abelian varieties over the complex numbers. Then, we have the following diagram:

$$\begin{tikzcd} A_g \arrow[r, hook] \arrow[d, equal] & \overline{A_g} \arrow[d, "\pi"', "\text{proper surjective}"] \\ A_g \arrow[r, hook] & A_g^* \end{tikzcd}$$

where

(1) A_g is the moduli space of g-dimensional principally polarized abelian varieties, which carries
- a *universal abelian variety*
$$\begin{tikzcd} A^u \arrow[r] & A_g \arrow[l, bend left, "e"] \end{tikzcd}$$
where e is a (left) section, and
- a *Hodge bundle* $\omega_{A_g} \coloneqq e^*(\Omega^g_{A^u/A_g})$.

(2) $\overline{A_g}$ is the toroidal compactification of A_g, which carries
- a *universal semiabelian variety*
$$\begin{tikzcd} \overline{A^u} \arrow[r] & \overline{A_g} \arrow[l, bend left, "e"] \end{tikzcd}$$
where e is a (left) section, and
- a *Hodge bundle* $\omega_{\overline{A_g}} \coloneqq e^*(\Omega^g_{\overline{A^u}/\overline{A_g}})$.

(3) A_g^* is the minimal compactification of A_g, which carries an ample line bundle $\mathcal{O}(1)$, which extends ω_{A_g}.

The basic fact is that the two compactifications $\overline{A_g}$ and A_g^* are compatible:

Fact 4.1. $\pi^*\mathcal{O}(1) \cong \omega_{\overline{A_g}}$.

We will show today that if $K/\mathbf{Q}$ is a finite extension, and if A/K is an abelian variety, then we can define its *Faltings height* $h_F(A) \in \mathbf{R}$. We first give the idea for its construction:

Idea 4.2. For A a semistable principally polarized abelian variety, the *Faltings height* $h_F(A)$ is the "degree" of the classifying map
$$\operatorname{Spec}(\mathcal{O}_K) \longrightarrow \overline{A_g}$$
for the Néron model $\operatorname{Ner}(A)/\mathcal{O}_K$.

The issue is that $\operatorname{Spec}(\mathcal{O}_K)$ is not proper, and so the usual notion of degree doesn't make sense.

The application of the Faltings height in the following:

Theorem 4.3. *Fix $g \geq 1$, a composite number $m \geq 3$, and a constant $C > 0$. Then, the set*

$$\left\{ \begin{array}{c} \text{Principally polarized abelian varieties} \\ A/K \text{ of dimension } g\text{, with level structure} \\ (\mathbf{Z}/m)^{2g} \xrightarrow{\sim} A[m] \text{ and Faltings height } h_F(A) < C \end{array} \right\} \text{ is finite.}$$

Remark 4.4. Faltings actually showed a slightly different statement. Instead of assuming that the abelian varieties have level structure, he assumes they have semistable models. Our version has the advantage that the moduli spaces are actually schemes.

We also give a remark as to why you should expect the Faltings height to appear here. First, you have to have some restrictions on the abelian varieties you are counting; see Example 3.4 from Brandon's talk, where he mentioned that all quadratic twists of an elliptic curve have the same Faltings height. To get rid of issues like this, we need to bound the ramification somehow.

We can also think of this statement as the arithmetic analogue of a previous result of Faltings, where $\operatorname{Spec} \mathcal{O}_K$ is replaced by a complex curve, the Faltings height is replaced by the degree of the map to $\overline{A_g}$, and the condition on the set of abelian varieties is some specification of where the semiabelian variety has good reduction. You essentially need an analogue of the Tate conjecture in the complex setting to make this make sense.

To show the finiteness statement, we could try to show things directly about the structure of A_g, but this requires us to understand well the different compactifications of A_g. Instead, we will spend time talking about heights and logarithmic heights, which will give us enough information to prove Faltings's result.

There is a formalism of heights that exists very generally: any time you have a projective variety over a number field, you have a naïve height. The idea is that we will make the Faltings height similar enough to the naïve one to get the finiteness statements we need.

Let $K/\mathbf{Q}$ be a finite extension, and let M_K be the set of all absolute values on K extending the standard absolute values on $\mathbf{Q}$ (so $|p|_v = 1/p$). Given $v \in M_K$, set $\|\cdot\|_v = |\cdot|_v^{[K_v:\mathbf{Q}_v]}$.

4.1 Heights on $\mathbf{P}^n$

Definition 4.5. The *height* of a K-point in $\mathbf{P}^n$ is defined by the function

$$\begin{aligned} H_K\colon \mathbf{P}^n(K) &\longrightarrow \mathbf{R} \\ [x_i] &\longmapsto \prod_{v \in M_K} \max\{\|x_i\|_v\} \end{aligned}$$

Example 4.6. If $K = \mathbf{Q}$, and $P = [x_i]$, $x_i \in \mathbf{Z}$, $\gcd(x_i) = 1$, then $H_K(P) = \max\{|x_i|\}$.

Remark 4.7.

(1) This is well-defined: it does not depend on the homogeneous coordinates, by the product formula.

(2) This works pretty well under field extension, except you have to keep track of the degree of the extension since we are using $\|\cdot\|$ in the definition: if L/K is finite, then

$$H_L|_{\mathbf{P}^n(K)} = H_K^{[L:K]}.$$

As a result of this, if you rescale everything, you can "make all of these compatible," that is, you get a well-defined function

$$H\colon \mathbf{P}^n(\overline{\mathbf{Q}}) \longrightarrow \mathbf{R}$$

such that $H|_{\mathbf{P}^n(K)} = H_K^{1/[K:\mathbf{Q}]}$. You can also define the *logarithmic height* $h = \log H\colon \mathbf{P}^n(\overline{\mathbf{Q}}) \to \mathbf{R}$.

Observation 4.8. If you fix $C > 0$, then the set

$$\{P \in \mathbf{P}^n(\mathbf{Q}) \mid H(P) < C\} \text{ is finite.}$$

You can bootstrap this result to get a similar result for other number fields:

Theorem 4.9. *Fix $C > 0$ and $K/\mathbf{Q}$. Then, the set*

$$\{P \in \mathbf{P}^n(K) \mid H_K(P) < C\} \text{ is finite.}$$

Essentially what you do is reduce to the case of $\mathbf{P}^1$: the coefficients of the minimal polynomial of a point $[1 : x]$ determine the absolute value of x, and then you use the previous result.

Using this height function on projective space, you can get one on arbitrary projective varieties. But we can do something a bit better.

4.2 Heights on projective varieties

We can make a definition that does not depend on the embedding.

Let $V/\overline{\mathbf{Q}}$ be a projective variety, and consider a map $f\colon V \to \mathbf{P}^n$. We can pullback the height function on $\mathbf{P}^n$ to get a map on $V(\overline{\mathbf{Q}})$: $h_f\colon V(\overline{\mathbf{Q}}) \to \mathbf{R}$, $x \mapsto h(f(x))$. This is obviously well-defined, but it only depends on the pullback of $\mathcal{O}(1)$ as a line bundle, *not* on the specific linear system you are using. More precisely,

Proposition 4.10. *Say $f\colon V \to \mathbf{P}^n$ and $g\colon V \to \mathbf{P}^m$ are such that $f^*\mathcal{O}(1) \cong g^*\mathcal{O}(1)$. Then $h_f = h_g + O(1)$, that is, $|h_f - h_g|$ is bounded.*

This means that while h_f and h_g don't literally define the same function, they are bounded by each other by a constant amount.

This Proposition implies that if we start with a globally generated line bundle, then you get a well-defined height function out of it, up to some constant term. We want to say that this operation gives a group homomorphism from line bundles to functions. To do that, we need to ignore bounded functions somehow.

Definition 4.11. $\mathcal{H}(V) = \{\text{all function } h\colon V(\overline{\mathbf{Q}}) \to \mathbf{R}/\text{bounded functions}\}$.

The Proposition then says that there is a well defined map

$$\begin{aligned} \{\text{globally generated line bundles on } V\} &\longrightarrow \mathcal{H}(V) \\ L &\longmapsto h_L \end{aligned}$$

which Silverman calls the "Height Machine" [Sil86, Thm. 3.3].

This construction has some nice properties:

Properties 4.12.

1. $L \mapsto h_L$ sends tensor products to addition: the image of X under the map defined by a tensor product of two globally generated line bundles has coordinates which are products of the coordinates in the image of X under each line bundle individually, since taking a tensor product of line bundles corresponds to a Segre embedding. Multiplying coordinates corresponds to adding log heights. Thus, you get a group homomorphism

$$\begin{aligned} \operatorname{Pic}(V) &\longrightarrow \mathcal{H}(V) \\ L &\longmapsto h_L = h_A - h_B \end{aligned}$$

   by writing a line bundle as a difference of very ample line bundles, i.e., $L = A \otimes B^{-1}$ for A, B very ample.
2. This function $L \mapsto h_L$ is compatible with pullback (look at ample line bundles again).
3. If L is ample, and $C > 0$, and $K/\mathbf{Q}$ finite, V defined over K, then

$$\{x \in V(K) \mid h_L(x) < C\} \text{ is finite.}$$

Proof. If L is very ample, reduce to $\mathbf{P}^n$. In general, use linearity (up to $O(1)$): $h_{L^{\otimes n}} = n \cdot h_L$. $\square$

To connect the Height Machine with the Faltings height, we need to reinterpret everything in terms of metrics on line bundles, as Arakelov did.

4.3 Metrized line bundles on $\mathcal{O}_K$

Arakelov's insight was that if you keep track of the places at infinity, then you can define the degree of a line bundle. This corresponds to a metric on the bundle. The upshot is that you don't need properness to get degrees.

Let $K/\mathbf{Q}$ be a finite extension, where $\mathcal{O}_K$ is the ring of integers.

Definition 4.13. A *metrized line bundle* on $\mathrm{Spec}(\mathcal{O}_K)$ is an invertible $\mathcal{O}_K$-module M, plus the data of a norm $|\cdot|_v$ on $M \otimes K_v$ for each $v \in M_K^\infty$, the set of infinite places.

If you have a metrized line bundle, you can get a degree out of it, called the Arakelov degree.

Definition 4.14. If $(M, |\cdot|_v)$ is a metrized line bundle, we define the *Arakelov degree* as follows:

$$\deg(M, |\cdot|_v) = \deg(M) \coloneqq \log(\#M/(\mathcal{O}_K \cdot m)) - \sum_{v \in M_K^\infty} \log\|m\|_v,$$

where $0 \neq m \in M$ is any nonzero element in M.

Idea 4.15. We have the following chart of analogies:

$\mathrm{Spec}(\mathcal{O}_K) \cup M_K^\infty$	a complete curve C/K
a metrized line bundle M	a line bundle L
$\deg(M, \lvert\cdot\rvert_v)$	$\deg(L) = \dim_k(L/\mathcal{O}_C \cdot f)$ for $0 \neq f \in H^0(C, L)$

Remark 4.16. For v a finite place, $M \otimes \mathcal{O}_{K_v} \subset M \otimes K_v$ provides a norm $\|\cdot\|_v$ on $M \otimes K_v$, with unit ball $M \otimes \mathcal{O}_{K_v}$. Because of this, you can rewrite the Arakelov degree as follows:

$$\deg(M) = \sum_{v \in M_K} - \log\|m\|_v$$

which measures how far m is from generating the lattice. Scaling m by a global function gives the same degree by the product formula.

The most important example of a metrized bundle for us is the Hodge bundle.

Example 4.17. Let A/K be an abelian variety, and let $\mathrm{Ner}(A)/\mathcal{O}_K$ be the Néron model. This is a proper smooth group scheme of dimension g, and the Néron model is still smooth, but not proper. Then, you get the Hodge bundle $\omega_A = e^*(\Omega^g_{\mathrm{Ner}(A)/\mathcal{O}_K}) \in \mathrm{Pic}(\mathcal{O}_K)$. The claim is that this lifts to a metrized line bundle.

We just have to figure out what you do at the infinite places, which correspond to A/K. But in this case, all you have to do is look at the hermitian metric on $\omega_A \otimes_K \mathbf{C}$ for all $K \hookrightarrow \mathbf{C}$, which gives a norm via integration:

$$|\alpha| = \frac{1}{2^g} \int_{A(\mathbf{C})} |\alpha \wedge \overline{\alpha}|. \tag{3}$$

This structure is what we use to define the Faltings height.

Now if $\mathrm{Ner}(A)$ is semiabelian, we set the *Faltings height* to be $h_F(A) = \deg(\omega_A)$, where ω_A has the structure of a metrized line bundle by using this metric (3) at the infinite places.

The relationship between the naïve height and the Faltings height is actually quite easy.

4.4 Heights via metrized line bundles

We now generalize the notion of a metrized line bundle on a number field to a variety over that number field.

Let $K/\mathbf{Q}$ be a number field, V/K a projective variety, and L a line bundle on V, which we will often assume is ample but not for now.

Then, if $x \in V$, the fiber L_x of L over x is a vector space, which is an element of $\mathrm{Pic}(\kappa(x))$. What we need to do is to figure out what a metrized line bundle on V is.

Definition 4.18. Fix a place $v \in M_K$ (in practice, it will be an infinite place). A *v-adic metric* on L is a norm $|\cdot|_v$ on L_x for all $x \in V(K_v)$ which varies continuously in x in the v-adic topology, where we recall that the v-adic topology is the weakest topology such that for all local sections $f \in H^0(U, L|_U)$ on an open set $U \subset V$, the function $U(K_V) \to \mathbf{R}$ sending $x \mapsto |f_x|_v$ is continuous.

If we have such data at every place v, we say that L is a *metrized line bundle*.

You are supposed to think of this number $|f_x|_v$ as the v-adic distance from the zero locus of f.

Example 4.19. If $V = A_g$, $L = \omega_{A_g}$ is the Hodge bundle, and $v \in M_K^\infty$, you get a canonical $|\cdot|_v$ on L_x for all $x \in A_g(K_v)$ (as before).

Now there are two things going on: given a point in $\mathbf{P}^n$, we have a height. Given a point in a variety with a metrized line bundle, we have another notion. The point is that the two are actually the same! This is where the connection starts.

The following Lemma is very crucial but very fun. There are not many choices of metrics on a line bundle.

Lemma 4.20 (Comparison)**.** *Fix $v \in M_K^\infty$, $L \in \operatorname{Pic}(V)$, V projective. Suppose $|\cdot|_v, |\cdot|'_v$ are two v-adic metrics on L. Then, they are the same up to constants: there exist $C_1, C_2 > 0$ such that*

$$C_1|\cdot|_v \leq |\cdot|'_v \leq C_2|\cdot|_v,$$

and so the asymptotic behavior is the same.

Proof. For each $x \in V(K_v)$, choose a nonzero vector $f_x \in L_x$ non-zero. Then, the ratio $|f_x|_v/|f_x|'_v$ is actually well-defined (independent of f_x), since the norms have to work well with multiplication by scalars in $\kappa(x)$. Then, you get a continuous map $F\colon V(K_V) \to \mathbf{R}$ given by the formula $x \mapsto |f_x|_v/|f_x|'_v$, since you can pick f that works on an open subset just by trivializing the line bundle. Now you are more or less done: $V(K_V)$ is compact since its points are the K_V-points of a projective variety, so this map has bounded image. You then get the assertion of the Lemma. □

Corollary 4.21. *Fix v-adic metrics on $\mathcal{O}_{\mathbf{P}^n}(1)$ for all $v \in M_K^\infty$. Let $x \in \mathbf{P}^n(K)$ be given, with naïve height $h(x)$. The point x gives a map $\overline{x}\colon \operatorname{Spec}(\mathcal{O}_K) \to \mathbf{P}^n$ obtained by the valuative criterion, which turns $\overline{x}^*\mathcal{O}_{\mathbf{P}^n}(1)$ into a metrized line bundle on $\mathcal{O}_K$. Then, the naïve height and the Arakelov degree of $\overline{x}^*\mathcal{O}_{\mathbf{P}^n}(1)$ satisfy*

$$\deg(\overline{x}^*\mathcal{O}_{\mathbf{P}^n}(1)) = h(x) + O(1).$$

We apply this in the case where $\mathbf{P}^n$ is replaced by A_g, and the left-hand side is replaced by the Faltings height.

Proof. Use the Lemma, and calculate both sides. □

If A_g were compact, then the same arguments would work. Since it isn't, we have to generalize the theory so far to include open varieties.

4.5 Heights on open varieties

Let V/K be a projective variety, $Z \subset V$ a closed subvariety, and $U = C \setminus Z$ the open dense complement. Let $v \in M_K^\infty$.

So far, we don't have a good notion of heights for non-compact semiabelian varieties. What we will say is that basic finiteness results are still okay if heights don't increase too quickly as you go toward the boundary.

We first need some sort of distance function that tells you how far you are from a point in Z.

Definition 4.22. A *logarithmic distance function* is a map $d_Z\colon U(K_V) \to \mathbf{R}_{\geq 0}$ that locally looks like the log of a distance, i.e., if $f_1, \ldots, f_r$ are the local defining equations of Z on some open $W \subset V$, then

$$|d_Z(x) - \log^+ \min_j |f_j(x)|_v^{-1}|$$

extends to a bounded function on W.

Everyone uses a different normalization; we follow [Sil86]. $\log^+$ is zero when it otherwise would not be defined.

Remark 4.23. d_Z always exists, and any choices are $O(1)$ apart from each other.

The goal now is to figure out what it means to grow logarithmically as we go toward the boundary.

Definition 4.24. Fix V, Z, U as before, and pick $L \in \mathrm{Pic}(V)$. Fix a metric $|\cdot|'_v$ on $L|_U$. This metric has *logarithmic singularities* if for any other metric $|\cdot|_v$ that exists on the entirety of L, the difference of the two metrics grows logarithmically, that is, there exist constants $C_1, C_2 > 0$ such that

$$\max\{|\cdot|_v/|\cdot|'_v,\ |\cdot|'/|\cdot|_v\} \leq C_1 \cdot (d_Z + 1)^{C_2}$$

as functions on $U(K_v)$.

With this definition, the key result is the following finiteness result for the open case (see [Sil86]; the proof is easy once you know the definition):

Theorem 4.25 (Faltings)**.** *Fix V, Z, U, L as above. Assume L is very ample, and fix a v-adic metric $|\cdot|'_v$ on $L|_U$ with log singularities along the boundary $V \setminus U$ (for all $v \in M_K^\infty$). Then,*

$$\{x \in U(K) \mid h_{L,|\cdot|'_v} < C\} \text{ is finite.}$$

In the case the boundary is empty, this is the same theorem as before. The point is that you can still get boundedness in our open situation, when the metric grows logarithmically as you go toward the boundary.

We do want to connect this back to A_g.

Proof of basic finiteness theorem for abelian varieties. We have the diagram

$$\begin{tikzcd} A_g \arrow[r, hook] \arrow[d, equal] & \overline{A_g} \arrow[d, "\pi"] \\ A_g \arrow[r, hook] & A_g^* \end{tikzcd}$$

We need two facts from [Del85]:

Facts 4.26.

a) $(\mathcal{O}(1)|_{A_g}, |\cdot|^{\mathrm{can}}_{A_g})$ has log singularities along $A_g^* \setminus A_g$. This you prove by looking at the explicit construction using the Siegel upper half plane, when doing the toroidal compactification.

b) $\pi^*\mathcal{O}(1) = \omega_{\overline{A_g}}$.

(a) plus the formalism of heights implies

$$\{x \in A_g(K) \mid h_{\mathcal{O}(1)|_{A_g},|\cdot|'_v} < C\} \text{ is finite.}$$

We are not done, since we need to know that the pullback of the line bundle in the h above actually pulls back to the Hodge bundle.

(b) implies that given $x\colon \mathrm{Spec}(K) \to A_g$, with extension $\overline{x}\colon \mathrm{Spec}(\mathcal{O}_K) \to A_g^*$ obtained via the valuative criterion, we have that $\overline{x}^*\mathcal{O}(1) = \omega_A$ using Gabber's Lemma 3.22. □

5 October 13: The Tate–Raynaud theorem (Kannappan Sampath)

References for this talk are [Tat67; Fon82; Bin12].

5.1 The Hodge–Tate decomposition

We first state the following theorem of Faltings, which gives a Hodge-like decomposition in the p-adic setting, called the Hodge–Tate decomposition:

Theorem 5.1 (Faltings)**.** *If K is a p-adic field, and X is a non-singular projective variety over K, then letting $\mathbf{C}_p = \widehat{\overline{K}}$, we have the following G_K-invariant isomorphisms:*

$$(\mathbf{C}_p(j) \otimes H^i_{ét}(X_{\overline{K}}, \mathbf{Q}_p))^{G_K} \xrightarrow{\sim} \begin{cases} H^{i-j}(X, \Omega^j_X) & \text{if } 0 \le j \le i \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

When X is an abelian variety over K with good reduction, this is a theorem due to Tate in [Tat67]; Raynaud then used the theory of semistable reduction to extend Tate's result to all abelian varieties. Fontaine in [Fon82] found an easier proof of Raynaud's result, which is what we will present today.

Let X be an abelian variety over K. In this special case, to show the isomorphism (4), it suffices to show the $i = 1$ case since

$$\bigoplus_{i \ge 0} \left(\bigoplus_{j=0}^{i} H^{i-j}(X, \Omega^j_X) \right) \simeq \bigwedge^* \left(H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega_X) \right).$$

Since étale cohomology can be described in terms of Tate modules, the Tate–Raynaud theorem takes the following form:

Theorem 5.2 (Tate–Raynaud)**.** *Let K be a p-adic field, and let X be an abelian variety over K. Then, letting $\mathbf{C}_p = \hat{\overline{K}}$, we have the following two G_K-equivariant isomorphisms:*

$$(\mathbf{C}_p \otimes T_p(X)^*)^{G_K} \xrightarrow{\sim} H^1(X, \mathcal{O}_X) \tag{5}$$

$$(\mathbf{C}_p(1) \otimes T_p(X)^*)^{G_K} \xrightarrow{\sim} H^0(X, \Omega^1_X) \tag{6}$$

We will first show that in fact, it suffices to show (6):

Theorem 5.3. *Let X be an abelian variety over K. If* (6) *holds, then so does* (5)*.*

The reason why we would expect such a result to hold is because for abelian varieties, we have a nice duality theory: if X' denotes the dual abelian variety, then we have an isomorphism

$$H^1(X, \mathcal{O}_X) \xrightarrow{\sim} H^0(X', \Omega^1_{X'})^* \tag{7}$$

and an isomorphism

$$T_p(X) \xrightarrow{\sim} T_p(X')^*(1) \tag{8}$$

induced by the Weil pairing. Thus, (5) for X' implies

$$(\mathbf{C}_p(-1) \otimes T_p(X))^{G_K} \xrightarrow{\sim} H^0(X, \Omega^1_X)^*,$$

which is very close to (6), except that we have an extra dual on the right-hand side, and a missing dual on the left-hand side.

5.2 Proof of Theorem 5.3

To prove that (6) does indeed imply (5), we start with some computations of certain Galois cohomology groups, due to Tate.

Theorem 5.4 (Tate)**.** *We have the following descriptions of continuous Galois groups:*

$$H^0(G_K, \mathbf{C}_p(i)) = \begin{cases} K & i = 0 \\ 0 & i \ne 0 \end{cases} \qquad H^1(G_K, \mathbf{C}_p(i)) = \begin{cases} K & i = 0 \\ 0 & i \ne 0 \end{cases}$$

This implies an orthogonality result for different Tate twists of $\mathbf{C}_p$:

Corollary 5.5. *If $i \ne j$, then $\operatorname{Ext}^k_{\mathbf{C}_p[G_K]}(\mathbf{C}_p(i), \mathbf{C}_p(j)) = 0$. In particular, there are no G_K-equivariant homomorphisms $\mathbf{C}_p(i) \to \mathbf{C}_p(j)$ if $i \ne j$.*

We also need the following:

Fact 5.6. If V is an F-representation, and F/K is a field extension, then we have an injection

$$V^{G_K} \otimes F \hookrightarrow V.$$

Using the orthogonality result in Corollary 5.5, and Fact 5.6, we can now show Theorem 5.3.

Proof of Theorem 5.3. Set $V_{\mathbf{C}_p}(X) = \mathbf{C}_p \otimes_{\mathbf{Z}_p} T_p(X)$; this is a $\mathbf{C}_p$-representation. Then, by applying Fact 5.6 to $V_{\mathbf{C}_p}(X)^*(1)$, we have an injection

$$\begin{tikzcd}
(\mathbf{C}_p(1) \otimes T_p(X)^*)^{G_K} \otimes \mathbf{C}_p \arrow[r, hook] \arrow[d, "(6)\otimes\mathbf{C}_p"', "\wr"] & V_{\mathbf{C}_p}(X)^*(1) \arrow[d, equal] \\
H^0(X, \Omega^1_X) \otimes \mathbf{C}_p \arrow[r, hook] & V_{\mathbf{C}_p}(X)^*(1)
\end{tikzcd} \tag{9}$$

where as written, the second row is obtained by using the isomorphism (6). Now by applying $\mathrm{Hom}(-, \mathbf{C}_p(1))$, we get a surjection

$$\mathrm{Hom}(V_{\mathbf{C}_p}(X)^*(1), \mathbf{C}_p(1)) \twoheadrightarrow H^0(X, \Omega^1_X)^* \otimes_K \mathbf{C}_p(1)$$

We have isomorphisms

$$\mathrm{Hom}(V_{\mathbf{C}_p}(X)^*(1), \mathbf{C}_p(1)) \cong V_{\mathbf{C}_p}(X) \cong V_{\mathbf{C}_p}(X')^*(1),$$

where the second is by (8). The injection (9) for X' instead of X gives an injection

$$\begin{tikzcd}
H^0(X', \Omega^1_{X'}) \otimes \mathbf{C}_p \arrow[r, hook] \arrow[d, "(7)\otimes\mathbf{C}_p"', "\wr"] & V_{\mathbf{C}_p}(X')^*(1) \arrow[d, equal] \\
H^1(X, \mathcal{O}_X)^* \otimes \mathbf{C}_p \arrow[r, hook] & V_{\mathbf{C}_p}(X')^*(1)
\end{tikzcd}$$

by applying (7). Now combining this injection with the surjection above, we have a sequence

$$0 \longrightarrow H^1(X, \mathcal{O}_X)^* \otimes_K \mathbf{C}_p \longrightarrow V_{\mathbf{C}_p}(X) \longrightarrow H^0(X, \Omega^1_X)^* \otimes_K \mathbf{C}_p(1) \longrightarrow 0,$$

which is short exact: you can check that these maps form a complex, and exactness follows by a dimension count. This sequence must split by the statement about Ext's in Corollary 5.5, and so we have an isomorphism

$$V_{\mathbf{C}_p}(X) \simeq H^0(X, \Omega^1_X)^* \otimes_K \mathbf{C}_p(1) \oplus H^1(X, \mathcal{O}_X)^* \otimes_K \mathbf{C}_p.$$

Taking duals, we get the Hodge–Tate decomposition for $H^1_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$:

$$V_{\mathbf{C}_p}(X)^* \simeq H^0(X, \Omega^1_X) \otimes_K \mathbf{C}_p(-1) \oplus H^1(X, \mathcal{O}_X) \otimes_K \mathbf{C}_p$$

This implies $(\mathbf{C}_p \otimes T_p(X)^*)^{G_K} = (V_{\mathbf{C}_p}(X)^*)^{G_K} \simeq H^1(X, \mathcal{O}_X)$, which is exactly the isomorphism (5). $\square$

5.3 Proof of the Tate–Raynaud Theorem 5.2

The main goal of this subsection is to prove the following:

Theorem 5.7. *Let X be an abelian variety over K of dimension g. Then, there exists a K-linear injection*

$$\rho_X \colon H^0(X, \Omega^1_X) \hookrightarrow \mathrm{Hom}_{\mathbf{Z}_p[G_K]}(T_p(X), \mathbf{C}_p(1))$$

which is functorial in X and is canonical.

This will suffice to show the Tate–Raynaud Theorem 5.2:

Corollary 5.8. *Theorem 5.7 implies the Tate–Raynaud Theorem 5.2.*

Proof of Corollary. By Theorem 5.3, it suffices to show (6) is an isomorphism. The Weil pairing gives a pairing

$$V_p(X)^*(1) \times V_p(X')^*(1) \longrightarrow \mathbf{C}_p(1),$$

By using the injections (9) for X and X', this gives a pairing

$$(V_p(X)^*(1)^{G_k} \otimes \mathbf{C}_p) \times (V_p(X')^*(1)^{G_k} \otimes \mathbf{C}_p) \longrightarrow \mathbf{C}_p(1),$$

under which the two spaces on the left are orthogonal, since $H^0(G_K, \mathbf{C}_p(1)) \otimes \mathbf{C}_p = 0$ by Theorem 5.4. The direct sum of the spaces on the left has dimension $\leq 2g$, but each have dimension $\geq g$, and so they must both be of dimension g. □

Now to show Theorem 5.7, we want to construct a G_K-equivariant pairing

$$H^0(X, \Omega^1_X) \times T_p(X) \longrightarrow \mathbf{C}_p(1), \tag{10}$$

which will induce the morphism ρ_X. The way we will do so is to define a pairing on an integral model $\mathfrak{X}$ of X involving the module $\Omega := \Omega_{\mathcal{O}_{\overline{K}/K}}$ (which satisfies $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Q}_p, \Omega) \simeq V_p(\Omega) \simeq \mathbf{C}_p(1)$ by a calculation of Fontaine [Fon82, §2]), and then show that this construction does not depend on the choice of integral model. Note that Beilinson has a two-page proof the isomorphism $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Q}_p, \Omega) \simeq V_p(\Omega) \simeq \mathbf{C}_p(1)$ in [Bei12, pp. 718–719].

Since abelian varieties are projective, we can find an integral model $\mathfrak{X} \subseteq \mathbf{P}^n_{\mathcal{O}_K}$, which is a proper flat scheme over $\mathcal{O}_K$, such that the generic fiber X_η is isomorphic to X; denote $i\colon X \hookrightarrow \mathfrak{X}$ to be the inclusion of the generic fiber.

Now given $u\colon \mathrm{Spec}(\mathcal{O}_{\overline{K}}) \to \mathfrak{X}$ and $\omega \in H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K})$, let $u^*(\omega) \in \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ be the pullback. This gives a pairing

$$\langle -, - \rangle \colon \mathfrak{X}(\mathcal{O}_{\overline{K}}) \times H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \longrightarrow \Omega$$

which is $\mathcal{O}_K$-linear in the second variable, and is G_K-equivariant in the first variable: $\langle g \cdot u, w \rangle = g\langle u, \omega \rangle$. We want to say that it is additive under the left-hand side, where the additivity is with respect to the abelian variety structure on the generic fiber $X_\eta(\overline{K}) \simeq \mathfrak{X}(\mathcal{O}_{\overline{K}})$, but this doesn't quite work: you need to pass to a submodule.

Theorem 5.9. *There exists $r \geq 0$ such that*

$$\mathfrak{X}(\mathcal{O}_{\overline{K}}) \times p^r H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \longrightarrow \Omega$$

is additive in the first variable.

Remark 5.10. Let $\psi\colon X \times_K X \to \mathbf{P}^n_K$ be a projective embedding. Let $\mathfrak{Y}$ denote the schematic closure of $X \times_K X$ under the map

$$X \times_K X \xrightarrow{(\mathrm{id}, m)} \mathbf{P}^n_K \hookrightarrow \mathbf{P}^n_{\mathcal{O}_K}$$

This $\mathfrak{Y}$ has the property that the projection maps $\mathrm{pr}_1, \mathrm{pr}_2 \colon X \times_K X \to X$ and the multiplication map $m\colon X \times_K X \to X$ all extend to maps $\mathrm{pr}_{1,\mathfrak{X}}, \mathrm{pr}_{2,\mathfrak{X}}, m_{\mathfrak{X}} \colon \mathfrak{Y} \to \mathfrak{X}$, by taking suitable projections out of $\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}$ in the diagram below:

$$\begin{tikzcd}
X \times X \arrow[r, "(\mathrm{id}, m)"] \arrow[d, "\sigma"'] & X \times X \times X \arrow[r] \arrow[d, "i \times i \times i"] & \mathbf{P}^n_K \arrow[r] \arrow[d] & K \arrow[d] \\
\mathfrak{Y} \arrow[r] & \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \arrow[r] & \mathbf{P}^n_{\mathcal{O}_K} \arrow[r] & \mathcal{O}_K
\end{tikzcd}$$

Remark 5.11. The phenomenon observed in the previous remark is in fact more general. If $f_i\colon S \to T$ is a finite family of morphisms between $\mathcal{O}_K$-schemes, such that both S and T are both proper and flat over $\mathcal{O}_K$, then their schematic closures $\mathfrak{S}$ and $\mathfrak{T}$ are such that these maps f_i extend to maps $\tilde{f}_i \colon \mathfrak{S} \to \mathfrak{T}$.

Proof of Theorem 5.9. Consider the map

$$\begin{aligned} H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) &\longrightarrow H^0(\mathfrak{Y}, \Omega^1_{\mathfrak{Y}/\mathcal{O}_K}) \\ \omega &\longmapsto \omega' = m_{\mathfrak{X}}^*\omega - \mathrm{pr}_{1,\mathfrak{X}}^*\,\omega - \mathrm{pr}_{2,\mathfrak{X}}^*\,\omega \end{aligned}$$

We have that

$$\begin{array}{ccccccc} H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) & \longrightarrow & H^0(\mathfrak{Y}, \Omega^1_{\mathfrak{Y}/\mathcal{O}_K}) & \xrightarrow{\sigma^*} & H^0(X \times X, \Omega^1_{X\otimes X/\mathcal{O}_K}) & \xrightarrow{\sim} & K \otimes_{\mathcal{O}_K} H^0(\mathfrak{Y}, \Omega_{\mathfrak{Y}/\mathcal{O}_K}) \\ & & \tilde{\omega} & \longmapsto & & & 1 \otimes \tilde{\omega} \end{array}$$

Under this composition, $H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K})$ maps to 0, hence the image of $H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K})$ in $H^0(\mathfrak{Y}, \Omega_{\mathfrak{Y}})$ is torsion. By coherence, this torsion is bounded, and we can kill it using p^r.

We can now show additivity in the first variable. Let $u_1, u_2 \in \mathfrak{X}(\mathcal{O}_{\overline{K}})$, and v_1, v_2 their corresponding points in $X(\overline{K})$. Let $v_X = (v_1, v_2)$. Let v be the corresponding point in $\mathfrak{Y}$. This means $u_1 = \mathrm{pr}_{1,\mathfrak{X}} \circ v$, and $u_2 = \mathrm{pr}_{2,\mathfrak{X}} \circ v$, and $u_1 + u_2 = m_{\mathfrak{X}} \circ v$. If $\omega \in p^r H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K})$, then

$$(u_1 + u_2)^*(\omega) = v^* \circ m_{\mathfrak{X}}^*(\omega) = v^*(\mathrm{pr}_{1,\mathfrak{X}}^*\,\omega + \mathrm{pr}_{2,\mathfrak{X}}^*\,\omega) = u_1^*\omega + u_2^*\omega. \qquad \square$$

The upshot of Theorem 5.9 is that we now have an $\mathcal{O}_K$-linear map

$$p^r H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \to \mathrm{Hom}_{\mathbf{Z}[G_K]}(X(\overline{K}), \Omega)$$

which we will use to define ρ_X.

Sketch of Theorem 5.7. Now recall that $V_p(X) = \mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{Q}_p, X(\overline{K})[p^\infty])$. Using the map

$$\mathrm{Hom}_{\mathbf{Z}[G_K]}(X(\overline{K}), \Omega) \longrightarrow \mathrm{Hom}(V_p(X), V_p(\Omega)),$$

we have a map

$$p^r H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}}) \longrightarrow \mathrm{Hom}(V_p(X), V_p(\Omega)).$$

Extending scalars, we get a K-linear map

$$\rho_{\mathfrak{X},X} \colon H^0(X, \Omega^1_X) \longrightarrow \mathrm{Hom}(V_p(X), \mathbf{C}_p(1)).$$

To finish the proof, you check that $\rho_{\mathfrak{X},X}$ is independent of $\mathfrak{X}$, and that $\rho_{\mathfrak{X},X}$ is injective. $\square$

6 October 20: Background on p-divisible groups (Valia Gazaki)

We mostly follow [Tat67]. A good reference for the one-dimensional theory (especially for formal groups) is [Sil09], and we refer to [Ser79] for some properties of the discriminant.

The main goal of today is to prove Proposition 6.10, which gives a description of discriminant ideals associated to p-divisible groups. Before this, we have to do some background on finite group schemes and p-divisble groups.

6.1 Finite group schemes

Let R be a commutative ring. We do not put any hypotheses on R for now; later, we will assume R is complete and noetherian.

Definition 6.1. A *finite (flat) group scheme* over R is a scheme $\Gamma = \operatorname{Spec} A$, where A is a locally free R-module of finite rank, and Γ has the structure of a group scheme (i.e., there exists a comultiplication map $\mu\colon A \to A \otimes_R A$, etc.). If A has rank m over R, we say Γ is of *order m*, and denote $\operatorname{ord}\Gamma = m$.

We will always assume them to be commutative.

Examples 6.2.

(1) If Γ is a usual finite abelian group of order n, we can construct a finite flat group scheme $\Gamma = \operatorname{Spec} A$ of order n, by setting A to be the ring of R-valued functions on Γ. Then, the comultiplication is given by identifying $A \otimes_R A$ with R-valued functions on $\Gamma \times \Gamma$, so $\mu\colon A \to A \otimes_R A$ can be defined by $\mu(f)(s,t) = f(st)$. A key example is $\Gamma = \mathbf{Z}/n$.

(2) We can also define the finite flat group scheme

$$\mu_m = \operatorname{Spec}\left(\frac{R[x]}{(x^m - 1)}\right)$$

where the comultiplication is given by $x \mapsto x \otimes x$. This finite flat group scheme has order $\operatorname{ord} \mu_m = m$.

6.1.1 Duality

Definition 6.3. Let $G = \operatorname{Spec} A$ be a finite flat group scheme over R. We can then define the *dual group scheme* $G' = \operatorname{Spec} A'$, where $A' = \operatorname{Hom}_{R\text{-}\mathsf{Mod}}(A, R)$.

Example 6.4. $(\mu_m)' \cong \mathbf{Z}/m\mathbf{Z}$, and vice versa.

6.1.2 Short exact sequences

A sequence

$$0 \longrightarrow G' \xrightarrow{i} G \xrightarrow{j} G'' \longrightarrow 0$$

of finite flat group schemes over R is a *short exact sequence* if i is an exact closed immersion (here, exactness means that i identifies G' with the categorical kernel of j), and j is faithfully flat. In this case, $\operatorname{ord}(G) = \operatorname{ord}(G') \cdot \operatorname{ord}(G'')$.

Remark 6.5. Given $G \xrightarrow{j} G'' \to 0$, we can define $G' = j^{-1}(\text{neutralizer of } G')$, which has the required property for kernels. Constructing G'' from an exact sequence $0 \to G' \xrightarrow{i} G$ is more subtle.

6.1.3 Connected and étale groups

Now let R be a local, complete, Noetherian ring with residue field k. Then, given $G = \operatorname{Spec} A$, we define $G^{\text{ét}} = \operatorname{Spec} A^{\text{ét}}$, where $A^{\text{ét}} \hookrightarrow A$ is the maximal étale subalgebra of A. Then, there is a faithfully flat surjection $\operatorname{Spec} A \to \operatorname{Spec} A^{\text{ét}} \to 0$, and letting $G^0 = \ker(\operatorname{Spec} A \to \operatorname{Spec} A^{\text{ét}}) = \operatorname{Spec} A^0$ where A^0 is the local quotient of A such that the coidentity $A \to R$ factors through A^0, we see that G^0 is connected, and we have a short exact sequence

$$0 \longrightarrow G^0 \xrightarrow{i} G \xrightarrow{j} G^{\text{ét}} \longrightarrow 0$$

Facts 6.6.

(1) G is connected if and only if $G = G^0$. In this case, $\operatorname{ord} G$ is a power of the residue field characteristic. Thus, if k is of characteristic 0, then every finite flat group scheme is étale.

(2) The functors $G \mapsto G^0$ and $G \mapsto G^{\text{ét}}$ are both exact.

6.2 p-divisible groups

From now, R is a complete Noetherian local ring with residue field k of chracteristic $p > 0$.

Definition 6.7. A *p-divisible group* over R of *height* $h \geq 0$ is an inductive system (G_ν, i_ν) of (commutative) finite flat group schemes over R, such that

(i) Each G_ν is a finite group of order $p^{\nu h}$;

(ii) For every $\nu \geq 0$, there exists an exact sequence

$$0 \longrightarrow G_\nu \xrightarrow{i_\nu} G_{\nu+1} \xrightarrow{p^\nu} G_{\nu+1}$$

This is the scheme-theoretic analogue of the fact that in the world of abelian groups, letting $G_\nu = \mathbf{Z}/p^v$, we have $\varinjlim G_\nu = \mathbf{Q}_p$.

Remark 6.8. By iteration, there exists a closed immersion $i_{\nu,\mu}\colon G_\nu \to G_{\nu+\mu}$, which fits into the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & G_\nu \arrow[r, "i_{\nu,\mu}"] & G_{\nu+\mu} \arrow[r, "p^\nu"] \arrow[dr, dashed, "j_{\nu,\mu}"'] & G_{\nu+\mu} \\ & & & G_\mu \arrow[u, hook, "i_{\mu,\nu}"'] \end{tikzcd}$$

for all $v, \mu \geq 0$, and so there is a short exact sequence

$$0 \longrightarrow G_\nu \xrightarrow{i_{\nu,\mu}} G_{\nu+\mu} \xrightarrow{j_{\nu,\mu}} G_\mu \longrightarrow 0.$$

Examples 6.9.

(1) If A is an abelian scheme over R, then $A \overset{p^\nu}{\to} A$ and $A[p^\nu] = \ker(p^\nu)$, then we have a p-divisible group $A(p) \coloneqq (A[p^\nu], i_\nu)$.

(2) If $\mathbf{G}_m/R$, then we have $\mathbf{G}_m \overset{p^\nu}{\to} \mathbf{G}_m$, and so $\mathbf{G}_m(p) \coloneqq (\mu_{p^\nu}, i_\nu)$ is a p-divisible group.

We can now state the main result we want to prove:

Proposition 6.10. *If $G = (G_\nu, i_\nu)$ is a p-divisible group of height h over R, and $G_\nu = \operatorname{Spec} A_\nu$ where A_ν is a R-module via a finite map $R \hookrightarrow A_\nu$, the discriminant ideal* $\operatorname{disc}(A_\nu/R)$ *is generated by $p^{\nu n p^{h\nu}}$, where $n = \dim G$.*

We will later reduce to the case where G_ν are all connected, in which case they are related to formal Lie groups, for which we can compute things using differential forms. We therefore begin with some preliminaries on formal Lie groups.

6.3 Relations with formal Lie groups

Definition 6.11. An *n-dimensional formal Lie group* over R is a family $F = (F_i(\vec{X}, \vec{Y}))$ of n power series in $2n$ variables (so $F_i(\vec{X}, \vec{Y}) \in R[\![X_1, \dots, X_n, Y_1, \dots, Y_n]\!]$) that satisfies the axioms:

(*i*) $X = F(X, 0) = F(0, X)$;

(*ii*) $F(X, Y) = F(Y, X)$;

(*iii*) $F(X, F(Y, Z)) = F(F(X, Y), Z)$.

Even though X and Y are strictly speaking vectors, we will usually suppress the vector notation.

Note 6.12. By (*i*) and (*ii*), we get that $F(X, Y) = X + Y + \text{higher order terms}$. In particular, $F_i(\vec{X}, \vec{Y}) = X_i + Y_i + \text{higher order terms}$.

Definition 6.13. We define $X * Y = F(X, Y)$, and

$$\underbrace{X * X * \cdots * X}_{m \text{ times}} = [m]X.$$

Using this definition, $[m]$ defines a homomorphism $F \to F$, that is, we have the equality

$$[m](F(X, Y)) = F([m]X, [m]Y),$$

and $[m]$ corresponds to a ring homomorphism

$$R[\![X_1, \dots, X_n]\!] \xrightarrow{\psi} R[\![X_1, \dots, X_n]\!]$$

where $\psi(X_i)$ is the ith coordinate of $[m]X$.

Remark 6.14. Suppose $(m, p) = 1$, then $[m]$ is invertible (see [Sil09, Prop. 2.3] for the one-dimensional case).

6.3.1 Case $m = p^v$

Definition 6.15. If Γ is a formal Lie group with $F(X, Y)$ a formal group law, we say Γ is *divisible* if $[p]$ is an isogeny, that is,

$$R[\![X_1, \ldots, X_n]\!] \xrightarrow{\psi} R[\![X_1, \ldots, X_n]\!]$$

makes $R[\![X_1, \ldots, X_n]\!]$ into a free module of finite rank over itself.

The following is very important, and we will use it.

Remark 6.16. If Γ is divisible, then $\Gamma_p := \operatorname{Spec} A$ where $A := \frac{R[\![X_1,\ldots,X_n]\!]}{(\psi(X_i))}$ is a connected finite flat group scheme over R.

Proof of Remark. First, the formal group law induces a comultiplication and group scheme structure on Γ_p, since the formal group law can be thought of as a map $R[\![\vec{X}]\!] \to R[\![\vec{Y}, \vec{Z}]\!]$, and modding out by $\psi(X_i)$ induces a comultiplication.

Next, $R[\![X_1, \ldots, X_n]\!] \xrightarrow{\psi} R[\![X_1, \ldots, X_n]\!]$ gives the target a finite free module structure over itself, hence $R \hookrightarrow A$ is a finite extension.

Finally, we claim A is local (and hence Γ_p is connected). Denote $\mathfrak{m}$ to be the maximal ideal of R. Since $R \overset{\phi}{\hookrightarrow} A$ is finite, it is integral, and so it satisfies Going-Up. This implies $\phi^{-1}(\max) = \mathfrak{m}$, and so every maximal ideal of A contains pA. Now $[p]X = pX+$ higher order terms (e.g., if $\dim \Gamma = 1$, then $[p]X = pf(X) + g(X)$ where $g(X) = aX^n + \cdots$ with a a unit). This implies that X is in every maximal ideal of A. Thus, A has a unique maximal ideal.

Finally, setting $\Gamma_\nu = \operatorname{Spec} A_\nu$ where $A_\nu = R[\![X_1, \ldots, X_n]\!]/(\psi^\nu(X_i))$ form an inductive system, so these Γ_ν form a p-divisible group $\Gamma(p)$, where each Γ_ν is connected. $\square$

Proposition 6.17. *The functor $\Gamma \to \Gamma(p)$ which sends divisible formal Lie groups over R to connected p-divisible groups over R is an equivalence of categories.*

We won't have time to go through this proof carefully. Full-faithfulness follows by the fact that R is p-adically complete. Essential surjectivity is the harder part: the idea is to create a projective system of rings that end up being a power series ring. We may describe some steps at the end.

Definition 6.18. If G is any p-divisible group over R, $G = (G_\nu, i_\nu)$, we can define

$$0 \longrightarrow G_\nu^0 \xrightarrow{i} G_\nu \xrightarrow{j} G_\nu^{\text{ét}} \longrightarrow 0$$

and we can define a new p-divisible group $(G_\nu^0, i_\nu) =: G^0$, a connected p-divisible group. Similarly, we can define $G^{\text{ét}} = (G_\nu^{\text{ét}}, \cdot)$. We define the dimension of (G_ν^0, i_ν) to be the dimension of its associated formal Lie group, and define $\dim G = \dim G^0$.

6.4 Proof of Proposition 6.10

Before we prove Proposition 6.10, we remind ourselves of the definition of the discriminant.

Definition 6.19. Let $R \hookrightarrow A$ be a R-algebra homomorphism which realizes A as a finite R-module. Then, $\operatorname{disc}(A/R)$ is the ideal of R generated by $\det(\operatorname{Tr}(\alpha_i\alpha_j)_{ij})$ for any basis $\alpha_1, \ldots, \alpha_n \in A$, where n is the rank of A over R.

Proof of Proposition 6.10. Let G be a p-divisible group, and consider the short exact sequence

$$0 \longrightarrow G^0 \longrightarrow G \longrightarrow G^{\text{ét}} \longrightarrow 0$$

where $G = (G_\nu, i_\nu)$, $G_\nu = \operatorname{Spec} A_\nu$, and for all ν, we have

$$0 \longrightarrow G_\nu^0 \longrightarrow G_\nu \longrightarrow G_\nu^{\text{ét}} \longrightarrow 0.$$

Since the discriminant behaves well with short exact sequences (i.e., if $0 \to H' \to H \to H'' \to 0$ is a short exact sequence of finite flat group schemes over R, we have $\operatorname{disc}(H) = \operatorname{disc}(H')^{\operatorname{ord}(H'')} \cdot \operatorname{disc}(H'')^{\operatorname{ord}(H')}$), we may assume G_ν is connected. This is because the discriminant of $G^{\text{ét}}$ is 1 (the discriminant measures ramification).

Now let $G_\nu = \operatorname{Spec} A_\nu$ be connected, so that there exists a corresponding divisible formal Lie group Γ such that $A_\nu \simeq R[\![X_1, \ldots, X_n]\!]/(\psi^\nu(X_i))$, where n is the dimension of G. From now on, denote $A = R[\![X_1, \ldots, X_n]\!]$, and denote A' to be the copy of A such that there is a finite injection $A' \overset{\phi}{\hookrightarrow} A$, where $\phi = \psi^\nu$. It therefore suffices to show that $\operatorname{disc}(A/A') = p^{n\nu p^{h\nu}}$.

For $A' \overset{\phi}{\hookrightarrow} A$, take Ω to be the formal module of differential forms on A, and Ω' that on A'. Ω is a free A-module generated by dX_i, and dX_i' generate Ω'. Then, $\bigwedge^n \Omega$ is free of rank 1 over A, with generator θ, and $\bigwedge^n \Omega'$ is free of rank 1 over A', with generator θ'. We then have $d\psi^\nu\colon \bigwedge^n \Omega' \to \bigwedge^n \Omega$ defined by $\theta' \mapsto a\theta$ for some a. We claim that $a = p^{n\nu}$. This follows since Ω has a basis of invariant differential forms ω_i, and similarly Ω' has one with ω_i', where invariance means that if $A \overset{\epsilon}{\hookrightarrow} A \hat{\otimes} A$ defines a group law $\epsilon_*(\omega_i) = \omega_i \otimes \omega_i$.

Claim. *$d\psi^\nu(\omega_i') = p^\nu \omega_i$.*

The idea is that $d\psi^\nu(\omega_i') = \omega_i \circ [p]^\nu$, and so taking a derivative gives the correct number of powers of p.

The final step uses the trace map $\operatorname{Tr}\colon \bigwedge^n \Omega \to \bigwedge^n \Omega'$, which satisfies the following properties:

1. Tr is A'-linear;
2. $\alpha \mapsto [\omega \mapsto \operatorname{Tr}(\alpha\omega)]$ gives an A'-module isomorphism $A \xrightarrow{\sim} \operatorname{Hom}_A(\bigwedge^n \Omega, \bigwedge^n \Omega')$;
3. for all $\alpha \in A$, and all $\omega' \in \bigwedge^n \Omega'$, the equation

$$\operatorname{Tr}(\alpha\, d\psi^\nu(\omega')) = \operatorname{Tr}_{A/A'}(\alpha) \cdot \omega'$$

holds.

Finally, $\operatorname{Tr}(\alpha p^{n\nu}\theta') = \operatorname{Tr}_{A/A'}(\alpha)\theta'$ so $p^{n\nu} \operatorname{Tr}(\alpha\theta') = \operatorname{Tr}_{A/A'}(\alpha)\theta'$ by A'-linearity, which implies $\operatorname{Tr}_{A/A'}(\alpha) \in (p^{n\nu})$. We conclude that $\operatorname{disc}(A_\nu/R) \subseteq p^{n\nu p^{h\nu}}$.

For the reverse inclusion, we use that $\operatorname{disc}(A_\nu/R) = N_{A_\nu/R}(\mathcal{D}_{A_\nu/R})$, where $\mathcal{D}$ is the different ideal. We would then need to show $p^{n\nu} \in \mathcal{D}^*_{A_\nu/R}$. □

7 November 3: The behavior of the Faltings height under isogeny (Andrew Snowden)

A good reference for this material is [Lev11] from the Stanford seminar.

Let K be a number field, and let A/K be an abelian variety. We fix $G \subset A[\ell^\infty]$ an ℓ-divisible group, where we denote $G_n = G[\ell^n]$, and $A_n = A/G_n$.

The main goal for today is to show the following:

Theorem 7.1. *If A is everywhere semistable, then $h(A_n)$ is eventually constant as a function of n.*

We only really care that it is bounded.

Remark 7.2. Faltings [Fal86, Thm. 2] states that $h(A_n)$ is constant, but this is not quite true. See [Fal86, Erratum].

We briefly recall why we are interested in this statement. Fixing an ℓ-divisible group G in $A[\ell^\infty]$ gives a Galois subrepresentation W of the rational Tate module. Theorem 7.1 combined with the finiteness statement in Theorem 4.3 will imply there exists a sequence $n_1 < n_2 < \cdots$ such that $A_{n_i} \cong A_{n_j}$. We can use these isomorphisms to build interesting isogenies of A:

$$u_i\colon A \longrightarrow A_{n_1} \xrightarrow{\sim} A_{n_i} \longrightarrow A$$

and the sequence $\{u_i\} \in \operatorname{End}(A) \otimes \mathbf{Q}_\ell$ will have a convergent subsequence accumulating to some $u \in \operatorname{End}(A) \otimes \mathbf{Q}_\ell$, which will satisfy the property that $W = u(V_\ell(A))$. This is the necessary ingredient in showing Faltings's isogeny theorem $\mathbf{E}$; see §1.2.

7.1 Outline of proof of Theorem 7.1

We now give an overview of the proof of Theorem 7.1. We will mainly concentrate on the case where $K = \mathbf{Q}$, and A where has good reduction at ℓ. In this case, the sequence $h(A_n)$ is actually constant. The field doesn't matter too much, but the good reduction hypothesis is actually pretty strong: we will talk about this later.

Step I. *There is the following easy formula relating the heights of A and A_n:*

$$h(A_n) - h(A) = \log\left(|0^*\Omega^1_{G_n/\mathbf{Z}_\ell}|\,\ell^{-\frac{1}{2}nh}\right), \tag{11}$$

where $h = \mathrm{ht}(G)$, $0\colon \mathrm{Spec}(\mathbf{Z}_\ell) \to G_n$ is the zero section of $G_n \to \mathrm{Spec}(\mathbf{Z}_\ell)$, and $\Omega^1_{G_n/\mathbf{Z}_\ell}$ is the module of relative Kähler differentials.

There isn't much input in this step; it is very easy. This is the main reason to use Faltings height, since the naïve height doesn't have such a nice formula.

Step II. *The cardinality appearing in the height formula* (11) *in Step I is given by*

$$|0^*\Omega^1_{G_n/\mathbf{Z}_\ell}| = \ell^{dn},$$

where d is the dimension of (the formal group associated to) $G_{\mathbf{Z}_\ell}$.

The input to this step is Tate's theorem (Proposition 6.10) on $\mathrm{disc}(G_n)$ from Valia's talk.

To show the difference (11) is actually zero, we then need to show that $h = 2d$; the general statement over an arbitrary field is that

$$[K : \mathbf{Q}]h = 2\sum_{v|\ell}[K : \mathbf{Q}_\ell]d_v,$$

where $d_v = \dim G_{\mathcal{O}_{K_v}}$. This is the key formula we need to prove Theorem 7.1.

Remark 7.3. We pause and talk about why this is so cool. The equality $h = 2d$ in particular shows that heights of sub-ℓ-divisible groups in $A[\ell^\infty]$ must be even. This is very particular to the situation we are in, where we have a global model for our abelian variety over $\mathbf{Z}$:

- There exist odd height ℓ-divisible groups over $\mathbf{Z}$: e.g., $\mathbf{Q}_\ell/\mathbf{Z}_\ell$, μ_{ℓ^∞}.
- There exist odd height ℓ-divisible groups in $B[\ell^\infty]$ if $B/\mathbf{Q}_\ell$ is an abelian variety. One way is to take an ordinary elliptic curve, in which case

$$T_\ell(B) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

So we get a one-dimensional subgroup of $T_\ell(B)$, and thus, a height 1 ℓ-divisible subgroup of $B[\ell^\infty]$. This means that it will be important in our proof that the ℓ-divisible group lives in the ℓ-torsion of an abelian variety, and that the abelian variety has a globally defined model over $\mathbf{Z}$.

We return to the outline of the proof of Theorem 7.1. The next step is to introduce a third invariant k, which we will show in later steps is equal to $\frac{1}{2}h$ and d.

Step III. *Recall $V_\ell(G)$ is the rational Tate module, whose dimension is the height of G. Then,*

$$\det(V_\ell(G)) = \psi \cdot \chi_\ell^k, \tag{12}$$

where ψ is a finite order character of $G_{\mathbf{Q}}$, χ_ℓ is the ℓ-adic cyclotomic character, and $k \in \mathbf{Z}$.

The input in proving this is some simple p-adic Hodge theory and class field theory for $\mathbf{Q}$.

We now focus on k, and try to relate it to h and d.

Step IV. *$k = d$.*

The input is Tate's theorem/the Hodge–Tate decomposition (Theorem 5.2) for $V_\ell(G)$ from Kannappan's talk. Note this step only really uses local information of A.

Step V. *$k = \frac{1}{2}h$.*

This where we use global information, namely, the Weil conjectures for the abelian variety $A_{\mathbf{F}_p}$, for $p \neq \ell$.

7.2 Step I: Formula for change of height under isogeny

We first want to prove the easy formula (11) relating heights. Tildes will always denote Néron models.

Proposition 7.4. *Let $\phi\colon A \to B$ be an isogeny of degree n, where A and B are abelian varieties over a number field K, with semistable reduction at places over n. Then,*

$$h(B) - h(A) = \frac{1}{[K:\mathbf{Q}]} \log|0^*\Omega^1_{G/\mathcal{O}_K}| - \frac{1}{2}\log(n),$$

where $G = \ker(\phi\colon \widetilde{A} \to \widetilde{B})$.

Before we get into the proof, there are quite a few things here that must be true for the statement to even make sense. For example, $0^*\Omega^1_{G/\mathcal{O}_K}$ must be finite: this is true since it is finitely generated over $\mathcal{O}_K$, and the map is étale at almost all points. This also means that the only places we care about are those divisible by n. That means we can enlarge our ground field K, since the Faltings height doesn't change, and the right-hand side is easily seen to not, either. This implies the following:

Remark 7.5. To show Proposition 7.4, we may enlarge K. We can therefore assume that A and B have semistable reduction everywhere.

Recall 7.6. We recall the definition of $h(A)$: pick $\omega \in \Gamma(A, \Omega^g_{A/K})$, where $g = \dim(A)$. Then, if $i\colon K \to \mathbf{C}$ is an embedding of our field, we can set

$$|\omega|_i = \int_{A(\mathbf{C})} \omega \wedge \overline{\omega},$$

which is a real number, and has the property that for $a \in K$, $|a\omega|_i = i(a)\overline{i(a)}|\omega|_i$. Our definition for Faltings height uses this. Denote $M = \Gamma(\widetilde{A}, \Omega^g_{\widetilde{A}/\mathcal{O}_K})$, which is a rank one projective $\mathcal{O}_K$-module. Then,

$$h(A) = \frac{1}{[K:\mathbf{Q}]}\left(\sum_{i\colon K\to\mathbf{C}} \frac{1}{2}\log|\omega|_i - \log|M/\mathcal{O}_K\cdot\omega|\right)$$

for any non-zero $\omega \in M$.

Proof of Proposition 7.4. Let $M = \Gamma(\widetilde{A}, \Omega^g)$, and let $N = \Gamma(\widetilde{B}, \Omega^g)$, which are both rank one projective modules. We will assume without loss of generality that they are free, by possibly enlarging K. Now let ω be a generator of M, and ω' a generator of N. We then have that $\phi^*\omega' = a\omega$, where $a \in \mathcal{O}_K$ is nonzero. Note

$$|\omega|_i = \frac{1}{n}|\phi^*\omega|_i = \frac{i(a)\overline{i(a)}}{n}|\omega_i|,$$

by looking at how $A(\mathbf{C})$ is $\mathbf{C}^g$ mod a lattice, and seeing that A is defined by an index n sublattice of that defining B. We therefore have

$$\begin{aligned}
h(B) &= \frac{1}{2[K:\mathbf{Q}]}\sum_{i\colon K\to\mathbf{C}} \log|\omega'|_i \\
&= \frac{1}{2[K:\mathbf{Q}]}\sum_{i\colon K\to\mathbf{C}} (-\log(n) + \log(i(a)\overline{i(a)}) + \log|\omega|_i) \\
&= -\frac{1}{2}\log(n) + \frac{1}{[K:\mathbf{Q}]}\log|N_{K/\mathbf{Q}}(a)| + h(A)
\end{aligned}$$

Thus,

$$h(B) - h(A) = -\frac{1}{2}\log(n) + \frac{1}{[K:\mathbf{Q}]}\log|N_{K/\mathbf{Q}}(a)|.$$

It then remains to show that $|N_{K/\mathbf{Q}}(a)| = |0^*\Omega_{G/\mathcal{O}_K}|$. We have the following exact sequence for Arakelov forms:

$$\phi^*(\Omega^1_{\widetilde{B}/\mathcal{O}_K}) \longrightarrow \Omega^1_{\widetilde{A}/\mathcal{O}_K} \longrightarrow \Omega^1_{\widetilde{A}/\widetilde{B}} \longrightarrow 0$$

Pulling back via the 0-section, we obtain a short exact sequence

$$0 \longrightarrow 0^*\Omega^1_{\widetilde{B}/\mathcal{O}_K} \xrightarrow{\phi^*} 0^*\Omega^1_{\widetilde{A}/\mathcal{O}_K} \longrightarrow 0^*\Omega^1_{\widetilde{A}/\widetilde{B}} \longrightarrow 0,$$

where you need to check exactness on the left. Since $\phi^*\omega' = a\omega$, we have $\det \phi^* = a$. Thus, by taking top wedge powers of this short exact sequence, $|0^*\Omega^1_{\widetilde{A}/\widetilde{B}}| = |N_{K/\mathbf{Q}}a|$

Finally, we claim $0^*\Omega^1_{\widetilde{A}/\widetilde{B}} \cong 0^*\Omega^1_{G/\mathcal{O}_K}$. Conisder the diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d] & \widetilde{A} \arrow[d, "\phi"] \\ \mathrm{Spec}(\mathcal{O}_K) \arrow[r, "0"] & \widetilde{B} \end{tikzcd}$$

which is a fibre square. The claim then follows from the behavior of Ω^1 under base change. $\square$

Proposition 7.4 is now proved. Step I is an immediate consequence:

Proof of Step I. To get (11), take $B = A_n$ and let $K = \mathbf{Q}$. Since $\deg \phi = |G_n| = \ell^n$, we get

$$h(A_n) - h(A) = \log\left(|0^*\Omega^1_{G/\mathbf{Z}_\ell}|\,\ell^{-\frac{1}{2}nh}\right). \qquad \square$$

7.3 Step II: Computation of cardinality of $0^*\Omega^1_{G_n/\mathbf{Z}_\ell}$

Now we want to compute $|0^*\Omega^1_{G_n/\mathbf{Z}_\ell}|$. We will use the following general result about connected finite flat group schemes:

Lemma 7.7. *Suppose $H/\mathbf{Z}_\ell$ is a connected finite flat group scheme. Then,*

$$|0^*\Omega^1_{H/\mathbf{Z}_\ell}|^{\#H} = |\mathbf{Z}_\ell/\mathrm{disc}(H)|.$$

Proof. Let $H = \operatorname{Spec} R$, where R is a finite local $\mathbf{Z}_\ell$-algebra; note it is local since H is connected. Then, we have a homomorphism $R \to \mathbf{Z}_\ell$ corresponding to $0 \in H(\mathbf{Z}_\ell)$. Let $I \subset R$ be its kernel. Then,

$$I/I^2 = 0^*(\Omega^1_{H/\mathbf{Z}_\ell})$$

We have an isomorphism $\Omega^1_{H/\mathbf{Z}_\ell} \cong R \otimes_{\mathbf{Z}_\ell} I/I^2$, since $\Omega^1_{H/\mathbf{Z}_\ell}$ has a basis of translation-invariant one-forms, which we can think of as coming from I/I^2 (since H is connected). As a $\mathbf{Z}_\ell$-module, $R \cong \mathbf{Z}_\ell^{\#H}$, and so as a group, $|\Omega^1_{H/\mathbf{Z}_\ell}| = |I/I^2|^{\#H} = |0^*\Omega^1_{H/\mathbf{Z}_\ell}|^{\#H}$. We then have that

$$|\Omega^1_{H/\mathbf{Z}_\ell}| = |\mathbf{Z}_\ell/\mathrm{disc}(H)| \qquad \square$$

We can now use the Lemma to prove Step II.

Proof of Step II. Let G_n^0 be the connected part of G_n. Then,

$$0^*\Omega^1_{G_n/\mathbf{Z}_\ell} = 0^*\Omega^1_{G_n^0/\mathbf{Z}_\ell},$$

which implies

$$|0^*\Omega^1_{G_n/\mathbf{Z}_\ell}| = |0^*\Omega^1_{G_n^0/\mathbf{Z}_\ell}| = |\mathbf{Z}_\ell/\mathrm{disc}(G_n^0)|^{1/\#G_n^0} = (\ell^{dn\#G_n^0})^{1/\#G_n^0} = \ell^{dn},$$

where the second equality is by Lemma 7.7, and the third is by Tate's theorem (Proposition 6.10), which says that $\mathrm{disc}(G_n^0) = \ell^{dn\#G_n^0}$, where d is the dimension of the formal group of G^0. $\square$

7.4 Step III: Background on Hodge–Tate representations and formula (12)

Before we talk start the proof of that $h = 2d$, we give some preliminaries on p-adic Hodge theory, which might overlap with what Kannappan said (§5).

Let $K/\mathbf{Q}_\ell$ be a finite extension, and denote

$$\mathbf{C}_\ell = \widehat{\overline{K}}.$$

Then, we make the following definition:

Definition 7.8. Let V be a continuous representation of G_K, where V is a finite dimensional $\mathbf{Q}_\ell$-vector space. Then, we say V is *Hodge–Tate* if

$$V \otimes_{\mathbf{Q}_\ell} \mathbf{C}_\ell \cong \bigoplus_{n \in \mathbf{Z}} \mathbf{C}_\ell(n)^{\oplus h(n)}$$

for some $h(n) \in \mathbf{Z}$, where G_K acts on each factor on the left, and (n) denotes Tate twist. The n such that $h(n) \neq 0$ are called *Hodge–Tate weights*, which we say occur with *multiplicity* $h(n)$.

Facts 7.9.

- If X/K is a smooth projective variety, then $H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_\ell)$ is a Hodge–Tate representation.
- The category of Hodge–Tate representations is abelian, and closed under tensor products.
- Any subrepresentation or quotient representation of a Hodge–Tate representation is again Hodge–Tate.
- If G_K acts on V through a finite quotient, then V is Hodge–Tate of weight 0.
- A one-dimensional representation α of G_K is Hodge–Tate if and only if it is of the form $\alpha = \psi\chi_\ell^k$ for some k, where ψ is of finite order on I_K.

Proposition 7.10. *Let $\alpha\colon G_{\mathbf{Q}} \to \mathbf{Q}_\ell^*$ be a continuous character which ramifies at only finitely many places, such that $\alpha|_{G_{\mathbf{Q}_\ell}}$ is Hodge–Tate of weight k. Then, $\alpha = \psi\chi_\ell^k$, where $\psi\colon G_{\mathbf{Q}} \to \mathbf{Q}_\ell^*$ has finite order.*

Proof. First, we may replace α with $\frac{\alpha}{\chi_\ell^k}$, so now $\alpha|_{G_{\mathbf{Q}_\ell}}$ has weight zero. Thus, $\alpha|_{I_{\mathbf{Q}_\ell}}$ is of finite order. Then, since $\mathbf{Q}_\ell^*$ is abelian, α factors to give a character $\alpha\colon G_{\mathbf{Q}}^{\mathsf{ab}} \to \mathbf{Q}_\ell^*$, where by class field theory,

$$G_{\mathbf{Q}}^{\mathsf{ab}} = \hat{\mathbf{Z}}^\times = \prod_p \mathbf{Z}_p^\times,$$

where each factor at p is the inertial group at p. By assumption, since α ramifies at only finitely many places, α factors through some quotient of the form $\prod_{p|N} \mathbf{Z}_p^\times$ for some N. So we know that $\alpha|_{\mathbf{Z}_\ell^\times}$ is of finite order by the Hodge–Tate condition. It is also true that $\alpha|_{\mathbf{Z}_\ell^\times}$ is of finite order for $p \neq \ell$, since any homomorphism $\mathbf{Z}_p^\times \to \mathbf{Q}_\ell^\times$ is of finite order. Finally, this implies that α is of finite order. ☐

The takeaway from this concerns the Tate module of the p-divisible group we care about:

Proof of Step III. Let $G \subset A[\ell^n]$, where $A/\mathbf{Q}$ is an abelian variety. We know that $V_\ell(G)$, a subquotient of the Tate module of A, is a Hodge–Tate representation of $G_{\mathbf{Q}}$. Then, the determinant $\alpha = \det(V_\ell(G))$ of this representation is also Hodge–Tate of $G_{\mathbf{Q}}$. Proposition 7.10 implies that $\alpha = \psi\chi_\ell^k$, where k is the weight of $\alpha|_{G_{\mathbf{Q}_\ell}}$, and ψ is of finite order. ☐

7.5 Step IV: $k = d$

We basically just have to quote Tate's theorem.

Proof of Step IV. By Tate's theorem 5.2, the representation $V_\ell(G)|_{G_{\mathbf{Q}_\ell}}$ is Hodge–Tate with weights $0, 1$. The multiplicity of the weight 1 is d, the dimension of G. Writing this out, we have

$$V_\ell(G) \otimes \mathbf{C}_\ell = \mathbf{C}_\ell(0)^{\oplus h(0)} \oplus \mathbf{C}_\ell(1)^{\oplus d},$$

and so the determinant $\det(V_\ell(G))$ has Hodge–Tate weight d. By Step III, we conclude that $k = d$. ☐

7.6 Step V: $k = h/2$

So far we haven't used the global information; this is where it is used, in the form of the Riemann hypothesis part of the Weil conjectures for abelian varieties.

Proof of Step V. Recall that $V_\ell(G) \subset V_\ell(A)$ is a h-dimensional $G_{\mathbf{Q}}$-subrepresentation, where $h = \mathrm{ht}(G)$. Now let $p \neq \ell$ be a prime of good reduction for A. Then, $V_\ell(A)$ is unramified at p, and so $V_\ell(A_{\mathbf{F}_p})$ is a representation of $G_{\mathbf{F}_p} = \langle \mathrm{Frob}_p \rangle$. This representation is semisimple, and the eigenvalues of Frobenius are Weil numbers of weight 1 (that is, $|\cdot| = p^{1/2}$ under any complex embedding). The same statement is true for $V_\ell(G)$, which is a subrepresentation. Thus, $\det(V_\ell(G))$ Frobenius acts by a weight h Weil number, and so

$$|\alpha(\mathrm{Frob}_p)| = p^{h/2}.$$

On the other hand, we know from Step III that

$$|\alpha(\mathrm{Frob}_p)| = |\psi(\mathrm{Frob}_p)| \cdot |\chi_\ell(\mathrm{Frob}_p)|^k = p^k,$$

and so $k = \frac{1}{2}h$. □

7.7 Comments about general case

We assumed that A had good reduction at ℓ, so we first explain what happens when that is not the case. We used this when we assume that G and $A[\ell^\infty]$ are ℓ-divisible groups over $\mathbf{Z}_\ell$.

If A does not have good reduction, $\widetilde{A}[\ell^n]$ is a quasi-finite flat group scheme over $\mathbf{Z}_\ell$, which is not necessarily finite in general (look at the ℓ-torsion group scheme of an elliptic curve that has toric reduction). What you can do instead is to use the following structure theorem:

Fact 7.11. If $H/\mathbf{Z}_\ell$ is a quasi-finite flat group scheme, there is a canonical open and closed subgroup $H^f \subset H$, which is finite over $\mathbf{Z}_\ell$. This is functorial in H, and is the maximal thing with this property.

What we would like to do is to reduce to studying H^f instead of H. Note that the zero section is always contained in H^f, and so everything factors through. There is a slight problem: you need to show that given the system $\widetilde{A}[\ell^n]$ of quasi-finite flat group schemes over ℓ, taking $\{\widetilde{A}[\ell^n]^f\}$ gives an ℓ-divisible group. Brian Conrad proves this statement in one of this seminar talks [Con11b, Lem. 5.4]. However, this does not give a proof for G: $\{G_n^f\}_{n\geq 1}$ may not form an ℓ-divisible group. Faltings assumed this would, which is an error in his original argument; see [Fal86, Erratum]. But this is true if we replace A by A/G_n for sufficiently large $n \gg 0$.

8 November 10: Faltings's isogeny theorem (Takumi Murayama)

References for this talk are [Fal+92, Ch. IV; Lev11]. We also recommend looking back to §2 and [Lic10] for the analogous proof over finite fields.

Today, we will continue using the machinery we have built up so far to prove Faltings's isogeny theorem. As always, let K be a number field, and let G_K be the absolute Galois group of K.

Recall our main goal is to prove statements **E** and **D** from §1.1:

E. Faltings's isogeny theorem. *If A and B are two abelian varieties over K, then the natural map*

$$\mathrm{Hom}_K(A, B) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \longrightarrow \mathrm{Hom}_{G_K}(T_\ell(A), T_\ell(B)) \qquad (\star)$$

is an isomorphism for all primes ℓ.

D. Semisimplicity of the Tate module. *If A is an abelian variety over K, then $V_\ell(A)$ is a semisimple G_K-representation.*

The proof will be fairly similar to the proof of Tate's isogeny theorem [Tat66] for abelian varieties over a finite field in §2, but there are a couple of differences. The major one is that we need to find a way to use the following weaker finiteness results from the talks by Andrew (§7) and Bhargav (§4):

Theorem α (Theorem 7.1)**.** *Let A be an abelian variety over a number field K. Fix an ℓ-divisible group $G \subset A[\ell^\infty]$, and denote $G_n = G[\ell^n]$ and $A_n = A/G_n$. If A has* semistable reduction*, then $h(A_n)$ is eventually constant as a function of n.*

Theorem β (Theorem 4.3)**.** *Fix $g \geq 1$, a composite number $m \geq 3$, and a constant $C > 0$. Then, there exist only finitely many (up to K-isomorphism)* principally polarized *abelian varieties B of dimension g with*

(*a*) level structure $(\mathbf{Z}/m)^{2g} \xrightarrow{\sim} B[m]$, *and*

(*b*) *Faltings height $h_F(B) < C$.*

As Andrew pointed out in §1.2, and just as in §§2.3–2.4, it suffices to show the following:

Key Result 8.1 (cf. Key Result 1.3)**.** *If A is an abelian variety over a number field K, and $W \subset V_\ell(A)$ is a subrepresentation of the rational Tate module, then there exists $u \in \operatorname{End}_K(A) \otimes \mathbf{Q}_\ell$ such that $u(V_\ell(A)) = W$.*

The plan for today is to first discuss how to deduce statements **D** and **E** from Key Result 8.1. After a short discussion about finiteness statements and how to move to a setting where the finiteness results above apply, we will prove Key Result 8.1.

8.1 Reduction to Key Result 8.1

We first show Key Result implies **D**, following [Lic10, p. 9], since we did not show it when we proved Tate's isogeny theorem over finite fields in §2. Denote

$$E_\ell \coloneqq \operatorname{End}_K(A) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell \subset \operatorname{End}_{\mathbf{Q}_\ell}(V_\ell(A)).$$

Key Result 8.1 $\Rightarrow$ **D**. Let $W \subset V_\ell(A)$ be G_K-semistable; it suffices to show that there exists a G_K-stable complement W'. The right ideal

$$\mathfrak{a} \coloneqq \{u \in E_\ell \mid u(V_\ell(A)) \subset W\} \subset E_\ell,$$

is principally generated by some element u_0 such that $u_0^2 = u_0$, as are all right ideals in semi-simple algebras [Lev11, Prop. 4.4]. Now since there exists $u \in E_\ell$ such that $u(V_\ell(A)) = W$, we have that

$$u_0(V_\ell(A)) = u_0 E_\ell(V_\ell(A)) = \mathfrak{a}(V_\ell(A)) = W,$$

so u_0 is a projection operator on $V_\ell(A)$ with image W. Thus, $1 - u_0$ is a projection operator onto a direct complement W' of W, and $V_\ell(A)$ is therefore semisimple. □

Remark 8.2. The proof of the fact that all right ideals in a semi-simple algebra over a field k are principal is [Lev11, Prop. 4.4], and goes as follows. By decomposing the semi-simple algebra, you reduce to the case of central simple algebras over k. This is isomorphic to a matrix algebra $\operatorname{Mat}_n(D)$ for some central division algebras D over k. In this case, you can do an explicit matrix analysis.

We next remind everyone how to deduce **E** from Key Result 8.1. The first reduction is the same as in §2.3:

Lemma 8.3 (Lemmas 2.9 and 2.11)**.** *To show* **E***, it suffices to show that the natural map*

$$\operatorname{End}_K(A) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell \longrightarrow \operatorname{End}_{G_K}(V_\ell(A)) \tag{$\star\star\star$}$$

is a surjection.

Idea. Injectivity in $(\star)$ holds in general, so it suffices to show it is surjective. The morphism $(\star)$ is surjective if and only if $(\star) \otimes \mathbf{Q}_\ell$ is surjective since $\operatorname{cok}(\star)$ is torsion-free. To replace Hom with End, you apply the endomorphism statement with $A \times B$ replacing A. □

We can now deduce **E**. Since this is basically the same argument as in §§2.4–2.3, we simply give a sketch:

Key Result 8.1 ⇒ **E** *(Sketch).* Let C be the centralizer of E_ℓ in $\operatorname{End}_{\mathbf{Q}_\ell}(V_\ell(A))$. The centralizer C° of C equals E_ℓ by the double centralizer theorem [Jac89, Thm. 4.10], since E_ℓ is a semisimple algebra.

Now let $\alpha \in \operatorname{End}_{G_K}(V_\ell(A))$; we want to show that $\alpha \in C^\circ$. Consider any $d \in C$. Then, $d \oplus d$ commutes with everything in $\operatorname{End}_K(A \times A) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell$. In particular, by Key Result 8.1, there exists $u \in \operatorname{End}_K(A \times A) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell$ such that

$$u(V_\ell(A \times A)) = \{(x, \alpha x) \in V_\ell(A \times A)\} =: W,$$

and $(d \oplus d)$ commutes with u. By applying $(d \oplus d)$ to both sides of this equation, we have $(d \oplus d)W \subset W$, which implies $d\alpha = \alpha d$, i.e., $\alpha \in C^\circ$. Thus, $(\star\star\star)$ is surjective. □

So we see that it suffices to show Key Result 8.1.

8.2 Some comments on finiteness theorems

Before we get into the details of the proof of Key Result 8.1, we want to point out the major difference between Faltings's proof for number fields and Tate's proof for finite fields. The setup is basically the same:

Setup 8.4. Let $W \subset V_\ell(A)$ be a G_K-invariant subspace. Then, letting $U := W \cap T_\ell(A)$, for $n \geq 1$ we can define an ℓ-divisible subgroup $G \subset A[\ell^\infty]$ with levels

$$U/\ell^n U \hookrightarrow T_\ell(A)/\ell^n T_\ell(A) = A[\ell^n](\overline{K})$$

which is actually defined over K since W is G_K-invariant. We can then consider the subgroups $G_n = G[\ell^n]$ and the quotients $A_n = A/G_n$ appearing in Theorem α.

In Tate's proof, the steps thereafter are as follows:

Step 1. Use either the "strong" finiteness hypothesis

$\operatorname{Hyp}(K, A, \ell)$: There exist only finitely many (up to K-isomorphism) abelian varieties B such that there is a K-isogeny $B \to A$ of ℓ-power degree.

or the "weak" finiteness hypothesis

$\operatorname{Hyp}(K, A, d, \ell)$: There exist only finitely many (up to K-isomorphism) abelian varieties B such that
(*a*) there exists a K-isogeny $B \to A$ of ℓ-power degree;
(*b*) there exists a polarization $\lambda\colon B \to B^\vee$ where $\deg \lambda = d$.

to construct a sequence $n_1 < n_2 < \cdots$ such that $A_{n_i} \cong A_{n_j}$.

Step 2. Define isogenies

$$u_i\colon A \longrightarrow A_{n_1} \xrightarrow{\sim} A_{n_i} \longrightarrow A,$$

and then extract $u \in E_\ell$ as the limit of a convergent subsequence of these u_i that satisfies the conclusion of Key Result 8.1.

Step 2 looks like it will still work. However, for Step 1, Faltings proves neither of these hypotheses until *after* establishing the isogeny theorem (this is a consequence of the Shafarevich conjecture for abelian varieties, which is statement **C**). We want to instead use Theorems α and β.

Problems 8.5. The finiteness statements in Theorems α and β don't seem to apply, since
(1) A does not have semistable reduction;
(2) A_n don't come with level structure; and
(3) A, A_n aren't principally polarized.

To solve these problems, we notice that there is a bit more flexibility in our setup: we can fix (1) and (2) by taking field extensions, and fix (3) by using Zarhin's trick appropriately, and replacing Theorem α with a stronger statement. We will have to go in order.

8.2.1 Ensuring A has semistable reduction

We fix Problem 8.5(1).

Proposition 8.6. *Let A be an abelian variety over K. If Key Result 8.1 holds for $A_L = A \times_K L$, where L is a finite extension of K, then it holds for A.*

Proof. Let $W \subset V_\ell(A)$ be a G_K-invariant subspace, so there exists $u \in \mathrm{End}_L(A) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell$ such that $u(V_\ell(A)) = W$. Choose representatives $\{\sigma_i\} = G_K/G_L$, and let

$$u' = \frac{1}{[L:K]} \sum_i \sigma_i(u).$$

Since W is G_K-invariant, each $\sigma_i(u)$ satisfies the same property as u, and thus, so does u'. Galois descent of morphisms then implies $u' \in \mathrm{End}_K(A) \otimes \mathbf{Q}_\ell$. $\square$

By applying Grothendieck's semistable reduction theorem 3.15, we obtain

Corollary 8.7. *It suffices to show Key Result 8.1 for abelian varieties with semistable reduction.*

Note, however, in the rest of the proof we will use very often that we can pass freely to finite extensions.

8.2.2 Ensuring A_n have level structure

To fix Problem 8.5(2), the point is the following:

Proposition 8.8. *For m coprime to ℓ, there exists a finite field extension L of K such that every A_n has level m-structure.*

It suffices to produce level m-structure on A, since the m-torsion parts of A_n are isomorphic to those of A. But this happens we know after base extension to $\overline{K}$, so base changing to the fixed field L of the kernel of the representation

$$G_K \longrightarrow \mathrm{GL}_{2g}(\mathbf{Z}/m\mathbf{Z})$$

defined by acting on $A_{\overline{K}}[m]$ suffices, since the degree of the field extension L/K is bounded above by $|\mathrm{GL}_{2g}(\mathbf{Z}/m\mathbf{Z})|$.

8.2.3 Ensuring A, A_n don't have to be principally polarized

Now we fix Problem 8.5(3). This is a bit more involved. We want to show the following strengthening of Theorem β:

Theorem β^*. *Theorem β holds without assuming principal polarization.*

We claim it suffices to note the following:

Proposition 8.9 [Fal+92, Ch. IV, Prop. 3.7]**.** *If B is an abelian variety over K with semistable reduction, then $h(B^\vee) = h(B)$, where $B^\vee$ denotes the dual abelian variety to B.*

We basically use the tricks from §2.5.

Proposition 8.9 $\Rightarrow$ Theorem β^*. Using Proposition 8.9, we have

$$h((B \times B^\vee)^4) = 8 \cdot h(B).$$

Since $(B \times B^\vee)^4$ is principally polarized by Zarhin's trick, the number of K-isomorphism classes of $(B \times B^\vee)^4$ is finite by Theorem β. But each abelian variety $(B \times B^\vee)^4$ has only finitely many direct factors (up to K-isomorphism), so there are only finitely many isomorphism classes for B. $\square$

We now give the idea for Proposition 8.9. Note that Deligne just states this fact without proof [Del85, Rem. 1.22].

Idea of Proposition 8.9. Since A has semistable reduction, the Faltings height is invariant under taking finite field extensions [MilAV, p. 153], and so after possibly passing to a larger finite extension, it suffices to consider the case when B is isogenous to a principally polarized abelian variety. If B is actually *isomorphic* to a principally polarized abelian variety, then we are done, since $B \cong B^\vee$. Thus, it suffices to show that $h(B^\vee) - h(B)$ is an isogeny invariant. Since any isogeny can be factored (over another finite field extension) into ones of prime degree, we are reduced to showing that

$$h(B^\vee) - h(C^\vee) + h(C) - h(B) = 0,$$

when there is an isogeny $\varphi\colon B \to C$ of prime degree p. By using the formula for change of height under isogeny (Proposition 7.4), it suffices to show

$$[K : \mathbf{Q}] \cdot \log(p) = \log(|0^*\Omega^1_{G/\mathcal{O}_K}| \cdot |0^*\Omega^1_{G^\vee/\mathcal{O}_K}|),$$

where $G = \ker(\varphi\colon B \to C)$ and $G^\vee = \ker(\varphi^\vee\colon C^\vee \to B^\vee)$. After this, by completing at each place dividing p, it suffices to show

$$|0^*\Omega^1_{G_v/\mathcal{O}_{K,v}}| \cdot |0^*\Omega^1_{G_v^\vee/\mathcal{O}_{K,v}}| = |\mathcal{O}_{K,v}/p\,\mathcal{O}_{K,v}|.$$

The rest of the argument uses the decomposition of Tate modules T_ℓ into torsion and free parts (hinted at in §7.7), after which you do some diagram chasing involving formal schemes; see [Fal+92, Ch. IV, Prop. 3.7]. □

8.3 Proof of Key Result 8.1

We are now ready to prove Key Result 8.1. By Proposition 8.6, Corollary 8.7, and Proposition 8.8, we may assume that A has semistable reduction and has level m-structure for some composite $m \geq 3$ coprime to ℓ. Note that most of the argument is due to Tate, and could have been presented in §2.4.

Proof of Key Result 8.1. Let A_n as in the Setup 8.4. Since A has semistable reduction, by Theorem α, the Faltings height $h(A_n)$ of the A_n are bounded uniformly by some constant C. By Theorem β^*, this implies there is a sequence $n_1 < n_2 < \cdots$ such that $A_{n_i} \cong A_{n_j}$. We then define isogenies

$$u_i\colon A \xrightarrow{f_{n_1}^{-1}} A_{n_1} \xrightarrow[\sim]{v_i} A_{n_i} \xrightarrow{f_{n_i}} A,$$

where we note f_{n_i} is of order $\ell^{n_i \dim X}$, and satisfies

$$f_{n_i}(T_\ell(A_n)) = W \cap T_\ell(A) + \ell^n T_\ell(A) =: X_n.$$

Viewed in $\operatorname{End}_{\mathbf{Q}_\ell}(V_\ell(A))$, each u_i maps X_{n_1} onto $X_{n_i} \subset X_{n_1}$, since

$$u_i(X_{n_1}) = u_i f_{n_1}(T_\ell(A_{n_1})) = f_{n_i} v_i T_\ell(A_{n_1}) = f_{n_i} T_\ell(A_{n_i}) = X_{n_i}.$$

Since by definition $X_{n_i} \subset X_{n_1}$, this says the u_i all preserve the lattice X_{n_1} in $T_\ell(A)$. Thus, the u_i all lie in a compact subspace $\operatorname{End}_{\mathbf{Z}_\ell}(X_{n_1}) \cap E_\ell \subset E_\ell \coloneqq \operatorname{End}_K(A) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell$. By compactness, passing to a subsequence of the n_i, the sequence u_i converge to a limit $u \in \operatorname{End}_{\mathbf{Z}_\ell}(X_{n_1}) \cap E_\ell$. Now consider

$$U \coloneqq W \cap T_\ell(A) = \bigcap_{i \in I} X_{n_i}.$$

Since $u_i(T_{n_1}) = T_{n_i}$ as in (8.3), every $x \in U$ is a limit $\lim_{i\in I} u_i(x_i)$ of $x_i \in T_{n_1}$. Passing to a convergent subsequence x_j of these x_i gives that x is the limit of $u(\lim_{j\in J} x_j)$, and so $u(T_\ell(A)) = W \cap T_\ell(A)$, and so $u(V_\ell(A)) = W$. □

Remark 8.10. In [Fal86, §5; Lev11, §4], because of the choice of moduli space, a variant of Theorem β is used, where there is no level structure involved. In [Fal+92, p. IV], Schappacher decides to use a similar variant of Theorem β^*, where again no level structure is involved. If we wanted to assume abelian varieties were principally polarized in our finiteness theorem, then we would have to deal with another reduction step in the proof of Key Result 8.1, which involves considering maximally isotropic subspaces W, first, similarly to how the proof of Tate's isogeny theorem went for the weak finiteness hypothesis in §2.6. The disadvantage to this approach is that you have to reprove a portion of Zarhin's trick at the end to reduce the case of a general subspace to the case of a maximally isotropic one; see [Fal+92, IV, n° 4.5]. On the other hand, this method would avoid Proposition 8.9, which we had to sketch.

8.4 Consequences

We list some consequences of Faltings's isogeny theorem **D** and **E**. Recall that we have nice descriptions of when two abelian varieties are isomorphic in the finite field case (Corollary 2.2); there, we used zeta functions, and so you would want some statement involving (Hasse–Weil) L-functions. The L-function for A is defined to be

$$L(A,s) = \prod_v \frac{1}{\det(1-(N_v)^{-s}\cdot F_v \mid T_\ell(A)^{I_v})} = \prod_v L_v(A,s),$$

where ℓ can be any prime not dividing v, N_v is the cardinality of the residue field at v, F_v is the Frobenius element at v, and $I_v \subset G_K$ is the inertia subgroup at v. Then, we have the following:

Corollary 8.11. *Let A_1, A_2 be abelian varieties over K. Then, the following are equivalent:*

(i) *A_1 and A_2 are isogenous;*
(ii) *For all ℓ, $V_\ell(A_1) \cong V_\ell(A_2)$ as G_K-modules;*
(iii) *For some ℓ, $V_\ell(A_1) \cong V_\ell(A_2)$ as G_K-modules;*
(iv) *$L_v(s,A_1) = L_v(s,A_2)$ for almost all places v of K;*
(v) *$L_v(s,A_1) = L_v(s,A_2)$ for all places v of K.*

We already did $(i) \Leftrightarrow (ii)$ in the proof of Corollary 2.2.

Proof. $(i) \Leftarrow (ii)$. Note that $f\colon A_1 \to A_2$ is an isogeny if and only if $T_\ell(f)$ has full rank, i.e., $\det T_\ell(f) \neq 0$.

$(ii) \Rightarrow (iii)$ is clear.

$(iii) \Rightarrow (i)$. Suppose $\varphi\colon V_\ell(A_1) \to V_\ell(A_2)$ is an isomorphism of G_K-modules. Choose n such that $\ell^n\varphi \in \mathrm{Hom}(T_\ell(A_1), T_\ell(A_2))$. By the isogeny theorem, this comes from $\mathrm{Hom}_K(A_1,A_2) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell$, and can be approximated by elements of $\mathrm{Hom}_K(A_1,A_2)$. Since $\det(\ell^n\varphi) \neq 0$, these approximations will also have nonvanishing determinant, and this way you can get an isogeny.

$(iii) \Rightarrow (v)$. L-factors can be read off of the Tate module for ℓ not dividing v, and so isomorphisms for all ℓ mean that we have the same L-factors.

$(v) \Rightarrow (iv)$ is clear.

$(iv) \Rightarrow (iii)$. By Čebotarev density, the set of all F_v for all but finitely many v is dense in G_K, and so we know the characteristic polynomial of $g \in G_K$. It is a general fact that a semisimple representation is determined by the characteristic polynomial, and the $V_\ell(A_1), V_\ell(A_2)$ are semisimple by statement **D**. ☐

Corollary 8.12. *Let A be an abelian variety over K. Then, there are only finitely many isomorphism classes of abelian varieties B over K such that for all ℓ, $T_\ell(A) \cong T_\ell(B)$.*

Sketch. By assumption and the isogeny theorem **E**, there exists an isogeny $A \to B$ with degree prime to ℓ for all ℓ. As before, we can freely extend the ground field, and therefore assume A and all B's have semistable reduction and have level structure. By choosing the isogenies above correctly, there exists an N such that for every prime number ℓ and all B, there exist isogenies $\phi\colon A \to B$ for which the greatest power of ℓ in $\deg\phi$ divides N (see [Fal+92, V, Lem. 3.2]). Finally, [Fal86, Rem.] says that

$$\exp(2[K:\mathbf{Q}]\cdot(h(B)-h(A))) \in \mathbf{Q},$$

whose numerator and denominator divide a certain power of N. Applying Theorem β^*, we are done. ☐

9 November 17: Raynaud's theorem on finite flat group schemes (Valia Gazaki)

References for this talk are [Tat97; Sno13, Lec. 7].

9.1 Statement of Raynaud's Theorem and initial reductions

The goal for today is the following:

Theorem 9.1 (Raynaud)**.** *Suppose $e < p-1$, and let G_0 be a finite group scheme over K. Then, unique prolongation holds for G_0.*

We first need to explain everything in the statement.

Setup 9.2. $K/\mathbf{Q}_p$ is a finite field extension, R is a ring of integers, k is the residue field of R, and e is the ramification index of K.

Reminders 9.3.

(1) *By a finite flat group scheme $G = \operatorname{Spec} A$ over R, we mean that A is a free R-mod which is also a finite flat K-algebra, together with a comultiplication morphism, etc.. We denote* $\operatorname{ord}(A) = \operatorname{rk}_R A$.

(2) *There exist kernels and cokernels for morphisms between finite flat group schemes: a sequence*

$$0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0$$

is exact if $G' \hookrightarrow G$ is closed and exact, and $G \to G''$ is finite and flat.

(3) *Given G, there exists a Cartier dual $G^\vee = \operatorname{Spec} A^\vee$, where $A^\vee = \operatorname{Hom}_{R\text{-}\mathsf{Mod}}(A, R)$.*

(4) *If Γ is a finite abelian group (e.g., a continuous $\operatorname{Gal}(\overline{K}/K)$-mod), then $\Gamma = \operatorname{Spec} A$, where $A = \{$functions $\Gamma \to K\}$, is a constant group over K.*

Definition 9.4. If G_0/K is a finite group scheme, a *prolongation* of G_0 is a finite flat group scheme G/R such that $G \otimes_R K \xrightarrow{\sim} G_0$. We say that G_0/K satisfies *UP* or *unique prolongation* if any two prolongations are isomorphic.

Remark 9.5. The bound on e is strict: $\mu_p \subset K$, $e \geq p-1$, then

$$(\mu_p)_K \xrightarrow{\sim} (\mathbf{Z}/p)_K,$$

but there does not exist a nontrivial morphism $(\mu_p)_R \to (\mathbf{Z}/p)_R$, since the former is connected and the latter is étale.

We start with an outline of the proof of Theorem 9.1:

Step 1. Some reductions. The main reduction is to the case of simple groups.

Step 2 (Heart of proof)**.** The structure theorem for Raynaud F-module schemes.

Step 3. Prove the theorem by hand for the objects that show up in the structure theorem.

Properties 9.6 (of prolongations)**.**

(1) Let $G_0 = \operatorname{Spec} A_0$, where A_0 is a finite Hopf algebra over K. If $G = \operatorname{Spec} A$ is a prolongation, then A is a finite R-subalgebra of A_0, closed under comultiplication, and spans all of A_0 (this implies A is an order of A_0; the two properties together imply closure under the antipode).

(2) If $f\colon A_1 \to A_2$ is a map of prolongations (in particular, it is compatible with reduction to K), then $f\colon A_1 \hookrightarrow A_2$.

(3) Prolongations of A_0 are partially ordered by inclusion. Any two prolongations have a supremum and an infimum.

Definition 9.7. $G_1 \geq G_2 \iff A_2 \hookrightarrow A_1 \iff \exists G_1 \to G_2$.

Then, $A_1 \cdot A_2 = \sup(A_1, A_2)$, and the infimum is given by Cartier duality.

(4) We have the following:

Proposition 9.8. *If G_0 has a prolongation, then there exists G^+ maximal and G^- minimal, which are unique*

Proof. It suffices to show the ascending chain condition: if there is a sequence $A_1 \subset A_2 \subset \cdots$ of prolongations, then it stabilizes. You can think of this sequence as an ascending chain of orders of A_0. This stabilizes because A_0 has a maximal order. Thus, the chain above stabilizes, and there exists a maximal A^+. By Cartier duality, you get a minimal A^-. They are unique by taking products, as in (3). □

There is a map $A^- \hookrightarrow A^+$, and so you get a map $G^+ \to G^-$.

We start with some preliminary reductions, corresponding to Step 1 above.

Reduction 0. It suffices to show $f\colon G^+ \xrightarrow{\sim} G^-$.

Reduction 1. It suffices to show f is an isomorphism after base change.

Reduction 2. It suffices to show unique prolongation for simple groups.

Proof of Reduction 2. It suffices to show that UP behaves well under short exact sequences:

$$0 \longrightarrow G_0' \longrightarrow G_0 \longrightarrow G_0'' \longrightarrow 0$$

We want to show that if UP holds for G_0', G_0'', then it holds for G_0.

Suppose G_0 has a prolongation G. Then, consider the scheme-theoretic closure G' of G_0' in G. This is a prolongation of G_0'. Take $G/G' = G''$. This will be a prolongation of G_0''. Moreover, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & G' \arrow[r] \arrow[d, dashed, "\simeq"] & G \arrow[r] \arrow[d] & G'' \arrow[r] \arrow[d, dashed, "\simeq"] & 0 \\
0 \arrow[r] & H' \arrow[r] & H \arrow[r] & H'' \arrow[r] & 0
\end{tikzcd}$$

and the middle morphism is an isomorphism by the five lemma. □

9.2 Raynaud F-module schemes

We start with a preliminary discussion. Let G_0/K be a simple group of p-power order. Let $V = G_0(\overline{K})$. Then, V is an irreducible $\mathbf{F}_p$-representation of $\mathrm{Gal}(\overline{K}/K) = G_K$ (here you need the fact that G_0 is simple). Since V is irreducible, by Schur's theorem, $\mathrm{End}_{G_K}(V)$ is a division algebra over $\mathbf{F}_p$. Then, $\mathrm{End}_{G_K}(V)$ is central over some finite extension F of $\mathbf{F}_p$. Thus, $\mathrm{End}_{G_K}(V) \in \mathrm{Br}(F) = 0$, and so $\mathrm{End}_{G_K}(V) = F$. Thus, V can be thought of as an absolutely irreducible F-linear representation of G_0.

Now suppose $k = \overline{k}$ by passing to K^{ur}. Then, G_K is equal to the inertia group I_K, and we have a short exact sequence

$$1 \longrightarrow I^w \longrightarrow I_K \longrightarrow I^t \longrightarrow 0,$$

where I^w is the wild subgroup (which is a pro-p group), and I^t is the tame quotient. Then, $V^{I_w} \neq 0$ (by using the orbit-stabilizer theorem), and so $V^{I_w} = V$ (since the fact that I_w is a normal subgroup of I_K implies the invariants form a subrepresentation), i.e., there is no wild action.

Now consider V as an F-vector space. Then, we must have $\dim_F V = 1$, since if it were larger, we would get nontrivial subrepresentations, contradicting its irreducibility.

This suggests the following definition:

Definition 9.9.

(1) An *F-module scheme* over K or R, for a finite field F of p-power order, is a finite group scheme G, equipped with a ring homomorphism $F \to \mathrm{End}(G)$ (so it is an F-vector space object). The image of t will be denoted by $[t]$.

(2) A *Raynaud F-module scheme* also satisfies that $\dim_F G(\overline{K}) = 1$. We will frequently refer to them as *Raynauds* for short.

Proposition 9.10. *Suppose $k = \overline{k}$, and G_0/K is a finite simple group scheme of p-power order. Then, G_0 is canonically a Raynaud F-module scheme for some F.*

Reduction 3 (no hypothesis on k)**.** Suppose UP holds for every Raynaud over K^{ur}. Then, UP holds for any finite group over K.

Proof of Reduction 3. Suppose there exists a prolongation G of G_0. From one of the previous reductions, it suffices to show $G^+ \to G^-$ is an isomorphism, and moreover to show that $G^+ \otimes \mathcal{O}_{K^{\mathrm{ur}}} \to G^- \otimes \mathcal{O}_{K^{\mathrm{ur}}}$ is an isomorphism. Thus, we may assume $k = \overline{k}$; moreover, it suffices to show this for simple groups.

Case 1. If G_0 has p-order, then G_0/K^{ur} is Raynaud, so $\simeq$ holds by assumption.

Case 2. If G_0 has order coprime to p, then it is étale, and so UP holds automatically. □

Remark 9.11. If G_0 is Raynaud over K, and there exists a prolongation G, it is not necessarily true that G is Raynaud over R. However, G^+, G^- will be, since F acts by automorphisms, so will preserve maximal objects.

9.3 The structure theorem for Raynaud F-module schemes

Let F be a finite field of order $q = p^r$. Assume k contains the $q-1$ roots of unity.

Definition 9.12. A character $\chi \colon F^\times \to R^\times$ is *fundamental* if the composite map $F^\times \to R^\times \to k^\times$ extends to an embedding $F \hookrightarrow k$. So χ will just be a power of the Frobenius k/F. If χ' is another fundamental character, then $\chi' = \chi^{p^m}$ for some $m \in \mathbf{Z}$. Now order fundamental characters in some index set I, such that $\chi_i^p = \chi_{i+1}$, so that I becomes a torsor of $\mathbf{Z}/r\mathbf{Z}$. If $\mu \colon F^\times \to R^\times$ is any character, then $\mu = \prod_{i \in I} \chi_i^{a_i}$. This is a unique expression if $0 \le a_i < p$, and not all $a_i = 0$. We will write

$$\mu = \prod_i \chi_i^{\mu(i)}.$$

Example 9.13. If $\mu = 1$, then $\mu(1) = p - 1$.

We start with an initial analysis of the structure of an arbitrary Raynaud F-module scheme. Let $G = \operatorname{Spec} A$ be a scheme, which is a Raynaud F-module over R. Then, $G_{\overline{K}}$ is a constant group scheme for some F, and $G(\overline{K}) = F$ by the Raynaud assumption. Note $G_{\overline{K}} = \operatorname{Spec} A_{\overline{K}}$, where $A_{\overline{K}} = \{\text{functions } F \to \overline{K}\}$. Let $\mu \colon F^\times \to k^\times$ be a character. This extends to a character $\varepsilon_\mu \colon F \to \overline{K}$, where $\varepsilon_\mu(0) = 0$. This character lives in $\overline{A}_K$. We denote $\varepsilon_i = \varepsilon_{\chi_i}$, and let I be the augmentation ideal of A ($I \overset{ker}{\to} A \overset{0}{\to} R$).

Remark 9.14. Let $t \in F$. Then $[t] \colon A \to A$ descends to a map $[t] \colon I \to I$, all of whose eigenvalues are defined over R, since we assumed the $q-1$ roots of unity are in R. The characters μ are then defined over R as well.

Now for any $t_1, t_2 \in F$, the induced morphisms $[t_1], [t_2]$ commute, and so there exists a basis of common eigenvectors:

$$I = \bigoplus_{\mu \text{ char}} I_\mu$$

where $I_\mu = \{x \in I \mid [t] \cdot x = \mu(t) \cdot x \ \forall t \in F^\times\}$.

Claim 9.15. $\mathrm{rk}_R(I_\mu) = 1$.

It suffices to show $\dim_{\overline{K}}(I_\mu \otimes \overline{K}) = 1$. This is easy, since you can show ε_μ spans $I_\mu \otimes \overline{K}$: if f is in this space, then $[t]f(s) = \mu(t)f(x)$, and then by looking at the equation $f(ts) = \mu(t)f(s)$ for $s = 0$ and $s = 1$.

So now choose a generator X_μ of I_μ over R. By the relation on the fundamental characters $X_i = X_{\chi_i}$, its p-power lies in $I_{X_{i+1}}$, so that there exists $\delta_i \in R$ such that $X_i^p = \delta_i X_{i+1}$. We want to show that these are the only relations that we get, so that we have a nice structure theorem for Raynaud F-module schemes.

Theorem 9.16 (Structure theorem)**.** *Let $G = \operatorname{Spec} A$ be a Raynaud F-module. Then,*

$$A \simeq \frac{R[X_i]_{i \in I}}{(X_i^p - \delta_i X_{i+1})}$$

where $v(\delta_i) \le e$.

Our initial analysis shows there is a morphism from the right-hand side to A, but there are three more things to show: first, that the X_i generate everything; second, there are no more relations; and third, the valuation statement.

For a fundamental expansion $\mu = \prod \chi_i^{\mu(i)}$, define

$$X^\mu = \prod_{i \in I} X_i^{\mu(i)} \in I_\mu.$$

Take also $\varepsilon^\mu = \prod \varepsilon_i^{\mu(i)} = \varepsilon_\mu$.

Let $\operatorname{Spec} B = G^\vee$ (Cartier dual, not vector space dual). Then, $B = \operatorname{Hom}_{A\mathsf{Mod}}(A, R)$, and base changing to $\overline{K}$ gives $B_{\overline{K}} = \operatorname{Hom}_{\overline{K}}(A_{\overline{K}}, \overline{K}) \simeq \overline{K}[F]$.

Notation 9.17.

(1) If $t \in F$, then denote $\{t\}$ to be the corresponding generator of $\overline{K}[F]$. Then, $\{t\} \cdot \{s\} = \{t + s\}$.

(2) Also we have an F-vector space structure $[t] \colon \overline{K}[F] \to \overline{K}[F]$, where $[t]\{s\} = \{ts\}$.

Remark 9.18. $G^\vee$ is an F-module scheme (by reversing the arrows). $B_{\overline{K}}$ is one-dimensional over F, and so it is also Raynaud. Now if J is the augmentation ideal of B, then $J = \bigoplus_\mu J_\mu$.

Claim 9.19. *$\operatorname{rk}_\mu J_\mu = 1$, and $F_\mu \otimes \overline{K}$ is spanned by the following elements:*

- *If $\mu \neq 1$,*

$$e_\mu = \frac{1}{q-1} \sum_{t \in F^\times} \mu^{-1}(t)\{t\}.$$

- *If $\mu = 1$,*

$$e_1 = -1 + \frac{1}{q-1} \sum_{t \in F^\times} \{t\}.$$

(This is not hard.)

Now the X_i generate I_i, where $X_i = c_i \cdot \varepsilon_i$ in $\overline{K}$. Then, we choose $Y_i = c_i^{-1} \cdot e_i$. These X_i, Y_i give a basis of J_{χ_i}.

Remark 9.20. There is a pairing $A \times \operatorname{Hom}_R(A, R) \to R$ by evaluation. Base changing to $\overline{K}$ gives a pairing $A_{\overline{K}} \times B_{\overline{K}} \to \overline{K}$, where the former is functions and the latter is $\overline{K}[F]$, given by $\langle f, \{t\} \rangle = f(t)$.

Claim 9.21. $\langle \varepsilon_i, e_j \rangle_{\overline{K}} = \delta_{ij}$.

Set $Y^\mu = \prod_{i \in I} Y_i^{\mu(i)} \in J_\mu$. A key step is then to compute $W_\mu = \langle X^\mu, Y^\mu \rangle$ and $W_i = \langle X_i^p, Y_i^p \rangle$.

If you write these expressions down, you obtain

$$W_\mu = \langle \varepsilon_\mu, e^\mu \rangle \quad W_i = \varepsilon_\mu(e^\mu).$$

These descriptions do not depend on G, and so to understand them, it suffices to consider the case when G is the constant group scheme F over R. In this case, you can show the following:

Key Proposition 9.22.

(1) $W_\mu \equiv \prod \mu(i)! \bmod p$.

(2) $W_i \equiv -p \bmod p^2$.

We can now prove the Structure Theorem 9.16.

Proof of the Structure Theorem 9.16. We proceed in steps.

Step 1. Show that X_i generate A as an R-algebra.

$\langle X^\mu, Y^\mu \rangle$ is a unit, and $X^\mu \in I^\mu$ implies it is a generator of I_μ.

Now $\langle X^\mu, Y^\nu \rangle = 0$ for $\mu \neq \nu$, and so by combining this with the fact shown above, we have that if $f(X_i)$ is any other relation in A with degree $< p$, then we can take any monomial to give a character μ, and $\langle f(X_i), Y^\mu \rangle = \alpha \langle X^\mu, Y^\mu \rangle$ must be both zero and nonzero, a contradiction. Thus,

$$A = \frac{R[X_i]}{X_i^p - \delta_i X_{i+1}}.$$

Step 2. Show $v(\delta_i) \leq e$.

Choose $Y_i = c_i^{-1} e_i$, and so $Y_i^p = \gamma_i \cdot Y_{i+1}$. Using that $W_i = \langle X_i^p, Y_i^p \rangle = \delta_i \gamma_i \equiv -p \bmod p^2$, and the fact that $\delta_i, \gamma_i \in R$, we obtain that $v(\delta_i) \leq e$. □

There is also a converse for this: the ring above can define a Raynaud group scheme. The idea is to base change to $\overline{K}$, and write $\delta_i = c_i^p / c_{i+1}$, in which case you get an isomorphism $A_{\overline{K}}$ the function algebra. You then need to show things are closed under (co)multiplication.

Conclusion 9.23. *A Raynaud F-module is always of the form G_δ, where $\delta = (\delta_i)_{i \in I}$ and $v(\delta_i) \leq e$.*

We can then describe maps between Raynauds:

Easy Property 9.24. *A morphism $G_\delta \to G_{\delta'}$ of Raynauds corresponds to $(\alpha_i)_{i \in I}$ such that $\alpha_{i+1} \delta_i = \alpha_i^p \delta_i'$.*

Finally, we can prove Raynaud's theorem 9.1 by hand for Raynaud F-module schemes, which by our previous reductions give the general theorem for all finite group schemes.

Proof of Raynaud's theorem 9.1. First, show that if $e < p - 1$, then a map $G_\delta \to G_{\delta'}$ of Raynauds must be a isomorphism. Then, apply this to $G^+ \to G^-$. Now choose α_i such that $v(\alpha_i)$ is maximal in the situation of the Easy Property above for this morphism. Then,

$$v(\text{lhs}) \leq v(\alpha_i) + v(\delta_i) \leq v(\alpha_i) + e.$$

Also,

$$v(\text{lhs}) \geq v(\alpha_i^p) = p v(\alpha_i).$$

These two imply $v(\alpha_i) = 0$, and so f is an isomorphism. □

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