Wednesday, October 16, 2024 2:10 AM

<u>Goal</u>

- Shafarevich Conj for a.v:

  Fix K, g, S: fin set of pl of K.

  $\exists$ only fin. a.v. $/K$ of dim g, good red o/s S.

($\Rightarrow$ Shafa. for curves $\xRightarrow{\text{Parshin-Kodaira}}$ Mordell).

- Thm F (Semi-simplicity)

  $\forall$ A a.v. $/K$. ℓ pr. the G_K-rep

  $$V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \text{ is s.s.}$$

- Thm G (Tate Conj)

  A, B a.v. $/K$

  $$\mathrm{Hom}(A,B) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_\ell[G_K]}(T_\ell A, T_\ell B).$$

<u>Steps</u>.

Thm D: A/K $\exists$ only fin. B/K isog to A.

Thm E: K, g, S. $\exists$ only fin. isog classes of a.v. $/K$, dim g, good red o/s S.

$D + E \Rightarrow$ Shafa.

Main Tool from last semester

NP (Northcott property) (田 5/23) $\forall g, d \in \mathbb{N}, t \in \mathbb{R}$.
$\exists$ fin. A of dim g def'd / any K w/ $[K:\mathbb{Q}] \le d$ s.t.
$h(A) < t$
Faltings height.

This semester: 1) Prove D' (weaker version of D).

Same talk
2) $D' \Rightarrow F + G$ (Same as Tate's
S.S. Tate Conj. thm / finite fld)
3) $F + G \Rightarrow E$
(fin of # of isog cl).

(bound # isog cl w/ properties
isom cl of Gal reps
w/ properties)

4) $NP + F + G \Rightarrow D$.

K # field A/K a.v. $G \subset A[p^\infty]$ p-div gp of ht h

$G = \varinjlim_n G[p^n]$. $A_n := A/G[p^n]$

$A \to A_1 \to A_2 \to \cdots$ isogenies each of deg p^h.

<u>Thm D'</u> Supp A has semi-stable red. Then $\{A_n\}/_{\text{isom}}$ is finite.

By NP, STP: $\{h(A_n)\}$ bounded.

actually, $h(A_n)$ is eventually const.

Need to study how $h(A_n)$ changes.

In general:

<u>Lem.</u> $\varphi : A \to B$ isog betw s. st ab var $/K$.

$\mathcal{G} = \ker(\mathcal{A} \to \mathcal{B})$

Néron models $/\mathcal{O}_K$

Then

$h(B) - h(A) = \frac{1}{\ }\ \log\left|e^*\Omega^1_{\mathcal{G}/}\right| - \frac{1}{2}\log(\deg\varphi)$

$$h(B) - h(A) = \frac{1}{[K:\mathbb{Q}]} \log \left| e^* \Omega^1_{\mathcal{G}/\mathcal{O}_K} \right| - \frac{1}{2} \log(\deg \varphi).$$

N.B. $\mathcal{G}/\mathcal{O}_K$ is quasi-finite, flat

but may not be finite.

Nevertheless, $e^* \Omega^1_{\mathcal{G}/\mathcal{O}_K}$ is a finite-card. $\mathcal{O}_K$-mod.

card is div. only by primes dividing $\deg \varphi$.

Pf. By def'n of Faltings height. $\boxed{\Xi}$.

Apply to $A \to A_n$: let $\mathcal{G}_n = \ker(\mathcal{A} \to \mathcal{A}_n)$.

$$h(A_n) - h(A) = \frac{1}{[K:\mathbb{Q}]} \log \left| e^* \Omega^1_{\mathcal{G}_n/\mathcal{O}_K} \right| - \frac{1}{2} \log p^{hn}.$$

For simp. $\boxed{K = \mathbb{Q}}$.

$$\left| e^* \Omega^1_{\mathcal{G}_n/\mathbb{Z}} \right| = \left| e^* \Omega^1_{\mathcal{G}_{n,\mathbb{Z}_p}/\mathbb{Z}_p} \right| \quad (*).$$

Technical issues.

① $\mathcal{G}_{n,\mathbb{Z}_p}$ may not be finite $/\mathbb{Z}_p$ (only q-fin, flat)

$\exists$ can. subgp $\mathcal{H}_n$, fin $/\mathbb{Z}_p$. same sp. fiber as $\mathcal{G}_{n,\mathbb{Z}_p}$. (flat)

$$(*) = \left| e^* \Omega_{\mathcal{H}_n/\mathbb{Z}_p} \right| .$$

② $\{\mathcal{H}_n\}_n$ may not be a p-div. gp.

Need a tricky argument to show $\{\mathcal{H}_n\}_n$ becomes p-div. if replace A by A/G_{n_0} for $n_0 \gg 0$.

$$(\mathcal{H}_n \rightsquigarrow \mathcal{H}_n / \mathcal{H}_{n_0})$$

For simp, we ass <u>A has good red. at p.</u>

$\Rightarrow$ $\{\mathcal{G}_{n,\mathbb{Z}_p}\}$ p-div. gp $/\mathbb{Z}_p$. denoted $\mathcal{G}$.

$d = \dim \mathcal{G}$.

Claim. $(*) = p^{dn}$

See Daxin Table 2.

Pf: $\mathcal{G}_{n,\mathbb{Z}_p} = \operatorname{Spec} A$. Tate: $v_p(\operatorname{disc} A) = dn \cdot p^{hn}$

General for f.f. gp: $(*) = |\mathbb{Z}_p/\operatorname{disc} A|^{1/\operatorname{rk} A}$, $\operatorname{rk} A = p^{hn}$

$= p^{dn}$ 図.

Remains to prove: $\boxed{d = \frac{1}{2} h}$

• Hodge-Tate reps.

$\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$. $K/\mathbb{Q}_p$ fin. $\Gamma_K = \mathrm{Gal}(\overline{K}/K) \curvearrowright \mathbb{C}_p$.

$\chi : \Gamma_K \to \mathbb{Z}_p^\times$ p-adic cycl. char.

$\forall \zeta$: p^n-th root of 1 in $\overline{\mathbb{Q}}_p$.

$\forall \sigma \in \Gamma_K$, $\sigma \cdot \zeta = \zeta^{\chi(\sigma)}$.

$\mathbb{C}_p(n) := \mathbb{C}_p \otimes \chi^n$ as G_K-rep.

Def. a cts f.d. $\mathbb{Q}_p$-lin rep V of Γ_K.

is called Hodge-Tate if

$V \otimes \mathbb{C}_p \cong \bigoplus \mathbb{C}_p(n)$'s. (allow repetition).

the n's that appear are called the HT wts of V.

Fact: • being HT is preserved by subquotients, $\otimes$, Λ^k, ...

• $\forall X$ sm proj var $/K$.

$H^i_{et}(X_{\overline{K}}, \mathbb{Q}_p)$ is HT-rep of Γ_K.

• 1-d rep $\alpha : \Gamma_K \to \mathbb{Q}_p^\times$ is HT $\Longleftrightarrow$

- 1-d rep $\alpha : \Gamma_K \to \mathbb{Q}_p^\times$ is HT $\Leftrightarrow$

  $\alpha = \psi \chi^k$. $\psi : \Gamma_K \to \mathbb{Q}_p^\times$ s.t. $\psi(I_K)$ finite.

  $k \in \mathbb{Z}$.

- (Tate, Dasah Talk 2)

  $\mathcal{G}$ p-div. gp $/ \mathcal{O}_K$.

  $$T_p(\mathcal{G}) := \varprojlim_n \mathcal{G}[p^n](\overline{K}) \circlearrowleft \Gamma_K.$$

  $V_p(\mathcal{G}) = T_p(\mathcal{G}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is HT,

  HT wts 0, 1.

  mult: $h-d$, d.

$\Rightarrow$ det $V_p(\mathcal{G})$ HT wt d.

Now, $G = \varinjlim G[p^n] \subset A[p^\infty]$ def'd over $\mathbb{Q}$.

$$V = T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \left(\varprojlim_n G[p^n](\overline{\mathbb{Q}})\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-rep. $\subset V_p(A)$.

Its restr. to $\Gamma_{\mathbb{Q}_p}$ is $V_p(\mathcal{G})$.

$\Rightarrow \det V$ is a 1-d $\mathbb{Q}_p$-rep of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

rest. to $\Gamma_{\mathbb{Q}_p}$ HT wt d.

Also, it's unram. at al. all prs.

<u>Lem</u> Any such rep α of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is of form

$\varphi \cdot \chi^d$. $\quad \varphi : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Q}_p^\times$ fin. order

$\chi : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_p^\times$ p-adic cycl.

Pf. Replace α by α/χ^d. WMA $d=0$.

$\Rightarrow \alpha|_{I_p}$ finite.

$$\alpha : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})^{ab} \longrightarrow \mathbb{Q}_p^\times$$

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})^{ab} = \hat{\mathbb{Z}}^\times = \underbrace{\prod_v \mathbb{Z}_v^\times}_{\text{Image of } I_v = \mathbb{Z}_v^\times}.$$

$\Rightarrow \alpha$ factors thru fin. prod $\prod_v \mathbb{Z}_v^\times$.

$\alpha|_{\mathbb{Z}_p^\times}$ fin. order.

$v \neq p \quad \alpha|_{\mathbb{Z}_v^\times}$ fin. order. automatically. $\boxtimes$.

$\nu \nmid p$ ch. $\mathbb{Z}_\nu^\times$ fin. order. automatically. $\boxed{}$.

Summary $V \subset V_p(A)$ subrep of $\Gamma_{\mathbb{Q}}$. $\dim = h$.

$$\det V = (\text{fin ord. char}) \cdot \chi^d.$$

<u>Pf of $d = \frac{1}{2}h$</u>.

Choose $\overline{\mathbb{Q}_p} \hookrightarrow \mathbb{C}$. $\rightsquigarrow |\cdot|$ on $\overline{\mathbb{Q}_p}$.

$\Rightarrow$ al. all pr ℓ, $|(\det V)(\mathrm{Frob}_\ell)|$

$$= |\chi^d(\mathrm{Frob}_\ell)| = |\ell^d|.$$

$$= \ell^d.$$

<u>Weil</u>: $\mathrm{Frob}_\ell \curvearrowright V_p(A) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}$ is s.s., every eigenval $\lambda \in \overline{\mathbb{Q}_p} \subset \mathbb{C}$ satisfies $|\lambda| = \ell^{\frac{1}{2}}$.

$\left(\Leftarrow \# A_{\mathbb{F}_\ell}(\mathbb{F}_{\ell^n}) \right.$ Weil bound. $\left. \right)$

(Special case of Weil conj.)

$$\Rightarrow \ell^d = \ell^{\frac{1}{2}\dim V} = \ell^{\frac{1}{2}h}$$

$\boxed{}$.