LECTURE NOTES ON KODAIRA-PARSHIN'S CONSTRUCTION

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Introduction

Kodaira-Parshin's construction is used to prove the following theorem.

Theorem 0.1. *Let K be a number field, S a finite of primes of K containing dyadic primes (i.e. primes containing 2), and C a geometrically connected complete nonsingular curve over K of genus g. Then there exists a finite extension L/K such that for every $P \in C(K)$, there exists a geometrically connected complete nonsingular curve C_P over L and a finite morphism*

$$\phi_P : C_P \to C \otimes_K L$$

with the following properties:

(1) C_P has good reduction outside the finite set

$$\{w : w \text{ is a prime of } L,\ w|v \text{ for some } v \in S\}.$$

(2) The genus $g(C_P)$ $P \in C(K)$ is bounded.
(3) ϕ_P is ramified exactly at P.

We also need the following theorem due to de Franchis.

Theorem 0.2. *Let C' and C be connected complete nonsingular curves over an algebraically closed field k. Suppose $g(C) \geq 2$. There are only finitely many non-constant morphisms $C' \to C$.*

0.3. **Shafarevich's conjecture implies Mordell's conjecture.** Apply Theorem 0.1 to a curve C of genus ≥ 2. By Shafarevich's conjecture and the properties (1) and (2) in Theorem 0.1, the isomorphism classes of C_P $(P \in C(K))$ form a finite set. By Theorem 0.2, for each isomorphism class X in this finite set, there are only finitely many non-constant morphisms $X \to C \otimes_K L$. Thus the set of isomorphism classes of pairs (C_P, ϕ_P) $(P \in C(K))$ is finite. For distinct $P_1, P_2 \in C(K)$, the pairs (C_{P_1}, ϕ_{P_1}) and (C_{P_2}, ϕ_{P_2}) are not isomorphic by the property (3) in Theorem 0.1. So $C(K)$ is finite.

1. Proof of Theorem 0.1

Remark 1.1. Let R be a discrete valuation ring with fraction field F and residue field κ. Suppose $\operatorname{char} \kappa \neq 2$. Let $f \in F^*$ and let π be a uniformizer. We can write $f = u\pi^r$ such that u is a unit and $r = v(f)$. We have

$$K[\sqrt{f}] = \begin{cases} K[\sqrt{u}] & \text{if } r \text{ is even,} \\ K[\sqrt{u\pi}] & \text{if } r \text{ is odd.} \end{cases}$$

One can check $K[\sqrt{u}]/K$ is an unramified extension and $K[\sqrt{u\pi}]$ is totally ramified. So $K[\sqrt{f}]$ is ramified if and only if $v(f)$ is odd.

Remark 1.2. Let C be a complete nonsingular curve over an algebraically closed field k with $\operatorname{char} k \neq 2$, f a nonzero element in the function field $K(C)$ of C, and C' the integral closure of C in $K(C)[\sqrt{f}]$. Write

$$(f) = 2D + P_1 \pm \cdots \pm P_n.$$

Then $C' \to C$ is ramified exactly over $P_1, \ldots, P_n$.

Remark 1.3. Let $C = \mathbb{P}^1_{\mathbb{Q}}$, and let $a, b \in \mathbb{Q} \cong \mathbb{A}^1(\mathbb{Q})$ be distinct rational numbers. Take

$$f_1 = \frac{x-a}{x-b}, \quad f_2 = p\frac{x-a}{x-b},$$

where p is a prime distinct from 2. We have

$$(f_1) = (f_2) = a - b$$

as divisors on C. Let C_i be the integral closure of C in $K[\sqrt{f_i}]$. Then both C_1 and C_2 are ramified exactly over a and b. The curve C_1 has good reductions at those prime $p \neq 2$ such that $\mathrm{sp}_p(a) \neq \mathrm{sp}_p(b)$, where

$$\mathrm{sp}_p : \mathbb{P}^1(\mathbb{Q}) \to \mathbb{P}^1(\mathbb{Z}/p)$$

is the specialization map of rational points for $\mathbb{P}^1_{\mathbb{Z}}$. But C_2 does not have good reduction at p.

Remark 1.4. We always work with a smooth proper algebraic curve $\pi : C \to \operatorname{Spec} K$ which is geometrically connected. Note that we have the following equivalence:

$$\begin{aligned} C \text{ is geometrically connected } &\Leftrightarrow \Gamma(C \otimes_K \bar{K}, \mathcal{O}_{C\otimes_K \bar{K}}) = \bar{K}, \\ &\Leftrightarrow \Gamma(C, \mathcal{O}_C) = K, \\ &\Leftrightarrow \mathcal{O}_{\operatorname{Spec} K} \cong \pi_* \mathcal{O}_C. \end{aligned}$$

A *good integral model* of C over R_S is a smooth proper morphism $\tilde{\pi} : \mathcal{C} \to \operatorname{Spec} R_S$ such that its base extension to K is $\pi : C \to \operatorname{Spec} K$. We claim that

$$\mathcal{O}_{\operatorname{Spec} R_S} \cong \tilde{\pi}_* \mathcal{O}_{\mathcal{C}},$$

so that by the Zariski connectedness theorem, the fibers of $\tilde{\pi}$ are geometrically connected smooth proper algebraic curves. Since $\tilde{\pi}$ is proper, $\tilde{\pi}_* \mathcal{O}_{\mathcal{C}}$ is a coherent $\mathcal{O}_{\operatorname{Spec} R_S}$-module. Let

$$A := \Gamma(\operatorname{Spec} R_S, \tilde{\pi}_* \mathcal{O}_{\mathcal{C}}) \cong \Gamma(\mathcal{C}, \mathcal{O}_{\mathcal{C}}).$$

Then A is an integral domain and a finite R_S-algebra. So A is integral over R_S. Let η be the generic point of $\operatorname{Spec} R_S$. Since the generic fiber $\tilde{\pi}$ is $\pi : C \to \operatorname{Spec} K$ and C is geometrically connected, we have

$$(\tilde{\pi}_* \mathcal{O}_{\mathcal{C}})_\eta \cong (\pi_* \mathcal{O}_C)_\eta \cong K.$$

We thus have $A \otimes_{R_S} K \cong K$. So the fraction field of A is also K. But R_S is integrally closed. So we have $R_S = A$, that is, $\mathcal{O}_{\operatorname{Spec} R_S} \cong \tilde{\pi}_* \mathcal{O}_{\mathcal{C}}$.

Lemma 1.5. *Let C be a complete geometrically connected nonsingular curve over a number field K, and let $2D + P_1 - P_2$ be a principal divisors on C such that $P_1, P_2 \in C(K)$. Choose $f \in K(C)^*$ such that*

$$(f) = 2D + P_1 - P_2.$$

Let C' be the integral closure of C in $K(C)[\sqrt{f}]$.

(i) *The morphism $C' \to C$ is ramified exactly at P_1, P_2.*

(ii) *Let S be a finite set of primes of K containing those v at which C has bad reduction, those v at which*

$$\mathrm{sp}_v(P_1) = \mathrm{sp}_v(P_2),$$

and those dyadic v. Extend C to a smooth proper morphism

$$\tilde{\pi} : \mathcal{C} \to \operatorname{Spec} R_S.$$

Let

$$2\mathcal{D} + \mathcal{P}_1 - \mathcal{P}_2$$

be the Zariski closure of $2D + P_1 - P_2$ in $\mathcal{C}$. Regard f as a rational function on $\mathcal{C}$. Then there exists a divisor $\mathfrak{d}$ of $\operatorname{Spec} R_S$ such that

$$(f) = 2\mathcal{D} + \mathcal{P}_1 - \mathcal{P}_2 + \pi^*(\mathfrak{d}).$$

If $\mathfrak{d} = 0$, then C' has good reduction outside S.

Proof. We have explained (i) and the last part of (ii) in the above Remarks 1.2 and 1.3. Since $(f) = 2D+P_1-P_2$, the generic fiber of $(f)-(2\mathcal{D}+\mathcal{P}_1-\mathcal{P}_2)$ is the zero divisor. So $(f)-(2\mathcal{D}+\mathcal{P}_1-\mathcal{P}_2)$ is a vertical divisor, that is, its irreducible components are contained in fibers of $\tilde{\pi}$ of closed points in $\operatorname{Spec} R_S$. The fibers of $\tilde{\pi}$ are smooth and connected by Remark 1.4. So $(f) - (2\mathcal{D} + \mathcal{P}_1 - \mathcal{P}_2)$ is of the form $\pi^*(\mathfrak{d})$. □

Lemma 1.6. *Let A be an abelian variety over a number field K. There exists a finite extension L of K such that $A(K) \subset 2A(L)$. If A can can be extended to an Abelian scheme $\mathcal{A} \to \operatorname{Spec} R_{A,S}$, then L can be taken to be unramified over S.*

Proof. By the Mordell-Weil theorem, $A(K)/2A(K)$ is a finite group. Choose a finite family $x_1, \ldots, x_n \in A(K)$ of representatives for the group $A(K)/2A(K)$. Choose L large enough so that $\frac{x_1}{2}, \ldots, \frac{x_n}{2}$ are defined in $A(L)$. □

1.7. **Proof of Theorem 1.** If $C(K) = \emptyset$, there is nothing to prove. Suppose $C(K)$ is not empty. We use a rational point to define a canonical morphism

$$\theta : C \to J,$$

where J is the Jacobian of C. The morphism $2 : J \to J$ is a finite étale morphism of degree 2^{2g}. Let $\phi : C' \to C$ be the base change of $2 : J \to J$ by θ. By Propositions 14 and 21 in Serre, *Algebraic groups and class fields theory*, C' is geometrically connected and ϕ' is finite étale of degree 2^{2g}.

Claim: C' has good reduction outside S. For any $P \in C(K)$ and $Q \in \phi^{-1}(P)$, the residue field $\kappa(Q)$ at Q is a finite extension of K unramified outside S.

Recall that $C \to \operatorname{Spec} K$ can be extended to a proper smooth relative curve $\mathcal{C} \to \operatorname{Spec} R_S$, and $2 : J \to J$ can be extended to the a finite étale morphism $2 : \mathcal{J} \to \mathcal{J}$ on the relative Jacobian $\mathcal{J}$ of $\mathcal{C}$. Let $\tilde{\phi} : \mathcal{C}' \to \mathcal{C}$ be the base change of $2 : \mathcal{J} \to \mathcal{J}$ via the canonical morphism $\mathcal{C} \to \mathcal{J}$. Then $\mathcal{C}'$ is smooth over $\operatorname{Spec} R_S$ and hence C' has good reduction outside S. Each $P \in C(K)$ defines an R_S-point $\tilde{P} : \operatorname{Spec} R_S \to \mathcal{C}$ (valuation criterion for the proper morphism $\mathcal{C} \to \operatorname{Spec} R_S$). Consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{C}' \times_{\mathcal{C}} \operatorname{Spec} R_S & \longrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{J} \\ \downarrow & & \downarrow & & \downarrow 2 \\ \operatorname{Spec} R_S & \xrightarrow{\tilde{P}} & \mathcal{C} & \longrightarrow & \mathcal{J}. \end{array}$$

The leftmost morphism $\mathcal{C}' \times_{\mathcal{C}} \operatorname{Spec} R_S \to \operatorname{Spec} R_S$ is finite étale with generic fiber $\phi^{-1}(P)$. So the residue fields of points in $\phi^{-1}(P)$ are finite extension of K unramified outside S. This finishes the proof of the claim. To go further, we need the following:

Hermite Theorem. *Let N be a positive integer, and let S be a finite set of primes of K. There are only finitely many extensions L/K unramified outside S with $[L : K] \leq N$.*

Given $P \in C(K)$ and $Q \in \phi^{-1}(P)$, we have $[\kappa(Q) : K] \leq 2^{2g}$. By the Hermite theorem, there exists a common finite extension L_1/K unramified outside S such that points in $\phi^{-1}(P)$ have coordinates in L_1 for all $P \in C(K)$. Let $P_1, P_2 \in \phi^{-1}(P)(L_1)$ be two distinct points. (Since L_1 contains the residue fields of all points in $\phi^{-1}(P)$, we have $|\phi^{-1}(P)(L_1)| = 2^{2g}$.) Denote the L_1-point $\operatorname{Spec} L_1 \to C$ above P by P_{L_1}. Let R_{S,L_1} be the integral closure of R_S in L_1. Extend P_1, P_2 and P_{L_1} to morphisms

$$\tilde{P}_1 : \operatorname{Spec} R_{S,L_1} \to \mathcal{C}', \quad \tilde{P}_2 : \operatorname{Spec} R_{S,L_1} \to \mathcal{C}', \quad \tilde{P}_{L_1} : \operatorname{Spec} R_{S,L_1} \to \mathcal{C}.$$

Consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{C}' \times_{\mathcal{C}} \operatorname{Spec} R_{S,L_1} & \longrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{J} \\ \downarrow & & \downarrow & & \downarrow 2 \\ \operatorname{Spec} R_{L_1,S} & \xrightarrow{\tilde{P}_{L_1}} & \mathcal{C} & \longrightarrow & \mathcal{J}. \end{array}$$

The leftmost morphism $\mathcal{C}' \times_{\mathcal{C}} \operatorname{Spec} R_{S,L_1} \to \operatorname{Spec} R_S$ is finite étale with generic fiber $\phi^{-1}(P_{L_1})$, and $\tilde{P}_1, \tilde{P}_2$ define sections of this finite étale morphism. These section and open and closed immersions and hence are disjoint. Thus $\mathrm{sp}_w(P_1)$ and $\mathrm{sp}_w(P_2)$ are distinct for all prime w lying above a prime v in S.

By Lemma 1.6, there exists a finite extension L_2/L_1 unramified over S such that $J'(L_1) \subset 2J'(L_2)$. So there exists a divisor D of $C' \otimes_{L_1} L_2$ such that $2D + P_1 - P_2$ is a principal divisor. Choose $f \in K(C' \otimes_{L_1} L_2)$ such that

$$(f) = 2D + P_1 - P_2.$$

Let $2\mathcal{D} + \mathcal{P}_1 - \mathcal{P}_2$ be the Zariski closure of $2D + P_1 - P_2$ in $\mathcal{C}' \otimes_{R_S} R_{S,L_2}$. Then as divisors on $\mathcal{C}' \otimes_{R_S} R_{S,L_2}$, we have

$$(f) = 2\mathcal{D} + \mathcal{P}_1 - \mathcal{P}_2 + \pi^* \mathfrak{d}$$

for a divisor $\mathfrak{d}$ on $\operatorname{Spec} R_{S,L_2}$. Let L be the Hilbert class field of L_2. Then the divisor $\mathfrak{d}$ in L_2 becomes a principle divisor (c) in L. Replace f by $c^{-1}f$. Then we have

$$(f) = 2\mathcal{D} + \mathcal{P}_1 - \mathcal{P}_2$$

as divisors in $\mathcal{C}' \otimes_{R_S} R_{S,L}$. Let C_P be the integral closure of $C' \otimes_K L$ in $K(C' \otimes_K L)[\sqrt{f}]$, and let $\phi_P : C_P \to C \otimes_K L$ be the composite

$$C_P \to C' \otimes_K L \xrightarrow{\phi} C \otimes_K L.$$

Then the properties (1) and (3) in Theorem 0.1 hold. Note that $\deg \phi_P = 2^{2g+1}$. By the Hurwitz formula, we have

$$2 - 2g(C_P) = 2^{2g+1}(2 - 2g) - (e_{P_1} - 1) - (e_{P_2} - 1).$$

where $e_{P_i} = 2$ are the ramification indices. It follows that $g(C_P)$ is bounded.

2. Proof of Theorem 0.2

For convenience, in this section, we work with an algebraically closed ground field k.

2.1. **Hilbert scheme.** Let X be a projective k-scheme. For any closed immersion $i : Y \to X$, define its *Hilbert polynomial* $P_Y(x) \in \mathbb{Q}[x]$ to be the polynomial such that

$$P_Y(n) = \dim_k H^0(Y, \mathcal{O}_Y(n))$$

for sufficiently large n. Let T be a connected k-scheme, and let $i : \mathcal{Y} \to X \times_k T$ be a closed immersion such that $\mathcal{Y}$ is flat over T. Then the Hilbert polynomial for the fiber

$$i_t : \mathcal{Y}_t \to X \otimes_k k(t)$$

is independent of P. (See Theorem III 9.9 of Hartshorne, *Algebraic Geometry*).

Fix a polynomial $P \in Q[x]$. Consider the functor

$$\mathrm{Hilb}_P : (k\text{-schemes}) \to (\text{sets})$$

so that for every k-scheme T, $\mathrm{Hilb}_P(T)$ is the set of closed subscheme $i : \mathcal{Y} \to X \times_k T$ such that $\mathcal{Y}$ is flat over T and that each fiber over a point in T has the Hilbert polynomial P. Grothendieck shows that this functor is representable by a k-scheme H_P of finite type, called the *Hilbert scheme.*

Let $i : Y \to X$ be a closed immeresion of X with the Hilbert polynomial P. It can be regarded as an element in $\mathrm{Hilb}_P(\mathrm{Spec}\, k)$, and hence a k-point in H_P. Denote by y the k-point in H_P corresponding to i. Let $T_y H_P$ be the Zariski tangent space of H_P at y, and let $\mathcal{I} = \ker(\mathcal{O}_X \to i_* \mathcal{O}_Y)$ be the ideal sheaf of i.

Proposition 2.2. *Notation as above. We have an isomorphism*

$$T_y H_P \cong \mathrm{Hom}_{\mathcal{O}_Y}(i^* \mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y),$$

Proof. Let $k[\epsilon]$ be the k-algebra generated by ϵ with the relation $\epsilon^2 = 0$. Then T_y can be identified with the set of closed subschemes $\mathcal{Y} \to X \otimes_k k[\epsilon]$ such that $\mathcal{Y}$ is flat over $k[\epsilon]$, and the base extension of $\mathcal{Y} \to X \otimes_k k[\epsilon]$ by

$$k[\epsilon] \to k, \quad \epsilon \mapsto 0$$

can be identified with $i : Y \to X$.

We study the affine version of this problem. Assume $i : Y \to X$ is given by

$$\mathrm{Spec}\, A/I \to \mathrm{Spec}\, A,$$

where I is an ideal of a ring A. Then T_y is the set of ideals J of $A[\epsilon]$ such that $A[\epsilon]/J$ is flat over $k[\epsilon]$ and

$$A[\epsilon]/J \otimes_{k[\epsilon]} k \cong A/I. \tag{2.2.1}$$

Let's prove this set is in one-to-one correspondence with the set $\mathrm{Hom}_{A/I}(I/I^2, A/I)$.

We have an exact sequence

$$0 \to k \xrightarrow{\epsilon} k[\epsilon] \to k \to 0,$$

where the arrow $k \xrightarrow{\epsilon} k$ is the map $a \mapsto a\epsilon$. Applying $A[\epsilon]/J \otimes_{k[\epsilon]} -$ to the above short exact sequence, the resulting sequence is still exact since $A[\epsilon]/J$ is flat over $k[\epsilon]$. Taking into account of (2.2.1), we get a short exact sequence

$$0 \to A/I \xrightarrow{\epsilon} A[\epsilon]/J \to A/I \to 0.$$

We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\epsilon} & A[\epsilon] & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A/I & \xrightarrow{\epsilon} & A[\epsilon]/J & \longrightarrow & A/I & \longrightarrow & 0. \end{array}$$

By the five lemma, it gives rise to an exact sequence

$$0 \to I \xrightarrow{\epsilon} J \to I \to 0.$$

Given $x \in I$, choose $x + a\epsilon \in J$ lifting x, where $a \in A$. The lift is not unique. Other lifts are of the form $x + (a+b)\epsilon$ with $b \in I$. We thus get a well-defined homomorphism

$$I \to A/I, \quad x \mapsto a + I.$$

One can check it induces a homomorphism

$$I/I^2 \to A/I.$$

Conversely, given a homomorphism $\phi : I/I^2 \to A/I$, define

$$J = \{x + \epsilon y : x \in I,\ y \in A,\ \phi(x) \equiv y \mod I\}.$$

Then J is an ideal of $A[\epsilon]$, $A[\epsilon]/J$ is flat over $k[\epsilon]$, and $A[\epsilon]/J \otimes_{k[\epsilon]} k \cong A/I$. □

Proposition 2.3 (Adjunction formula)**.** *Let C be a complete nonsingular curve embedded in a complete nonsingular surface X. Then*

$$2g(C) - 2 = C.(C + K_X),$$

where K_X is the canonical class of X.

Proof. Let $i : C \to X$ be the closed immersion and let $\mathcal{I}$ be the ideal sheaf of C. We have a short exact sequence

$$0 \to i^{-1}(\mathcal{I}/\mathcal{I}^2) \to i^*\Omega^1_X \to \Omega^1_C \to 0.$$

Let $\omega_X = \wedge^2 \Omega^1_X$. Then we have

$$i^*\omega_X \cong i^*\mathcal{I} \otimes_{\mathcal{O}_C} \Omega^1_C.$$

We thus have

$$i^*(\mathcal{L}(C) \otimes_{\mathcal{O}_X} \omega_X) \cong \Omega^1_C, \tag{2.3.1}$$

where $\mathcal{L}(C) \cong \mathcal{I}^{-1}$ is the invertible $\mathcal{O}_X$-module corresponding to C considered as a divisor on X. Taking degree on both sides of (2.3.1), we get $C.(C + K_X) = 2g(C) - 2$. □

2.4. **Proof of Theorem 2.** Let $\phi : C' \to C$ be a nonconstant morphism, let $X = C' \times_k C$, and let

$$\Gamma_\phi : C' \to X, \quad x \mapsto (x, \phi(x))$$

be the graph of ϕ. For any closed point P of C and any closed point P' of C', we have

$$\Gamma_\phi.(P' \times C) = 1, \quad \Gamma_\phi.(C' \times P) = |\phi^{-1}(P)| = d,$$

where $d = \deg(\phi)$. It follows that for any divisor $\mathfrak{d}$ of C and any divisor $\mathfrak{d}'$ of C', we have

$$\Gamma_\phi.(p_1^*\mathfrak{d}') = \deg(\mathfrak{d}'), \quad \Gamma_\phi.(p_2^*\mathfrak{d}) = d\deg(\mathfrak{d}), \tag{2.4.1}$$

where $p_1 : C' \times_k C \to C'$ and $p_2 : C' \times_k C \to C$ are the projections. We have

$$\Omega^1_X \cong p_1^*\Omega^1_{C'} \oplus p_2^*\Omega^1_C, \quad \omega_X \cong p_1^*\Omega^1_{C'} \otimes p_2^*\Omega^1_C.$$

So we have

$$K_X = p_1^*K_{C'} + p_2^*K_C.$$

Hence

$$\Gamma_\phi.K_X = \Gamma_\phi.(p_1^*K_{C'} + p_2^*K_C) = \deg K_{C'} + d\deg K_C = (2g'-2) + d(2g-2), \tag{2.4.2}$$

where g and g' are the geni of C and C', respectively. By the adjunction formula applied to the curve Γ_ϕ in X, we have

$$2g'-2=\Gamma_\phi.(\Gamma_\phi+K_X).$$

Combined with (2.4.2), we get

$$\Gamma_\phi^2=d(2-2g). \tag{2.4.3}$$

Since $g\geq 2$, we have $\Gamma_\phi^2<0$. Let $\mathcal{I}$ be the ideal sheaf of the closed immersion $\Gamma_\phi: C'\to X$. Recall that the *normal bundle* of Γ_ϕ in X is defined to be

$$\mathcal{N}=\mathcal{H}om_{\mathcal{O}_{C'}}(\Gamma_\phi^{-1}(\mathcal{I}/\mathcal{I}^2),\mathcal{O}_{C'})\cong\mathcal{H}om_{\mathcal{O}_{C'}}(\Gamma_\phi^*\mathcal{I},\mathcal{O}_{C'})\cong\Gamma_\phi^*(\mathcal{I}^{-1}).$$

We have

$$\Gamma_\phi^2=\deg\Gamma_\phi^*(\mathcal{L}(\Gamma_\phi))=\deg\Gamma_\phi^*(\mathcal{I}^{-1})=\deg\mathcal{N}.$$

So $\mathcal{N}$ is an invertible sheaf on C' of negative degree. It follows that $\Gamma(C',\mathcal{N})=0$. (Suppose D is a divisor on C' on such that $\mathcal{N}\cong\mathcal{L}(D)$. If we have a nonzero $f\in\Gamma(C',\mathcal{N})$, then $(f)+D\geq 0$ and $\deg(D)=\deg((f)+D)\geq 0$.) Let P be the Hilbert polynomial of the closed subscheme Γ_ϕ in X, and let y be the k-point in the Hilbert scheme H_P corresponding to Γ. By Proposition 2.2, we have

$$T_yH_P\cong\mathrm{Hom}_{\mathcal{O}_{C'}}(\Gamma^*(\mathcal{I}/\mathcal{I}^2),\mathcal{O}_{C'})\cong\Gamma(C',\mathcal{N})=0.$$

Those y is an isolated point in the Hilbert scheme H_P. But as a k-scheme of finite type, H_P has only finitely many isolated points. Thus there are only finitely many nonconstant morphism $\phi: C'\to C$ such that Γ_ϕ has a given Hilbert polynomial P.

Next we prove there are only finitely many possibilities for the Hilbert polynomial of Γ_ϕ. Let H be a very ample divisor on X so that $\mathcal{O}_X(1)=\mathcal{L}(H)$. By the Riemann-Roch theorem for the surface X, we have

$$\begin{aligned}\chi(X,(\Gamma_{\phi,*}\mathcal{O}_{C'})(n)) &= \chi(X,\mathcal{O}_X(n))-\chi(X,\mathcal{I}(n))\\ &= \chi(X,\mathcal{O}_X(n))-\Big(\frac{1}{2}(nH-\Gamma_\phi)(nH-\Gamma_\phi-K_X)+1+p_a\Big)\end{aligned}$$

From the last expression, to prove there are only finitely possibilities for the Hilbert polynomial, it suffices to show there are only finitely many possibilities for

$$\Gamma_\phi.H,\quad \Gamma_\phi.K_X,\quad \Gamma_\phi^2.$$

We can choose a very ample divisor of the form $H=p_1^*\mathfrak{d}'+p_2^*\mathfrak{d}$ for some very ample divisors $\mathfrak{d}'$ on C' and $\mathfrak{d}$ on C. By (2.4.1), (2.4.2) and (2.4.3), to prove there are only finitely many choice for the Hilbert polynomial of Γ_ϕ, it suffices to show there are only finitely many possibilities for the degree d of ϕ. This follows from the Hurwitz formula. Indeed, we have

$$(2-2g')=d(2-2g)-\sum_Q(e_Q-1)\leq d(2-2g),$$

where the summation is over the points Q in C' where ϕ is ramified. Since $g\geq 2$, we must have $d\leq\frac{2g'-2}{2g-2}$.

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