Overview of the proof

K : number field

Thm A (Mordell conj). Let X/K be a projective sm. curve. $g(X) \geq 2$

Then $|X(K)| < +\infty$.

Thm B (Shafarevich's conj) Let S be a finite set of places of K. Then given an integer $g \geq 1$. $\exists$ only finitely many curves over K of genus g, with good reduction outside S

Shafarevich $\Rightarrow$ Mordell

Thm (Parshin - Kodaira) : Given a curve X/K, with good red. outside S. Then $\exists$ finite ext. L/K, and a finite s.t. $\forall P \in X(K)$

$\exists$ curve C_P and a finite map $\varphi_P : C_P \to C_L$

s.t. (1) C_P has good reduction outside $S_L := \{ w \mid \begin{smallmatrix} \exists v \in S \\ w \mid v \end{smallmatrix} \}$

(2) C_P has bounded genus.

(3) φ_P is ramified only at P.

<u>Thm</u> (de Franchis) Let C' and C be curves over a field k, $g(C) \geq 2$.
Then $\exists$ finitely many non-constant maps $C' \to C$

<u>Reduction to counting of A.V.</u>

<u>Thm C</u> (Shafarevich conj. for A.V.)
Fix g K, S. Then there exist finitely many abelian varieties over K (up to isom), of dim. g with good reduction outside S.

<u>Thm C $\Rightarrow$ Thm B</u>:

$$\left\{\begin{matrix}\text{curves of genus} \\ g \geq 1 \text{ over } K\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{(P.P) AV. of dim.} \\ g\end{matrix}\right\}$$

$$C \longmapsto \mathrm{Jac}_C$$

- Torelli: this is injective;
- For a fixed A.V. A/K, $\exists$ finitely many polarizations $A \to A^\vee$ of fixed degree?

Finiteness Thm. of A.V.

Thm D. (Finiteness within one isogeneous class)
Fix A.V. A/K. Then $\exists$ only finitely many A.V. B/K isogenous to A.

Thm E (Finiteness of isogenous classes)
Fix $g>0$, K, S. Then finitely many isogenous classes of A.V. A/K of dim g with good red. outside S.

Ideas for Thm E: choose a prime ℓ. Consider

$$\left\{ \begin{matrix} \text{isogenous classes} \\ \text{of A.V. } /K \end{matrix} \right\} \longrightarrow \mathrm{Rep}_{G_K}(\mathbb{Q}_\ell)$$

$$A \longmapsto \rho_{A,\ell} : G_K \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(A))$$

$$V_\ell(A) := \left(\varprojlim_n A[\ell^n](\bar{K}) \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

2g-dim. $\mathbb{Q}_\ell$-v.s + cont. action by G_K.

Thm F: For any A/K, $\rho_{A,\ell}$ is semi-simple.

Thm G (Tate conj). Let A, B be A.V. over K.
Then the canonical map

$$\mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_\ell[G_K]}(T_\ell A, T_\ell B)$$

Thm F + Thm G $\Rightarrow$ Thm E.

A isogenous to $B \Leftrightarrow \rho_{A,\ell} \cong \rho_{B,\ell}$ as G_K-rep.

Reduces to showing. Given K, S, $d \geq 1$
$\exists$ only finitely many isom classes of ℓ-adic rep.

$$\rho : G_K \longrightarrow GL_n(\mathbb{Q}_\ell)$$

satisfying: (1) ρ is unramified outside S.

(2) $\forall\, v \notin S$, $\mathrm{tr}(\rho(\mathrm{Frob}_v)) \in \mathbb{Z}$.
and $|\mathrm{tr}(\rho(\mathrm{Frob}_v))| \leq n \cdot \sqrt{Nv}$.

(Chebotarev density)

Pf of Thm D, F, G: New ingredient: Faltings' height

A/K abelian var. of dim $g \geq 1$

Thm (Existence of Néron model). $\exists$ a smooth gp scheme $\mathcal{A}/\mathrm{Spec}(\mathcal{O}_K)$ separated of f.t. s.t.

(1) $\mathcal{A}_K = A$

(2) $\forall\ \mathcal{X}/\mathrm{Spec}(\mathcal{O}_K)$ smooth scheme, every map $X \to A$ extends uniquely to a morphism $\mathcal{X} \to \mathcal{A}$

<u>Def</u>: $\mathcal{A}$ is called the Néron model of A

<u>Rmk</u>: $\mathcal{A}$ is not necessarily proper.

Let: $e: \mathrm{Spec}(\mathcal{O}_K) \to \mathcal{A}$ be the unit section.

Define $\omega_{\mathcal{A}} := e^* \Omega^g_{\mathcal{A}/\mathcal{O}_K}$ line bundle on $\mathrm{Spec}(\mathcal{O}_K)$

- $M := \Gamma(\mathrm{Spec}(\mathcal{O}_K), \omega_{\mathcal{A}}) \leftarrow$ proj. $\mathcal{O}_K$-mod of rk 1.

- $\forall\ \tau: K \hookrightarrow \mathbb{C}$, $\omega \in \omega_{\mathcal{A}}$ define

$$\|\omega\|_\tau^2 := \frac{i}{2}\int_{A_\tau(\mathbb{C})} \omega \wedge \bar{\omega} \qquad dz \wedge d\bar{z} = -2i\, dx \wedge dy$$

<u>Define</u>: $h(A) := \frac{1}{[K:\mathbb{Q}]}\left(-\sum_{\tau: K \hookrightarrow \mathbb{C}} \log |\omega|_\tau + \log \left| M/_{\mathcal{O}_K \cdot \omega}\right|\right)$

<u>Thm H</u> (Northcott property). Fix K, $g > 0$, $C > 0$. Then $\exists$ only finitely many isom. classes of (P.P.)A.V. A/K of dim. g with $h(A) < C$.

<u>Rem</u>: By Zarhin's trick, one reduces to showing finiteness for PPAV.

Idea : Consider

S_g : moduli stack of P.P.A.V. of dim. g

$S_g \overset{\omega_A^{univ}}{\hookrightarrow} S_g^* \ (\omega_{A^*}^{univ,*})$ minimal compactification.

$\rightsquigarrow$ height function (Weil).

$$h_{\omega^*} : S_g(\bar{\mathbb{Q}}) \longrightarrow \mathbb{R}.$$

- $A/K \iff x : \mathrm{Spec}(K) \longrightarrow S_g$.
- Reduce Thm H to
  1. Compare $h(A)$ & $h_{\omega^*}(x)$.
  2. Show the Northcott property for h_{ω^*}

Once we have Thm H, to prove Thm D. it suffices to study the how $h(A)$ varies within one isogenous class.

Thm D' : Fix K. A/K with everywhere semi-stable reduction. p prime.
$G \subseteq A[p^\infty]$ be a p-divisible subgp.
$G_n := G[p^n]$. $A_n := A/G_n$.
Then the sequence $\{A_1, A_2, \cdots\}$ has only finitely many isom. of A.V. over K.

Idea: Need to study the variation of $h(A)$ within an isogenous class.

Lemma. $A \to B$ isogeny of A.V. of deg d. with semi-stable reduction at places over d. Then

$$h(B) - h(A) = \frac{1}{[K:\mathbb{Q}]} \log\left(e^{*}\Omega^{1}_{G/\mathcal{O}_K}\right) - \frac{1}{2}\log n.$$

$G := \ker(A \to B)$.

Cor: $H(A, B) := \exp\left(2[K:\mathbb{Q}]\,|h(B) - h(A)|\right)$ is an integer!

Key pt: Control $v_\ell(H(A, A_n))$.
(use p-adic Hodge theory, + p-divisible gps).

Thm D' $\Rightarrow$ Thm F + G: This argument was due to Tate (1966), same as the case of finite fields

Thm H + F. + G $\Rightarrow$ Thm D: Similar to the pf of Thm D'. We need to control
$$|h(A) - h(B)| \quad \text{for } B \text{ isogenous to } A$$

Prop (Horizontal Control). Let A be a P.P.A.V. over K with semi-stable reduction. Then $\exists$ integer N. s.t. if $A \xrightarrow{\varphi} B$ is an isogeny of degree prime to N. Then $h(A) = h(B)$.

Road map:

$$\left.\begin{array}{r} H \\ H \Rightarrow D' \Rightarrow F+G \end{array}\right\} \xRightarrow{\text{Horizontal}} D$$

$$F+G \Rightarrow E$$

$$\left.\begin{array}{l} D \\ E \end{array}\right\} \Rightarrow C \Rightarrow B \Rightarrow A$$

Lectures:

2. basics on A.V over finite fields.
3. Tate conj over finite fields.
4. Finiteness of isogenous classes.
   ($F+G \Rightarrow E$)
5. Heights and Arakelov Geometry.
   (Northcott property Weil's height function.
   relation with Arakelov's formulation.
   Northcott for open varieties (metric with log singularity))
6. Faltings height and Northcott on S_g.
   (Thm H)
7. Review of p-divisible gp and HT-decomp.

8. Variation of Faltings height under isogeny + Thm H $\Rightarrow$ Thm D'.

9. Faltings' isogeny Thm

$D' \Rightarrow F + G$.

10. Raynaud's Theorem and horizontal control. (Proof of Thm D)

11. Parshin-Kodaira's construction.

12. Torelli;