Abelian varieties.

§ Basic defs & facts.

S scheme. a <u>group scheme</u> over S is a scheme $G \to S$ together w/ <u>gp str</u>

$$\left.\begin{array}{lll} m: & G \times_S G \to G & \text{mult'n} \\ e: & S \to G & \text{neutral section} \\ i: & G \to G & \text{inversion} \end{array}\right\} \text{all } S\text{-morphs}$$

satis. usual axioms for a group.

Eg. $\forall$ S-sch T, $G(T)$ has gp str.
functorial in T.
(Yoneda).

Def. an <u>abelian scheme</u> $/S$ is a <u>proper</u> sm. gp sch. whose geom. fibers are connected.
(If $S = \operatorname{Spec} k$, say abelian variety).

<u>Thm</u> (Rigidity). A/S ab sch. G/S gp sch. separated$/S$
Any S-sch map $A \to G$ preserving neutral sections
"$1 \mapsto 1$" is a gp homo.

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<u>Cor</u>. ab schs are commutative. <u>Pf</u>. i is homo ∎.

<u>Write gp str as $+$.</u>

<u>Ex</u>. An <u>elliptic curve</u> $/S$ is a proper sm S-sch E, whose geom. fibers are conn. proj sm cvs of genus 1, together w/ distinguished section
$$e : S \to E.$$

<u>Fact</u>: E has unique str. of ab sch w/ e the neutral section.

$S = \operatorname{Spec} k$, $k = \bar{k}$. $\quad P + Q = R$

$$\iff [P] + [Q] \sim [R] + [e].$$

Fact ("Abel") this gives gp structure.

$$\left(\begin{array}{ccc} E & \xrightarrow{\sim} & \operatorname{Jac} E = \operatorname{Div}^0(E)/\sim \\ P & \mapsto & [P] - [e]. \end{array} \right)$$

<u>Ex</u>. Jacobian of a sm. proj. curve.

<u>Fact</u>. If S is spec of a field, (Weil) or normal, (Groth.)

Thm. If S is spec of a field, or normal, ...
then ab schs $/S$ are projective.
Not true in general.

We'll need "polarized ab. schs". By def'n they are (locally) proj. $/S$.

Thm (Chow?) $A/k=\bar{k}$ L ample line bdl $\Rightarrow$ L^3 very ample

§ Isogenies.

From now on, S is loc. noeth.

Def. $\varphi : A \to B$ hom. of ab. sch. is called an isogeny, if it's quasi-finite & surj.

Lem. isogenies are finite flat.

$\leadsto$ $\varphi : A \to B$ isog

$\ker(\varphi)$ is a finite flat gp sch $/S$. of locally constant order d (i.e. locally, $\mathcal{O}_{\ker(\varphi)}$ is a a finite free $\mathcal{O}_S$-mod. of rk d).

rep. by a gp sch for any hom of gps $/S$.

$d =: \deg(\varphi)$.

Rmk. $S = \mathrm{Spec}\, k$, $k = \bar{k}$, $\varphi : A \to B$ isog

1) $\Rightarrow \dim A = \dim B$

2) d = deg of ext of fn fields

3) If $d \neq 0$ in k

$\Rightarrow \ker(\varphi)(k)$ has d elts.

$\varphi : A(k) \to B(k)$ is d-to-1.

(finite étale).

Ex. $n \in \mathbb{Z} \setminus \{0\}$. $[n] : A \to A$, $x \mapsto \underbrace{x + \cdots + x}_{n}$

($n<0$, use $-$) is isog. $\deg = n^{2g}$. $g = \dim A$.

$\left(/\mathbb{C}. \quad \mathbb{C}^g/\Lambda \xrightarrow{n} \mathbb{C}^g/\Lambda \quad \text{kernel} \cong (\mathbb{Z}/n\mathbb{Z})^{2g} \right)$.

Pf later.

§ Line bdls.

$\boxed{A / k = \bar{k}}$

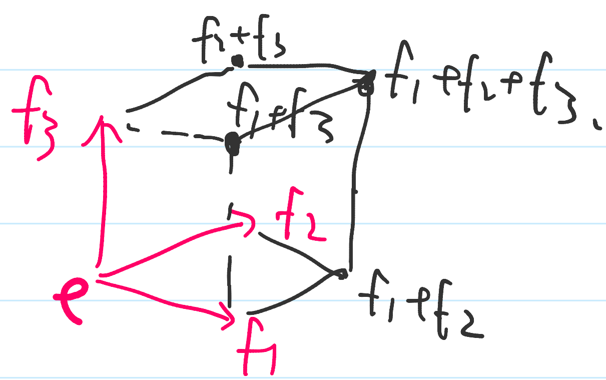
Thm of Cube:

$\mathcal{L}$ line bdl on A.

L line bdl on A.

$T \xrightarrow{f_i} A$ k-maps $i=1,2,3$.

Then
$$(f_1+f_2+f_3)^*L^{-1} \otimes (f_1+f_2)^*L \otimes$$
$$(f_1+f_3)^*L \otimes (f_2+f_3)^*L \otimes f_1^*L^{-1} \otimes f_2^*L^{-1} \otimes f_3^*L^{-1}$$
is trivial.



Cor. L on A, $[n]^*L \cong L^{n^2} \otimes \left(L \otimes [-1]^*L^{-1}\right)^{\frac{n-n^2}{2}}$.

Def. We call L symm if $[-1]^*L \cong L$.

Then $[n]^*L \cong L^{n^2}$.

Pf. $n=0$ or 1 ✓

Use induction to go up and down.

Apply Thm of cube to $T=A$

$f_1 = [n+1] \quad f_2 = [1] \quad f_3 = [-1]$.

(Exc) ☒

Pf that $[n]$ is isog of deg n^{2g}:

Reduce to case $S = \mathrm{Spec}\, k$, $k = \bar{k}$.

Take ample L on A (b/c A is proj.)

Replace L by $L \otimes [-1]^* L$.

$\Rightarrow$ WMA L is symmetric.

$\Rightarrow [n]^* L \cong L^{n^2}$. $(\star)$

If we know $[n]$ is isog, $(\star)$ gives

$\deg [n] = n^{2g}$ by standard tools

(Hilbert Poly or Intersection Theory)

To show: $Z := \ker [n]$ is finite.

$[n]^* L \cong L^{n^2}$, ample. so its restr. to Z

is ample. But this restr is trivial since

$[n] : Z \to \{e\} \subset A$. ☒.

<u>Thm of Square</u>. $a \in A(k)$, $t_a : A \to A$ transl.

by a. L l.b. on A. $\forall a, b \in A$.

Cor a. L l.b. on A. $\forall a, b \in A$,

$t_{a+b}^* L \otimes t_a^* L^{-1} \otimes t_b^* L^{-1} \otimes L$ is trivial.

$$\begin{tikzcd} t_a & t_{a+b} \arrow[l] \arrow[d, dash] \\ t_e = id \arrow[u] \arrow[r] & t_b \end{tikzcd}$$

Eg. define $\Lambda(L) : A(k) \to \mathrm{Pic}\, A = \{\text{l.b. on } A\}/\cong$

$a \mapsto t_a^* L \otimes L^{-1}$

then $\Lambda(L)$ is a homo.

Two Pfs 1) In thm of cube, take

$T = A$, $f_1 = t_e$, $f_2 = t_a$, $f_3 = t_b$.

2) $\mathrm{Pic}\, A$ has str. of **sep gp sch**$/k$. Check: $\Lambda(L) : 0 \mapsto 0$.

Rigidity $\Rightarrow$ $\Lambda(L)$ is homo.

Rmk.

A conn $\Rightarrow \Lambda(L)$ factors thru neutral conn. comp of $\mathrm{Pic}\, A$,

which is an ab. var! $A^\vee$ ($=:$ dual of A).

§ Dual ab. sch.

A/S. Define $\mathrm{Pic}_{A/S} : (S\text{-Schs}) \to (\text{Ab gps})$

$$T \longmapsto \left\{ (L,\rho) \;\middle|\; \begin{array}{l} L : \text{l.b. on } A_T := A \times_S T, \\ \rho : e_T^* L \xrightarrow{\sim} \mathcal{O}_T . \end{array} \right\} \Big/ \cong$$

$e_T : T \to A_T$ neutral section

((L,ρ) is called a rigidified line bundle).

ex. $T \to S$ a geometric point.

$$\mathrm{Pic}_{A/S}(T) = \mathrm{Pic}(A_T).$$

Thm. $\mathrm{Pic}_{A/S}$ is representable by a group scheme $/S$.

Let $\mathrm{Pic}^0_{A/S} = A^\vee$ be the max'l subgp sch. which has conn. geom. fibers. Then $A^\vee$ is an ab. sch $/S$.

Pf. Three constructions.

① Mumford [Abelian Varieties]. Explicitly construct $A^\vee$ by dividing A by a finite subgp (scheme)

works only for $S = \mathrm{Spec}\, k$. and specifically for A.V.

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② Grothendieck: projective methods, works for A/S (locally) projective

$\rightsquigarrow A^\vee$ is also projective (Rep. of $\mathrm{Pic}_{X/S}$. $X \to S$ proj, flat geom fibers integral)

③ General:

Artin: A/S abelian algebraic space

$\rightsquigarrow \mathrm{Pic}_{A/S}$, $A^\vee$ rep. by alg. sp.

Raynaud: Automatically, A, $\mathrm{Pic}_{A/S}$, $A^\vee$ are all schemes. A & $A^\vee$ are ab. schs.

Ref. [Faltings–Chai]. $\boxed{3}$.