## 1. Basic Facts and definitions

Let $S$ be a scheme. Recall that a *group scheme* over $S$ is a group object in $(\mathrm{Sch}/_S)$. More precisely,we have the following definition.

***Definition 1.1.*** — Let $S$ be a base scheme. A group scheme $G$ over $S$ is a scheme $G$ over $S$ with the following $S$-morphisms $m$, $e$, $i$:

$$\begin{aligned} m : G \times G &\longrightarrow G \\ e : S &\longrightarrow G \\ i : G &\longrightarrow G \end{aligned}$$

satisfying the usual group axioms for multiplicaiton, identity, and inversion. We also call $e$ the neutral section of the structure map.
Equivalently, we can also describe this definition by a concise category language, that is, we require that for any $S$-scheme $T$, the set $G(T) := Hom_S(T, G)$ is equipped with a group structure in a way functorial in $T$. Thus the functor of points defines a functor $G : (\mathrm{Sch}/_S) \longrightarrow Grp$.
By Yonneda lemma, we can see these two descriptions are indeed equivalent.

***Definition 1.2.*** — An *abelian scheme* over $S$ is a group scheme over $S$ which is proper, smooth and geometrically connected over $S$. When $S$ is the spectrum of a field, we also say *abelian variety.*

***Remark 1.3.*** — One can deduce that an abelian scheme is automatically commutative.

***Theorem 1.4* (Rigidity).** — *Let $A/S$ be an abelian scheme, and let $G$ be a separated group scheme over $S$. Any $S$-morphism $A \to G$ taking $e_A$ to $e_G$ is automatically a group homomorphism.*

***Corollary 1.5.*** — *The group structure of an abelian scheme $A/S$ is commutative.*

*Proof.* — The inverse map $i : A \to A$ is a group homomorphism, so $A$ is commutative. □

We shall always think of the group law on $A$ as addition.

***Example 1.6.*** — For an elliptic curve $E$ over $S$ (i.e. a proper smooth $S$-scheme $E/S$ such that each geometric fiber is a projective smooth connected curve of genus 1, together with a distinguished section $e : S \to E$), there is a unique group structure on $E$ such that $e$ is the neutral section. With this structure, $E$ is then an abelian scheme.

When $S = Spec(k), k = \bar{k}$ , the group structure is characterized by the condition:

$$P + Q = R \iff [P] + [Q] \overset{lin}{\sim} [R] + [e]$$

where $P, Q, R$ are points in $E$, and $e$ is the fixed zero point. By Abel's theorem the above condition indeed gives a well defined group structure (e.g., for given $P, Q$, such $R$ indeed exists). In fact, we have a group isomorphism $E \to \mathrm{Jac}(E) = \mathrm{Div}^0(E)/\sim$ defined by $[P] \mapsto [P] - [e]$ .

***Example 1.7.*** — Let $X$ be a smooth projective $S$-scheme whose geometric fibers are smooth projective connected curves. Then $\mathrm{Pic}^0_{X/S} = \mathrm{Jac}_{X/S}$ is an abelian scheme. When $S = \mathrm{Spec}\, k$, $k$ algebrically closed, we have $\mathrm{Jac}_{X/k} = \mathrm{Div}^0(X)/\sim$.

***Fact.*** — When $S$ is the spectrum of a field (Weil), or more generally $S$ is normal (Grothendieck), all abelian scheme over $S$ are projective, but not in general.

In this seminar, we only need projective abelian schemes (at least locally projective).

***Theorem 1.8* (Hard).** — *Let $S = \operatorname{Spec} k$, $k$ a field, $A/k$ an abelian variety. For any ample line bundle $\mathscr{L}$ on $A$, $\mathscr{L}^{\otimes 3}$ is very ample.*

***Example 1.9.*** — When $A = E$ is a elliptic curve, $\mathscr{O}(e)$ is ample $\Rightarrow \mathscr{O}(3e)$ is very ample. In fact, we have

$$\dim H^0(E, \mathscr{O}(ne)) = n.$$

by Riemannn–Roch, so in particular $\dim H^0(E, \mathscr{O}(3e)) = 3$. This is closely related to the construction of the Weierstrass equation.

Sketh of the idea: Denote $H^0(E, \mathscr{O}(ne))$ by $V_n$. We know that it is the vector space of rational functions in $k(E)$ which have a single pole at $e$ with order $\leq n$. We can choose functions $x, y \in k(E)$ such that $\{1, x\}$ is a basis of $V_2$ and $\{1, x, y\}$ is a basis of $V_3$. Since the dimension of $V_n$ is $n$, the orders of poles of $x$ and $y$ at $e$ are exactly 2 and 3 respectively.

We have $\dim V_6$=6, and it contains the seven functions:

$$1, x, y, x^2, xy, y^2, x^3$$

it follows that there is a linear relation:

$$A_1 + A_2x + A_3y + A_4x^2 + A_5xy + A_6y^2 + A_7x^3 = 0.$$

Then by some further arguments and transformation of the above equation, we will obtain the Weierstrass equation. The fact that $\mathscr{O}(3e)$ is very ample means that we indeed obtain an embedding of $E$ into the projective plane via $x, y$.

## 2. Isogeny

***Definition 2.1.*** — A morphism $\varphi : A \to B$ between abelian schemes is called an *isogeny* if it is quasi-finite and surjective.

***Lemma 2.2.*** — *Any isogeny is flat and finite.*

For any group morphism $f : G_1 \to G_2$ between group schemes over $S$, the functor $\operatorname{Ker}(f)$ is representable, and is a closed subgroup scheme of $G_1$ when $G_2$ are separated over $S$. (The morphism $\operatorname{Ker}(f) \to G_1$ is just the base change of $e_{G_2} : S \to G_2$ along $f$, and the latter is a closed immersion when $G_2$ is separated over $S$.) In particular, for $\varphi : A \to B$ an isogeny of abelian schemes, $\operatorname{Ker}(\varphi)$ is a commutative finite flat group scheme over $S$ since $\varphi$ is finite flat. We define the *degree* of $\varphi$ to be the order of $\operatorname{Ker}(\varphi)$. Here, for any finite flat group scheme $Z$ over $S$, locally on $S$, $\mathscr{O}_Z$ is a finite free $\mathscr{O}_S$-module of constant rank $d$. The integer $d$ is constant on each connected component of $S$, and we define it to be the rank of $Z$.

***Definition 2.3.*** — For any abelian scheme $A$ over $S$, any integer $n$, we define the homomorphism $[n] : A \to A$ to be the multiplication-by-$n$ map (with the group structure on $A$ regarded as addition).

***Example 2.4***. — Let $E$ be an elliptic curve over $\overline{\mathbb{F}_p}$. Take $\varphi = [p]$. This is an isogeny with $\deg(\varphi) = p^2$ but each fiber of $\varphi$ is a singleton (resp. has $p$ elements) for $E$ supersingular (resp. ordinary).

***Remark 2.5***. — Let $S = \operatorname{Spec} k$, $k$ algebrically closed. Let $\varphi : A \to B$ be an isogeny. We have the following facts:

(1) The existence of $\varphi$ implies that $\dim A = \dim B$.
(2) $\deg(\varphi) = [k(A) : k(B)]$, the extension degree of the function fields.
(3) Let $d = \deg(\varphi)$. Then $\operatorname{Ker}\varphi$ may have $< d$ points when $\operatorname{char} k > 0$, $\operatorname{char} k \mid d$.
If $\operatorname{char} k = 0$ or $\operatorname{char} k \nmid d$, then $\operatorname{Ker}\varphi$ is finite étale over $k$, so $\operatorname{Ker}\varphi$ has exactly $d$ points.

***Proposition 2.6***. — *Let $A/S$ be an abelian scheme, $n \in \mathbb{Z}$. $n \neq 0$. Then $[n] : A \to A$ is an isogeny of degree $n^{2\dim A}$.*

***Example 2.7***. — When $A$ is an abelian variety over $\mathbb{C}$, we have $A \simeq \mathbb{C}^g/\Lambda$, where $\Lambda$ is a lattice. The proposition above is then clear since

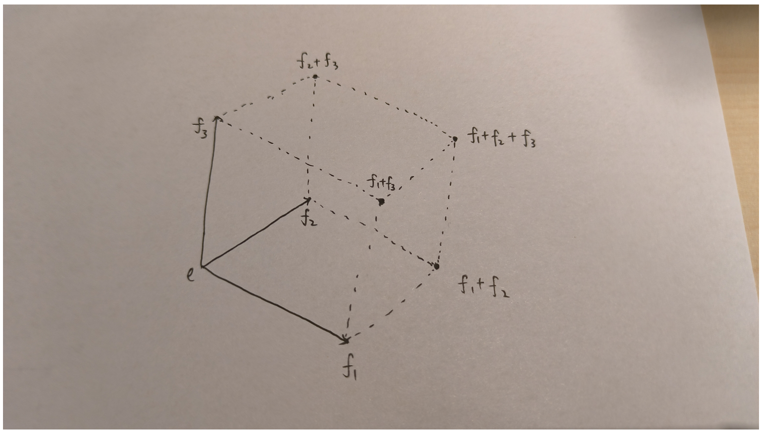
$$\operatorname{Ker}[n] \cong \frac{1}{n}\Lambda/\Lambda \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$$

## 3. Line Bundles

Let $k$ be an algebrically closed field, and let $A$ be an abelian variety over $k$.
Let $\mathscr{L}$ be a line bundle over $A$.

***Theorem 3.1* (Theorem of cube)**. — *Let $T$ be a $k$-scheme and let $f_1, f_2, f_3 : T \to A$ be $k$-morphisms. Then*

$$\begin{aligned} &f_1^*\mathscr{L} \otimes f_2^*\mathscr{L} \otimes f_3^*\mathscr{L} \\ &\otimes (f_1+f_2)^*\mathscr{L}^{-1} \otimes (f_1+f_3)^*\mathscr{L}^{-1} \otimes (f_2+f_3)^*\mathscr{L}^{-1} \\ &\otimes (f_1+f_2+f_3)^*\mathscr{L} \simeq \mathscr{O}_T. \end{aligned}$$




***Corollary 3.2***. — *For any $n \in \mathbb{Z}$, we have*

$$[n]^*\mathscr{L} = \mathscr{L}^{\otimes n^2} \otimes \underbrace{(\mathscr{L} \otimes [-1]^*\mathscr{L}^{-1})^{\frac{-n^2+n}{2}}}_{\textit{Junk term.}}$$

*Proof.* — For $n = 0$ or $1$ the claim is clear. We then prove the claim by induction, each time assuming the statement for $n = k, k + 1$ and prove it for $n = k - 1, k + 2$. For this induction step, take $f_1 = [n]$, $f_2 = [1]$, $f_3 = [-1]$ in the theorem of cube. □

***Definition 3.3.*** — For a line bundle $\mathscr{L}$ on $A$, we say $\mathscr{L}$ is *symmetric* if $\mathscr{L} \simeq [-1]^*\mathscr{L}$.

For such an $\mathscr{L}$, $[n]^*\mathscr{L} \simeq \mathscr{L}^{\otimes n^2}$.

*proof of Proposition 2.6.* — One easily reduce to $S = \operatorname{Spec} k$, $k$ algebrically closed.
1) If we knew $\varphi$ is an isogeny, take an ample line bundle $\mathscr{L}$ on $A$ (recall that in this case $A$ is projective). Then $\mathscr{L} \otimes [-1]^*\mathscr{L}$ is ample and symmetric. Replacing $\mathscr{L}$ by $\mathscr{L} \otimes [-1]^*\mathscr{L}$, we may assume $\mathscr{L}$ is symmetric. Then $[n]^*\mathscr{L} = \mathscr{L}^{\otimes n^2}$. By some standard tools (intersection theory, or Hilbert polynomials), we deduce that $\deg[n] = n^{2 \dim A}$.
2) Now we prove $[n]$ is an isogeny. It suffices to prove $Z = \operatorname{Ker}[n]$ is finite. With the same notation as in 1), $\mathscr{L}^{\otimes n^2}|_Z$ is ample. By $\mathscr{L}^{\otimes n^2} \cong [n]^*\mathscr{L}$, we obtain $\mathscr{O}_Z$ itself is ample. So $Z$ is quasi-affine and projective over $k$, which means $Z$ is finite over $k$ □

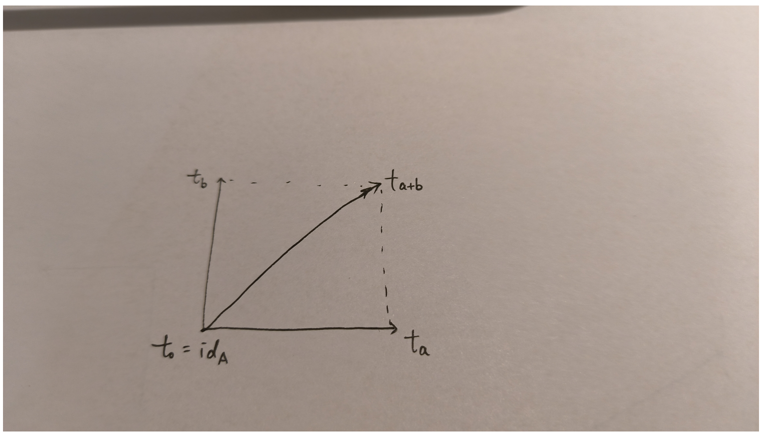
***Theorem 3.4* (Theorem of Square).** — *Let $\mathscr{L}$ be a line bundle over $A$. For any $a \in A$, denote by $t_a : A \to A$ the map $x \mapsto x + a$. For any $a, b \in A$, we have*

$$t_a^*\mathscr{L} \otimes t_b^*\mathscr{L} \simeq t_{a+b}^*\mathscr{L} \otimes \mathscr{L}.$$

*Equivalently, the map*

$$\Lambda(\mathscr{L}) : A \to \operatorname{Pic}(A), a \mapsto t_a^*(\mathscr{L}) \otimes \mathscr{L}^{-1}$$

*is a group homomorphism.*




*Proof.* — We give two proofs.
1) Use the Theorem of cube. Take $T = A$, $f_1 = a$, i.e., the composition $A \to \operatorname{Spec} k \to A$ where the second map is $a$, $f_2 = b$, and $f_3 = id_A$. Then

$$f_1 + f_2 + f_3 = t_{a+b}, f_1 + f_3 = t_a, f_2 + f_3 = t_b.$$

Theorem of cube immediately gives the result.
2) $\operatorname{Pic}(A)$ has a structure of separated group scheme over $k$. The map $\Lambda(\mathscr{L}) : A \to \operatorname{Pic}(A)$ is a map of schemes over $k$, preserving the neutral sections. So $\Lambda(\mathscr{L})$ is automatically a group homomorphism by rigidity. □

## 4. Dual abelian scheme

Let $A$ be an abelian scheme over $S$. We define the relative Picard functor:

$$\mathrm{Pic}_{A/S} : (\mathrm{Sch}/_S) \to \mathbf{Ab},$$

$$T \mapsto \{(\mathscr{L}, \rho) \mid \mathscr{L} \text{ is a line bundle on } A_T := A \times_S T,$$

$$\rho : e_T^* \mathscr{L} \xrightarrow{\sim} \mathscr{O}_T \text{ a“rigidification”}\}/\cong$$

where $e_T : T \to A_T$ is the neutral section.

***Theorem 4.1***. — *The functor* $\mathrm{Pic}_{A/S}$ *is representable by a commutative group scheme over* $S$*. Moreover, there exists a maximal open closed subgroup scheme*

$$\mathrm{Pic}^0_{A/S} =: A^\vee,$$

*such that each geometric fiber of* $A^\vee$ *is connected. Moreover,* $A^\vee$ *is an abelian scheme over* $S$*. We call* $A^\vee$ *the* dual abelian scheme *of* $A$.

***Remark 4.2***. — 1. In Mumford's *Abelian Varieties*, $A^\vee$ is constructed by dividing $A$ by a subgroup (only for $S = \operatorname{Spec} k$).

2. In FGA, Grothendieck uses projective methods to prove the representibility of the relative Picard functor $\mathrm{Pic}_{X/S}$ for arbitrary $X/S$ flat, projective, geometrically connected. (This works only for projective or at least locally projective abelian schemes. For such an $A$, the dual $A^\vee$ is also projective).

3. Artin's theory of representability by algebraic spaces/stacks. For any abelian algebraic space $A/S$, $\mathrm{Pic}_{A/S}$ is representable by an algebraic space, and $A^\vee = \mathrm{Pic}^0_{A/S}$ is also representable by an abelian algebraic space. This combined wit the following result of Raynaud gives the representability.

***Theorem 4.3* (Raynaud)**. — *Any abelian algebraic space A over S is automatically an abelian scheme. Moreover,* $\mathrm{Pic}_{A/S}$ *is a scheme, and* $A^\vee$ *is an abelian scheme.*

See [Faltings-Chai].