1. Dual abelian scheme

Let A/S be an abelian scheme. The relative Picard functor is defined as

$$\mathrm{Pic}_{A/S} : (\mathrm{Sch}/_S) \to \mathbf{Ab}, \quad T \mapsto \{(\mathscr{L}, \rho) \mid \mathscr{L} \text{ is a line bundle on } A_T,\ \rho : e_T^* \mathscr{L} \xrightarrow{\sim} \mathscr{O}_T\}/\cong,$$

such a pair $(\mathscr{L}, \rho)$ is called a rigidified line bundle.

Remark 1.1. — $\mathrm{Pic}_{A/S}(T) \xrightarrow{\sim} \mathrm{Pic}(A_T)/\pi_T^* \mathrm{Pic}(T)$, where $\pi_T : A_T \to T$ is the structure map. The isomorphism is given by

$$(\mathscr{L}, \rho) \mapsto \mathscr{L},$$

and the inverse is

$$\mathscr{L} \mapsto (\mathscr{L}^{\mathrm{rig}}, \rho),$$

where $\mathscr{L}^{\mathrm{rig}} := \mathscr{L} \otimes \pi_T^* e_T^* \mathscr{L}^{-1}$, so we have $\rho : e_T^* \mathscr{L}^{\mathrm{rig}} = e_T^* \mathscr{L} \otimes e_T^* \mathscr{L}^{-1} \simeq \mathscr{O}_T$.

Theorem 1.2. — *The functor* $\mathrm{Pic}_{A/S}$ *is representable by a commutative group scheme over S. There is a maximal open closed subgroup scheme* $\mathrm{Pic}^0_{A/S} = A^\vee$, *called the* dual *of A, such that each geometric fiber of $A^\vee$ is connected. Moreover, $A^\vee$ is an abelian scheme over S.*

On $A \times_S \mathrm{Pic}_{A/S}$, there is a universal rigidified line bundle $\mathscr{P}_A$, called the *Poincaré line bundle*, with a rigidification $(e \times \mathrm{id})^* \mathscr{P}_A \xrightarrow{\sim} \mathscr{O}_{\mathrm{Pic}_{A/S}}$, such that for each $T \to S$, and each rigidified line bundle $(\mathscr{L}, \rho)$ on A_T, there is a unique S-map $f : T \to \mathrm{Pic}_{A/S}$ such that

$$(\mathscr{L}, \rho) \text{ is the pullback of } \mathscr{P}_A \text{ along } A \times_S T \xrightarrow{(\mathrm{id}, f)} A \times_S \mathrm{Pic}_{A/S}.$$

Note: Let e be the neutral section for the group scheme $\mathrm{Pic}_{A/S}$. Then we have a canonical trivialization $(\mathrm{id}, e)^* \mathscr{P}_A \xrightarrow{\sim} \mathscr{O}_A$ by the moduli meaning of e, where $(\mathrm{id}, e) : A \to A \times_S \mathrm{Pic}_{A/S}$.

2. Homomorphism attached to a line bundle

Let A/S be an abelian scheme, and let $\mathscr{L}$ be a line bundle on A. Let $f_1, f_2 : T \to A$. Generally, $(f_1 + f_2)^*(\mathscr{L}) \not\simeq (f_1^* \mathscr{L}) \otimes (f_2^* \mathscr{L})$. We want to find the difference.

Universal case: $T = A \times_S A$, and f_1, f_2 are the projections. Then $f_1 + f_2 = m : A \times_S A \to A$ is the group law on A. Consider

$$\mathscr{M}(\mathscr{L}) = m^* \mathscr{L} \otimes f_1^* \mathscr{L}^{-1} \otimes f_2^* \mathscr{L}^{-1},$$

which is a line bundle on $A \times_S A$. Consider $A \times_S A$ as an abelian scheme over A by the second projection f_2, we have $\mathscr{M}(\mathscr{L})^{\mathrm{rig}} \in \mathrm{Pic}_{A/S}(A)$, i.e. we get a map $\Lambda(\mathscr{L}) : A \to \mathrm{Pic}_{A/S}$ such that $A \times_S A \xrightarrow{(\mathrm{id}, \Lambda(\mathscr{L}))} A \times_S \mathrm{Pic}_{A/S}$ pulls $\mathscr{P}_A$ back to $\mathscr{M}(\mathscr{L})^{\mathrm{rig}}$.

One can check that $\Lambda(\mathscr{L})$ preserves neutral sections. By rigidity, $\Lambda(\mathscr{L})$ is then a group homomorphism $A \to A^\vee \subset \mathrm{Pic}_{A/S}$. (Upshot: Start on a line bundle $\mathscr{L}$ on A, we get a group homomorphism $\Lambda(\mathscr{L}) : A \to A^\vee$).

Exercise. — $\Lambda(\mathscr{L})$ is additive in $\mathscr{L}$, i.e. $\Lambda(\mathscr{L}_1 \otimes \mathscr{L}_2^{\pm 1}) = \Lambda(\mathscr{L}_1) \pm \Lambda(\mathscr{L}_2)$.

Now we study $\Lambda(\mathscr{L})$ when $S = \operatorname{Spec} k$, k is an algebraically closed field. The map $\Lambda(\mathscr{L}) : A \to A^\vee$ is the same as last lecture, i.e.

$$A(k) \ni x \mapsto t_x^* \mathscr{L} \otimes \mathscr{L}^{-1} \in \operatorname{Pic}(A) = \operatorname{Pic}_{A/k}(k).$$

Proposition 2.1. — *If $\mathscr{L}$ is ample on A (such an $\mathscr{L}$ exists as A is projective), then $\Lambda(\mathscr{L})$ is an isogeny of degree $h^0(\mathscr{L})^2 \neq 0$. (For such $\mathscr{L}$, we know that $h^i(\mathscr{L}) = 0, \forall i > 0$) Such a $\Lambda(\mathscr{L})$ is called a* polarization.

In the above case, we think that $\Lambda(\mathscr{L})$ is the "most non-zero".

Corollary 2.2. — $\dim A = \dim A^\vee$.

Proposition 2.3. — *The following is equivalent.*

1) $\Lambda(\mathscr{L}) = 0$.
2) $\mathscr{L} \in \operatorname{Pic}_{A/k}(k)$ *lies in* $A^\vee$.
3) $\mathscr{L}$ *is algebraically equivalent to* 0*, which means that there exists a connected k-scheme U, two k-points u, v of U, and a line bundle $\widetilde{\mathscr{L}}$ on $A \times_k U$ such that the restrictions of $\widetilde{\mathscr{L}}$ on $A \times \{u\} \cong A$ and on $A \times \{v\} \cong A$ are $\mathscr{L}$ and $\mathscr{O}_A$ respectively.*

The equivalence of (2) and (3) is almost tautological, but the equivalence with (1) is quite non-trivial.

We then have an exact sequence (over $k = \bar{k}$)

$$0 \longrightarrow A^\vee \longrightarrow \operatorname{Pic}_{A/k} \xrightarrow{\Lambda(\cdot)} \operatorname{Hom}(A, A^\vee)$$

Question. — What's the image of $\Lambda(\cdot)$?

Answer: It is the group of symmetric homomorphisms $\operatorname{Hom}(A, A^\vee)^{\mathrm{Sym}}$ (defined later).

Example 2.4. — Let $A = E$ be an elliptic curve over k, k algebraically closed. Let $\mathscr{L}_1 = \mathscr{O}(e)$, e the neutral element of E. By Riemann-Roch, we have

$$h^0(\mathscr{L}_1) = 1 - g + \deg(\mathscr{L}_1) = 1.$$

So $\Lambda(\mathscr{L}_1)$ has degree 1, thus it is an isomorphism, called "*the canonical isomorphism*".

We now compute $\Lambda(\mathscr{L})$ for general $\mathscr{L}$. By the additivity of Λ, we only need to consider $\mathscr{L} = \mathscr{O}([P])$, $P \in E$. Then we have

$$\begin{aligned} \Lambda(\mathscr{L}) : E \to E^\vee, \quad x \mapsto & t_x^* \mathscr{L} \otimes \mathscr{L}^{-1} = \mathscr{O}([P-x]) \otimes \mathscr{O}(-[P]) \\ =& \mathscr{O}([P-x] - [P]) = \mathscr{O}([e] - [x]) = \Lambda(\mathscr{L}_1)(x). \end{aligned}$$

For the second last equality, we used the fact that $(P - x) + x = P$ if and only if $[P-x] + [x] \sim [P] + [e]$. So for general $\mathscr{L}$, $\Lambda(\mathscr{L}) = \deg(\mathscr{L})\Lambda(\mathscr{L}_1)$. We then have

$$\begin{array}{ccccccc} 0 \longrightarrow & E^\vee & \longrightarrow & \operatorname{Pic}(E) & \xrightarrow{\deg} & \mathbb{Z} & \longrightarrow 0 \\ & & & & \searrow^{\Lambda(\cdot)} & \big\downarrow {\scriptstyle n \mapsto n\Lambda(\mathscr{L}_1)} & \\ & & & & & \operatorname{Hom}(E, E^\vee) & \end{array}$$

3. More on duality

Recall that $\mathscr{P}_A$ is on $A \times_S A^\vee$, rigidified along both (e_A, id) and $(\mathrm{id}, e_{A^\vee})$. Viewing $\mathscr{P}_A$ as a family of line bundles on $A^\vee$ parameterized by A, we get $f : A \to A^{\vee\vee} \subset \mathrm{Pic}_{A^\vee/S}$.

Theorem 3.1. — *f is an isomorphism.*

Sketch of proof. — We may assume $S = \operatorname{Spec} k$, k algebraically closed. We have $\dim A = \dim A^\vee = \dim A^{\vee\vee} = g$. Let $F : A \times A^\vee \xrightarrow{(f,\mathrm{id})} A^{\vee\vee} \times A^\vee \xrightarrow{\sim} A^\vee \times A^{\vee\vee}$, where the last isomorphism is the swapping of the two factors. Note that $F^*(\mathscr{P}_{A^\vee}) = \mathscr{P}_A$.

Fact. — We have $\chi(\mathscr{P}_A) = (-1)^g = \chi(\mathscr{P}_{A^\vee})$ (difficult). Here $\chi(\cdot)$ denotes the Euler characteristic of a line bundle. In fact, $h^i(\mathscr{P}_A) = 0$ if $i \neq g$ and $h^g(\mathscr{L}) = 1$.

Fact. — For any isogeny $\lambda : B \to C$ between abelian varieties and any line bundle $\mathscr{L}$ on C, we have $\chi(\lambda^*\mathscr{L}) = \deg(\lambda)\chi(\mathscr{L})$.

If we knew f is an isogeny, then F is an isogeny with the same degree. By the facts above, $\deg F = \deg f = 1$, thus f is an isomorphism.

Thus it suffices to prove that f is an isogeny. Supposing not, then there is a finite subgroup $Z \subset \operatorname{Ker} f$, of order $d \geq 2$ (d coprime to $\operatorname{char} k$), and f factors through $A \to A/Z$, which is an isogeny of degree d. Then F also factors through an isogeny of degree d, and by $\mathscr{P}_A = F^*(\cdots)$ we have $d \mid \chi(\mathscr{P}_A)$, a contradiction. □

Construction. — Let $\varphi : A \to B$ be any homomorphism. We have $\varphi^\vee : B^\vee \to A^\vee$ by pullback of line bundles.

Lemma 3.2. — $(\varphi_1 \pm \varphi_2)^\vee = \varphi_1^\vee \pm \varphi_2^\vee$.

Proof. — We may assume $S = \operatorname{Spec} k$, k algebraically closed, and we only need to check the equality on k-points. Use the fact that a point $x \in B^\vee(k)$ corresponds to a line bundle $\mathscr{L}$ on B such that $\Lambda(\mathscr{L}) = 0$. The last condition implies that for any $\varphi_1, \varphi_2 : A \to B$, we have $\varphi_1^*\mathscr{L} \otimes \varphi_2^*\mathscr{L} \simeq (\varphi_1 + \varphi_2)^*\mathscr{L}$, i.e. $(\varphi_1 + \varphi_2)^\vee(x) = \varphi_1^\vee(x) + \varphi_2^\vee(x)$. Also it is easy to see that $(0 : A \to B)^\vee = (0 : B^\vee \to A^\vee)$, so we have the equality for the minus sign as well. □

Lemma 3.3. — *For any homomorphism of abelian schemes $\varphi : A \to B$, the following statements hold.*

1) Under the canonical isomorphism $A^{\vee\vee} \cong A$, $B^{\vee\vee} \cong B$, $\varphi^{\vee\vee}$ is the same as φ.
2) φ is an isogeny if and only if $\varphi^\vee$ is an isogeny. In this case, $\deg \varphi = \deg \varphi^\vee$.

Proof. — We may assume $S = \operatorname{Spec} k$, k algebraically closed. Let $F_1 = (\mathrm{id}, \varphi^\vee) : A \times B^\vee \to A \times A^\vee$, and $F_2 = (\varphi, \mathrm{id}) : A \times B^\vee \to B \times B^\vee$. Then $F_1^*\mathscr{P}_A = F_2^*\mathscr{P}_B$. This equality can be seen as a characterization of φ in terms of $\varphi^\vee$, and vice versa. It is then easy to see (1). To see (2), suppose $\varphi^\vee$ is not an isogeny. Then an argument similar to the proof of Theorem 3.1 shows that $\chi(F_1^*\mathscr{P}_A)$ is divisible by arbitrarily large integers, and hence equals 0. If φ is an isogeny, then $\chi(F_2^*\mathscr{P}_B) = \deg(\varphi)\chi(\mathscr{P}_B) \neq 0$, a contradiction. Similarly, if φ is not an isogeny, then $\varphi^\vee$ is not an isogeny. □

Lemma 3.4. — *Let $\varphi : A \to A^\vee$ be a homomorphism. The following are equivalent.*

1) $\varphi^\vee : A \cong A^{\vee\vee} \to A^\vee$ *is equal to* φ*, i.e.* φ *is* symmetric*.*
2) After an f.p.p.f. localization on S*,* $\varphi = \Lambda(\mathscr{L})$ *for some line bundle* $\mathscr{L}$ *on* A*.*
3) After an étale localization on S*,* $\varphi = \Lambda(\mathscr{L})$ *for some line bundle* $\mathscr{L}$ *on* A*.*

Proof. — (1) implies (2): Let $\mathscr{L}'$ be the pullback of $\mathscr{P}_A$ along $A \xrightarrow{\delta} A \times_S A \xrightarrow{(\mathrm{id},\varphi)} A \times_S A^\vee$. One checks that $\Lambda(\mathscr{L}') = \varphi + \varphi^\vee$. If (1) holds, then $\Lambda(\mathscr{L}') = 2\varphi$. As a general fact, for any line bundle $\mathscr{M}$ on A and any positive integer n, if $\Lambda(\mathscr{M}) \in n \cdot \mathrm{Hom}(A, A^\vee)$, then after fppf localization on S there exists an n-th root of $\mathscr{M}$. Applying this fact to our situation, we see that after fppf localization on S we may assume that $\mathscr{L}' = \mathscr{L}^{\otimes 2}$ for some $\mathscr{L}$. Then $2\Lambda(\mathscr{L}) = 2\varphi$, from which $\Lambda(\mathscr{L}) = \varphi$.

(2) implies (3). By (2), the choices of $\mathscr{L}$ such that $\varphi = \Lambda(\mathscr{L})$ is an $A^\vee$-torsor on S, for the fppf topology. But $A^\vee$ is smooth, so this $A^\vee$-torsor is étale locally trivial.

(3) implies (1). Whether φ is symmetric can be checked after étale localization on the base, so we may assume that $\varphi = \Lambda(\mathscr{L})$ for some line bundle $\mathscr{L}$ on A. We then leave it as an exercise for the reader to check that $\varphi = \varphi^\vee$. □

In the special case $S = \mathrm{Spec}\, k$, k algebraically closed, we have $\varphi : A \to A^\vee$ is symmetric if and only if $\varphi = \Lambda(\mathscr{L})$ for some line bundle $\mathscr{L}$ on A. In this case, $\mathscr{L} = \mathscr{L}_1 \otimes \mathscr{L}_2^{-1}$, with $\mathscr{L}_i$ ample, so $\varphi = \Lambda(\mathscr{L}_1) - \Lambda(\mathscr{L}_2)$, where $\Lambda(\mathscr{L}_i)$ are polarizations. Thus symmetric homomorphisms $A \to A^\vee$ are precisely those homomorphisms that can be written as the difference between two polarizations.

In this case, we have a short exact sequence

$$0 \longrightarrow A^\vee \longrightarrow \mathrm{Pic}_{A/k} \longrightarrow \mathrm{Hom}(A, A^\vee)^{\mathrm{Sym}} \longrightarrow 0$$

Remark 3.5. — For any base scheme S, we have a short exact sequence of abelian f.p.p.f. sheaves on $(\mathrm{Sch}/_S)$

$$0 \longrightarrow A^\vee \longrightarrow \mathrm{Pic}_{A/k} \longrightarrow \underline{\mathrm{Hom}}(A, A^\vee)^{\mathrm{Sym}} \longrightarrow 0$$

where $\underline{\mathrm{Hom}}(A, A^\vee)^{\mathrm{Sym}}$ is the sheaf defined by $T \mapsto \mathrm{Hom}(A_T, A_T^\vee)^{\mathrm{Sym}}$, and it is actually representable.

4. The isogeny category

Fix a field k (not necessarily algebraically closed). In the rest of this lecture, everything is over k.

Denote by $\mathcal{A}$ the category of abelian varieties over k. It is an additive category, $\mathrm{Hom}(A, B)$ is an abelian group for any $A, B \in \mathrm{Ob}\, \mathcal{A}$.

Let $\mathcal{A}^0$ be the isogeny category of $\mathcal{A}$. This means that $\mathcal{A}^0$ has the same objects of $\mathcal{A}$, and the morphisms between A, B in $\mathcal{A}^0$ are given by

$$\mathrm{Hom}(A, B)^0 = \mathrm{Hom}_{\mathcal{A}}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Observation. — The map $\mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}(A, B)^0$ is an injection, i.e. $\mathrm{Hom}_{\mathcal{A}}(A, B)$ is torsion-free: if the composition

$$A \xrightarrow{[n]} A \xrightarrow{\varphi} B$$

is zero, then $\varphi = 0$ as $[n]$ is an isogeny.

Observation. — For $\varphi \in \mathrm{Hom}(A, B)$, φ is an isogeny if and only if φ becomes an isomorphism in $\mathcal{A}^0$.

***Fact* (Poincaré Complete Reducibility)**. — $\mathcal{A}^0$ is semi-simple. In particular, for any $A \in \mathrm{Ob}\,\mathcal{A}$, $\mathrm{End}(A)^0 = \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a semi-simple $\mathbb{Q}$-algebra.

Concretely, for any abelian variety A over k, there are simple abelian varieties $A_1, \cdots, A_k$, mutually non-isogenous, and integers $n_1, \cdots, n_k \in \mathbb{Z}_{>0}$ such that A is isogenous to $A_1^{\times n_1} \times \cdots \times A_k^{\times n_k}$. So

$$\begin{aligned}\mathrm{End}(A)^0 &\simeq \mathrm{End}(A_1^{\times n_1} \times \cdots \times A_k^{\times n_k})^0 \\ &\simeq M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k),\end{aligned}$$

where $D_i = \mathrm{End}(A_i)^0$ are division algebras. We will show that D_i is finite dimensional over $\mathbb{Q}$.

Sketch of proof. — Suppose we have $i : B \hookrightarrow A$ an abelian subvariety. We want to find an abelian subvariety $C \subset A$ such that $B \cap C$ is finite, and $B \times C \to A$, $(b, c) \mapsto b + c$ is an isogeny. Take an ample line bundle $\mathscr{L}$ on A, with associated polarization $\Lambda(\mathscr{L}) : A \to A^\vee$. Let C' be the kernel of $A \xrightarrow{\Lambda(\mathscr{L})} A^\vee \xrightarrow{i^\vee} B^\vee$, and let $C = (C')^0_{\mathrm{red}}$.

The main point: $C \cap B \subset \mathrm{Ker}(B \xrightarrow{i} A \xrightarrow{\Lambda(\mathscr{L})} A^\vee \xrightarrow{i^\vee} B^\vee) = \mathrm{Ker}(\Lambda(i^*\mathscr{L}))$, thus it is finite since $i^*\mathscr{L}$ is an ample line bundle on B. □

5. Tate module

Fix a prime ℓ coprime to $\mathrm{char}\, k$. Let A be an abelian variety over k of dimension g. For any $n \geq 1$, denote by $A[\ell^n] = \mathrm{Ker}[\ell^n] : A \to A$. It is a finite étale group scheme of order ℓ^{2gn}. So $A[\ell^n](k^{\mathrm{sep}})$ is an abelian group of order ℓ^{2gn}.

Exercise. — $A[\ell^n](k^{\mathrm{sep}})$ is a free $\mathbb{Z}/\ell^n\mathbb{Z}$-module of rank $2g$. (Hint: we know the order of ℓ^m-torsion in this group, for all $m \leq n$.)

Definition 5.1. — The *Tate module* of A is

$$T_\ell(A) := \varprojlim_n A[\ell^n](k^{\mathrm{sep}}),$$

with the transition map $A[\ell^{n+1}](k^{\mathrm{sep}}) \to A[\ell^n](k^{\mathrm{sep}})$ given by $x \mapsto [\ell]x$. Then $T_\ell(A)$ is a finite free $\mathbb{Z}_\ell$-module of rank $2g$.

The Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k) = G_k$ acts continuously on $T_\ell(A)$. There is a natrual map

$$\Phi : \mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to \mathrm{Hom}_{\mathbb{Z}_\ell[G_k]}(T_\ell(A), T_\ell(B)).$$

***Conjecture 5.2* (Tate)**. — *If k is a finitely generated field, Φ is an isomorphism.*

Theorem 5.3. — *Let k be an arbitrary field, and let ℓ be a prime coprime to* $\mathrm{char}\, k$. *Let A, B be abelian varieties over k. Then*

$$\Phi : \mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to \mathrm{Hom}_{\mathbb{Z}_l}(T_\ell(A), T_\ell(B))$$

is injective. In particular, $\mathrm{Hom}(A, B)$ *is a finite free $\mathbb{Z}$-module, of rank* $\leq 4 \dim A \dim B$.

Sketch of proof. — *Step 1.* Suppose $M \subset \mathrm{Hom}(A,B)$ is a finitely generated $\mathbb{Z}$-submodule (so M is free of finite rank). Suppose M is saturated, i.e. $\mathbb{Q}M \cap \mathrm{Hom}(A,B) = M$. We will know $\Phi|_{M\otimes_{\mathbb{Z}}\mathbb{Z}_\ell}$ is injective.

Proof. — Take a $\mathbb{Z}$-basis $e_1, \cdots, e_r$ of M. Suppose $\sum_i e_i \otimes a_i \in \mathrm{Ker}(\Phi|_{M\otimes_{\mathbb{Z}}\mathbb{Z}_\ell})$, $a_i \in \mathbb{Z}_\ell$. For each i, let $a_i^{(k)} \in \mathbb{Z}$ be a sequence converging to a_i in the ℓ-adic topology, as $k \to \infty$. Then $\sum_i a_i^{(k)}\Phi(e_i) \xrightarrow{\ell\text{-adic}} \Phi(\sum_i e_i \otimes a_i) = 0$. For k large enough, LHS $\in \ell^{\max v_\ell(a_i)+1} \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell A, T_\ell B)$.

Observation. — If $f \in \mathrm{Hom}(A,B)$ is such that $\Phi(f) : T_\ell A \to T_\ell B$ lies in $\ell^N \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell A, T_\ell B)$, then $f \in \ell^N \mathrm{Hom}(A,B)$. (Proof: f kills $A[\ell^N]$.)

For $k >> 0$, we have $\sum_i a_i^{(k)} e_i \in \ell^{\max_i(v_\ell(a_i)+1)} \mathrm{Hom}(A,B)$, or equivalently $\sum_i a_i^{(k)} e_i \in \ell^{\max_i(v_\ell(a_i)+1)} M$ since M is saturated. But $v_\ell(a_i^{(k)}) = v_\ell(a_i)$ for $k >> 0$, a contradiction. ☐

Step 2. We show that $\mathrm{Hom}(A,B)$ is finitely generated. (Then we can take $M = \mathrm{Hom}(A,B)$ in Step 1 and finish the proof.) By Poincaré complete reducibility, we can reduce to the case where $A = B$, and A is simple. To prove $\mathrm{End}(A)$ is finitely generated, we only need to show that $\Phi : \mathrm{End}(A) \otimes \mathbb{Z}_\ell \to \mathrm{End}_{\mathbb{Z}_\ell}(T_\ell(A))$ is injective. Suppose not,then there is a finitely generated submodule M of $\mathrm{End}(A)$ such that Φ is not injective on $M \otimes \mathbb{Z}_\ell$. We then take M' to be the saturation of M inside $\mathrm{End}(A)$, i.e. $M' = \mathbb{Q}M \cap \mathrm{End}(A)$.

Fact. — There is a unique function $\deg : \mathrm{End}(A)^0 = \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ such that on every finite dimensional $\mathbb{Q}$-subspace of $\mathrm{End}(A)^0$ (of course, after the proof, we will know that $\mathrm{End}(A)^0$ is finite dimensional), deg is a homogeneous polynomial of degree $2g$, and moreover if $\varphi \in \mathrm{End}(A)$ is an isogeny, then $\deg(\varphi)$ is as before. (The proof of this fact is similar to the proof of $\deg([n]) = n^{2g}$, using Theorem of Cube + Intersection theory or Hilbert Polynomial).

Now
$$M' \cap \{v \in \mathbb{Q}M \mid |\deg(v)| < 1\} = \{0\},$$
because any non-zero $\varphi \in \mathrm{End}(A)$ is an isogeny (A simple) and hence satisfies $\deg(\varphi) \geq 1$. So M' is a discrete subgroup of the Euclidean space $\mathbb{R}M$, and so M' is finitely generated.

Applying Step 1 to M', we have Φ is injective on $M' \otimes \mathbb{Z}_\ell$, a contradiction with the choice of M. ☐

Remark 5.4. — In the notation of the above proof, for any fixed $\varphi \in \mathrm{End}(A)$, the function $P_\varphi : \mathbb{Z} \to \mathbb{Z}, n \mapsto \deg([n] - \varphi)$ is a monic polynomial $P_\varphi(T) \in \mathbb{Z}[T]$ of degree $2g$, and we have the following "independence of ℓ" result: For every prime ℓ coprime to $\mathrm{char}\, k$, $P_\varphi(T)$ is equal to the characteristic polynomial of φ acting on $T_\ell(A)$ (which is a priori a polynomial in $\mathbb{Z}_\ell[T]$). In particular, the determinant of φ acting on $T_\ell(A)$ is equal to $\deg(\varphi)$. Note that the determinant function on the $4g^2$-dimensional $\mathbb{Q}_\ell$-vector space $\mathrm{End}_{\mathbb{Z}_\ell}(T_\ell(A)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong M_{2g}(\mathbb{Q}_\ell)$ is indeed a polynomial of degree $2g$.