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Siegel moduli schemes and their compactifications (C.

$g \geq 1 \rightsquigarrow Sp_{2g} \subset GSp_{2g} / \mathbb{Z}$

$\forall$ ring R, $Sp_{2g}(R) = \left\{ \gamma \in M_{2g}(R) \,\middle|\, {}^t\gamma \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \gamma = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \right\}$

$\subset GSp_{2g}(R) = \left\{ \gamma \,\middle|\, \cdots = c(\gamma) \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}, \; c(\gamma) \in R^\times \right\}$

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(R) \iff \begin{aligned} & A\,{}^tB = B\,{}^tA \\ & C\,{}^tD = D\,{}^tC. \\ & A\,{}^tD - B\,{}^tC = I_g. \end{aligned}$$

$L = \mathbb{Z}^{2g}$, $\langle \cdot , \cdot \rangle : L \times L \to \mathbb{Z}$

$$(x, y) \mapsto {}^tx \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} y$$

$$Sp_{2g} = Sp(L, \langle , \rangle)$$

$$= \{ \gamma \in GL(L) \mid \langle \gamma x, \gamma y \rangle = \langle x, y \rangle \}.$$

$\rightsquigarrow$ exact sequence $1 \to Sp_{2g} \to GSp_{2g} \xrightarrow{c} \mathbb{G}_m \to 1$

Siegel half space

$$\mathbb{H}_g = \{ \Omega \in M_g(\mathbb{C}) \mid \Omega = {}^t\Omega,\ \mathrm{Im}\,\Omega > 0 \}$$

(positive definite)

$\circlearrowleft$

$$Sp_{2g}(\mathbb{R}) \ni \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\gamma\Omega = (A\Omega + B)(C\Omega + D)^{-1}.$$

$$Sp_{2g}(\mathbb{R})/K \xrightarrow{\sim} \mathbb{H}_g.$$

K max cpt of $Sp_{2g}(\mathbb{R})$.

$$\cong$$

$$GSp_{2g}(\mathbb{R})^+ / K\mathbb{R}_{>0}.$$

subgp of $GSp_{2g}(\mathbb{R})$ s.t. $c(\gamma) > 0$

$(GSp_{2g}\mathbb{Q},\ \pm\mathbb{H}_g)$ Siegel Shim. datum.

Schemes = Cat. of loc. noeth. Sch.

$\in$

S $\quad X/S$ abelian scheme

$\rightsquigarrow X^\vee/S = \mathrm{Pic}^0(X/S)$ dual ab sch.

Recall. a polarization of X/S is an S-morphism

$$\lambda : X \to X^\vee .$$

s.t. $\forall$ geom. pt $s \to S$,

$\lambda_s : X_s \to X_s^\vee$ is a polarization.

i.e. of the form $\varphi_{\mathcal{L}_s}$.

for some ample line bdl $\mathcal{L}_s / X_s$.

$\lambda_* \mathcal{O}_X$ loc. free $\mathcal{O}_{X^\vee}$ mod, constant rank over each conn. comp. of S.

$\rightsquigarrow$ deg of $\lambda = d^2$, $d \geq 1$

Principal polarization if $\deg \lambda = 1$ i.e. λ is isom.

Def. $\mathcal{A}_g$: Schs $\to$ Sets contravariant functor.

$$S \longmapsto \left\{ (X, \lambda)/S \text{ principally pol. ab. sch. } /S \text{ of rel dim } g. \text{ projective } /S \right\} /\simeq$$

More generally. $\forall$ integers $d \geq 1$, $n \geq 1$.

$\mathcal{A}_{g,d,n}$: Schs $\to$ Sets

; dim $X/S = g$

$$S \longmapsto \left\{ (X, \lambda, \eta) \;\middle|\; \begin{array}{l} \dim X_S = g \\ \deg \lambda = d^2. \\ \eta : X[n] \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^{2g} \\ \eta \text{ preserves pairings up to } (\mathbb{Z}/n\mathbb{Z})^\times. \end{array} \right\} \Big/ \cong$$

$\mathcal{A}_{g,1,1} = \mathcal{A}_g$.

$\mathbb{A}$: Schemes $\to$ Sets. Contravariant functor

a coarse moduli scheme of $\mathbb{A}$ is a scheme A & a morphism $F : \mathbb{A} \to h_A = \mathrm{Hom}(\cdot, A)$.

s.t.
a) $\forall$ morphism $G : \mathbb{A} \to h_X$ for some scheme X.
G factors thru F via a unique $A \to X$.

$$\begin{tikzcd} \mathbb{A} \arrow[r, "G"] \arrow[dr, bend right, "F"'] & h_X \\ & h_A \arrow[u] \end{tikzcd}$$

b) $\forall$ alg closed field k.

$F(\mathrm{Spec}\, k) : \mathbb{A}(k) \xrightarrow{\sim} A(k)$ bijection.

Thm (Mumford) $\forall g, d, n$, the coarse moduli sch $\mathcal{A}_{g,d,n}$ exists, faithfully flat $/ \mathbb{Z}[\frac{1}{n}]$.

quasi-proj $/ \mathbb{Z}[\frac{1}{n}]$

$\ldots, \oplus \mathcal{O}_{np})$ for any prime $p \nmid n$

via fact quasi-proj / $\mathbb{Z}[\frac{1}{n}]$.

Moreover, if $n \geq 3$, then $\mathcal{A}_{g,d,n}$ representable.

sm / $\mathbb{Z}[\frac{1}{nd}]$.

Pf $\begin{cases} \text{GIT.} \\ \text{Artin method (alg stacks)} \end{cases}$

cf. Faltings–Chai Chap 1 §4.

In the following, mainly discuss $\mathcal{A}_g$. $\mathcal{A}_{g,1,n}$ / $\mathbb{C}$.

Complex uniformization.

$$\left\{ (X, \lambda, (\alpha_i)_{1\leq i\leq g}) \;\middle|\; \begin{array}{l} X/\mathbb{C} \text{ ab var dim } g \\ \deg \lambda = 1 \\ \left.\begin{array}{l} \alpha_i \in H_1(X(\mathbb{C}), \mathbb{Z}) \\ \text{Symp basis} \end{array}\right\} \\ \langle \alpha_j, \alpha_k \rangle = \langle \alpha_{g+j}, \alpha_{g+k} \rangle \\ \qquad\qquad\qquad = 0 \\ \langle \alpha_j, \alpha_{g+k} \rangle = -\langle \alpha_{g+j}, \alpha_k \rangle \\ \qquad = \delta_{j,k}. \quad 1 \leq j,k \leq g. \end{array} \right.$$

$(X, \lambda, (\alpha_i)_{1\leq i\leq g})$ Pick Lie $X \xrightarrow{\sim} \mathbb{C}^g$ $\mathbb{C}$-lin isom

$$\rightsquigarrow \quad X(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}^g / \langle \alpha_1, \dots, \alpha_g \rangle, \quad \alpha_i \in \mathbb{C}^g.$$

$\Omega = (\alpha_1, \alpha_2, \dots, \alpha_g)$ period matrix

$= (\Omega_1, \Omega_2)$

Riemann rel.: $\Omega_2 {}^t\Omega_1 - \Omega_1 {}^t\Omega_2 = 0$

$$2i\left(\Omega_2 \overline{{}^t\Omega_1} - \Omega_1 \overline{{}^t\Omega_2}\right) > 0.$$

$$\{(X, \lambda, (\alpha_i))\} \xrightarrow{\sim} \mathbb{H}_g$$

$$(X, \lambda, (\alpha_i)) \mapsto \Omega = \Omega_2^{-1}\Omega_1.$$

Consider $\mathbb{H}_g \times \mathbb{C}^g \circlearrowleft \mathbb{Z}^{2g}$.

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} : (\Omega, z) \mapsto (\Omega, z + \Omega n_1 + n_2)$$

$$\rightsquigarrow \quad \mathcal{X}_g := \mathbb{Z}^{2g} \backslash \mathbb{H}_g \times \mathbb{C}^g.$$

$$\begin{tikzcd} \mathcal{X}_g \arrow[d] \\ \mathbb{H}_g \end{tikzcd}$$

holo. family of princ. pol. a.v. w/ symp basis of H_1.

$\mathbb{H}_g \times \mathbb{C}^g \curvearrowleft Sp_{2g}(\mathbb{Z}) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$

$$(\Omega, z) \longmapsto \left((A\Omega+B)(C\Omega+D)^{-1}, \; {}^t(C\Omega+D)^{-1} z \right),$$

$\rightsquigarrow Sp_{2g}(\mathbb{Z}) \ltimes \mathbb{Z}^{2g} \curvearrowright \mathbb{H}_g \times \mathbb{C}^g .$

$$\begin{tikzcd} \rightsquigarrow & Sp_{2g}(\mathbb{Z}) \backslash \mathfrak{X}_g \arrow[d] \\ & Sp_{2g}(\mathbb{Z}) \backslash \mathbb{H}_g . \end{tikzcd}$$

$\Rightarrow \; A_g(\mathbb{C}) \simeq Sp_{2g}(\mathbb{Z}) \backslash \mathbb{H}_g$ as complex anal. spaces.

For any $\Gamma \subset Sp_{2g}(\mathbb{Z})$ subgp of finite index get $\Gamma\backslash \mathbb{H}_g \to Sp_{2g}(\mathbb{Z})\backslash \mathbb{H}_g$.

$n \geq 3$.

$$\Gamma(n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z}) \;\middle|\; \begin{array}{l} A \equiv D \equiv I_g \mod n \\ B \equiv C \equiv 0 \mod n \end{array} \right\}$$

$$K(n) = \{ \gamma \in GSp_{2g}(\hat{\mathbb{Z}}) \mid \gamma \equiv I_{2g} \mod n \}$$

$$\Gamma(n) \cong K(n) \cap Sp_{2g}(\mathbb{Q})$$

(inside $GSp_{2g}(\mathbb{A}_f)$).

then

$$A_{g,1,n}(\mathbb{C}) \cong Sh_{K(n)}(GSp_{2g}, \pm\mathbb{H}_g)(\mathbb{C})$$

$$\cong \coprod_{(\mathbb{Z}/n\mathbb{Z})^\times} \Gamma(n)\backslash\mathbb{H}_g .$$

Siegel modular forms

recall: $Sp_{2g}(\mathbb{R})/K \xrightarrow{\sim} \mathcal{H}_g$. $K = \mathrm{Stab}(i I_g)$

$$K = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in M_{2g}(\mathbb{R}) \,\middle|\, \begin{array}{l} A\,{}^tB = B\,{}^tA \\ A\,{}^tA + B\,{}^tB = 1_g \end{array} \right\}$$

$$\xrightarrow{\sim} U_g(\mathbb{R})$$

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \longmapsto A + iB.$$

$$\Rightarrow K_{\mathbb{C}} \xrightarrow{\sim} GL_g(\mathbb{C}).$$

$$\rho : GL_g(\mathbb{C}) = K_{\mathbb{C}} \to GL(V_\rho)$$

fin. dim. rep.

$\Gamma \subset Sp_{2g}(\mathbb{Z})$ fin. index.

A Siegel <u>modular form</u> of weight ρ & level Γ is a hol. function. $f : \mathcal{H}_g \to V_\rho$ s.t.

1) $f(\gamma\Omega) = \rho(C\Omega + D)\cdot f(\Omega)$,

$\forall \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$

$\Omega \in \mathbb{H}_g$.

2) f is hol at all cusps if $g = 1$.

If $\rho = \det^k$. for some $k \in \mathbb{N}$,

$R_k(\Gamma) = \{$ Sieg. mod form of wt k, level $\Gamma\}$.

(Γ congr. subgp.)

<u>Thus</u>. 1) The graded $\mathbb{C}$-alg

$R(\Gamma) = \bigoplus_{k \in \mathbb{N}} R_k(\Gamma)$ is fin. gen. $/\mathbb{C}$.

2) $\operatorname{trdeg}_{\mathbb{C}} R(\Gamma) = \frac{g(g+1)}{2} + 1$.

3) $\forall k$, $\dim_{\mathbb{C}} R_k(\Gamma) < \infty$

$= O(k^{\binom{g+1}{2}})$.

4) $R(\Gamma)$ embeds $\Gamma\backslash\mathbb{H}_g$ into

$\mathrm{proj}(R(\Gamma))(\mathbb{C})$ as an open

dense subvar in Zar. top.

$$\begin{tikzcd} X = \mathfrak{X}_{g,\Gamma}(\mathbb{C}) \arrow[d, "\pi"] \\ T = \Gamma\backslash\mathbb{H}_g \end{tikzcd}$$

$\mathcal{E} = \pi_*(\Omega^1_{X/T})$

loc. free rk g.

$\omega_{g,T} = \wedge^g \mathcal{E}$.

then $R_k(\Gamma) = H^0(\Gamma\backslash\mathbb{H}_g, \omega_{g,T}^{\otimes k})$.

(If $g=1$, need extra cuspidal condition)

The minimal compactification.

<u>Goal</u>. Describe $\mathrm{proj}(R(\Gamma))(\mathbb{C})$ more explicitly.

$\circlearrowright \mathbb{H}_g^* = \{ (\gamma, \Omega) \mid \gamma \in Sp_{2g}(\mathbb{Q}),\ \Omega \in \mathbb{H}_r$

$Sp_{2g}(\mathbb{Q})$ for some $0 \le r \le g \} / \sim$

$(\gamma_1, \Omega_1) \sim (\gamma_2, \Omega_2) \Leftrightarrow$

- $r_1 = r_2 = r$,
- $\gamma_2^{-1}\gamma_1 \in N_r(\mathbb{Q}) = \left\{ \begin{pmatrix} A_{11} & 0 & B_{11} & * \\ * & A_{22} & * & * \\ C_{11} & 0 & D_{11} & * \\ 0 & 0 & 0 & {}^tA_{22}^{-1} \end{pmatrix} \right\}$

$$\xrightarrow{P_{r,g}} Sp_{2r}(\mathbb{Q})$$

$$\begin{pmatrix} A_{11} & B_{11} \\ C_{11} & D_{11} \end{pmatrix}.$$

- $\Omega_2 = P_{r,g}(\gamma_2^{-1}\gamma_1)\Omega_1$.

$\forall\, 0 \le r \le g$. $\mathbb{H}_r \hookrightarrow \mathbb{H}_g^*$

$\Omega \mapsto (1, \Omega)$.

image $= \mathcal{F}_r$.

N_r = normalizer of F_r.

$$r \in Sp_{2g}(\mathbb{Q}) \rightsquigarrow \gamma \cdot F_r = F \subset \mathbb{H}_g^*$$

"ratl bdry comp".

We def top on $\mathbb{H}_g^*$ as follows.

$\Omega = X + iY \in \mathbb{H}_g$.

$Y = {}^tB D B$ uniquely "Jacobi decomp".

$$D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_g \end{pmatrix} \quad d_i > 0 \qquad B = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}.$$

$\forall u > 0$, def <u>Siegel set</u> $\mathfrak{F}_g(u) \subset \mathbb{H}_g$.

$$= \left\{ X + iY \in \mathbb{H}_g \;\middle|\; \begin{array}{ll} |x_{ij}| < u & \forall i, j. \\ |b_{ij}| < u & \forall 1 \le i < j \le g. \end{array} \right\}$$

$$1 < n d_i$$

$$d_i < n d_{i+1} \qquad \forall\, 1 \le i \le g-1$$

$$\Rightarrow \bigcup_{n>0} \mathfrak{F}_g(n) = \mathbb{H}_g .$$

. $Sp_{2g}(\mathbb{Z}) \cdot \mathfrak{F}_g(n) = \mathbb{H}_g$ for n large enough. (*).

$\forall\, n > 0$, $\#\{\gamma \in Sp_{2g}(\mathbb{Z}) \mid \gamma \mathfrak{F}_g(n) \cap \mathfrak{F}_g(n) \neq \emptyset\} < \infty$.

Fix a suff. large n_0 s.t. (*) holds.

Set $\mathfrak{F}_g^* := \coprod_{r=0}^{g} \overline{\mathfrak{F}_r(n_0)}$.

where $\overline{\mathfrak{F}_r(n_0)} = \begin{cases} \text{closure of } \mathfrak{F}_r(n_0) \text{ in } \mathbb{H}_r. & 1 \le r \le g. \\ \text{pt.} & r = 0. \end{cases}$

Def top on $\mathcal{F}_g^*$:

$\forall\, \Omega \in \overline{\mathcal{F}_r(U_0)} \subset \mathcal{F}_g^*$.

a nbhd basis of Ω is

$$\Big\{ \bigcup_{r \le s \le g} W_{r,s}(U, c) \Big\}_{\substack{U \ni \Omega,\ c \in \mathbb{R}_{>0} \\ \text{op nbhd} \\ \text{in } \overline{\mathcal{F}_r(U_0)}}} .$$

$W_{r,s}(U, c)$

$$:= \left\{ \Omega = \begin{array}{cc} \left(\begin{array}{cc} \Omega_1 & * \\ * & * \end{array} \right) & \begin{array}{l} \}r \\ \}s-r \end{array} \\ \begin{array}{cc} \underbrace{\quad}_{r} & \underbrace{\quad}_{s-r} \end{array} & \end{array} = X + iY \in \mathcal{F}_s(U_0) \ \middle| \right.$$

$$\Omega_1 \in U, \ d_{r+1} > C \text{ in Jacobi decomp}$$

$$Y = {}^tB\, D\, B, \ D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_s \end{pmatrix} \Big\}$$

)

$\rightsquigarrow$

Def Satake top on $\mathbb{H}_g^*$:

$\forall x \in \mathbb{H}_g^*$.

a basis of nbhds of x is given by

$U \subset \mathbb{H}_g^*$ s.t.

(i) $\forall \gamma \in Sp_{2g}(\mathbb{Q})$. $\gamma U \cap F_g^*$ is an open nbhd of γx in F_g^* whenever $\gamma x \in F_g^*$.

b) $\forall \gamma \in Sp_{2g}(\mathbb{Z})$ s.t. $\gamma x = x$, then $\gamma U = U$.

$\rightsquigarrow$ Indep. of choice of U_0. nice properties --

Thm 3 For any $\Gamma \subset Sp_{2g}(\mathbb{Z})$ fin. index

1) $\Gamma \backslash \mathbb{H}_g^*$ has str. of compact

normal analytic sp.

Has natural finite stratification w/ strata loc. closed analytic subspaces of form

$$\Gamma' \backslash \mathbb{H}_r . \quad 0 \le r \le g .$$

2) $\Gamma \backslash \mathbb{H}_g^* \xrightarrow{\sim} \mathrm{Proj}(R(\Gamma))(\mathbb{C})$ isom of anal. sp.

thus projective.

- $\Gamma_g = Sp_{2g}(\mathbb{Z})$, $\quad \Gamma_g \backslash \mathbb{H}_g^* = \coprod_{r=0}^{g} \Gamma_r \backslash \mathbb{H}_r$

$\Gamma_r = Sp_{2r}(\mathbb{Z})$.

- For some $k > 0$, $\omega_{g,\Gamma}^{\otimes k}$ extends to $\Gamma \backslash \mathbb{H}_g^*$.

Toroidal Compactification.

Γ fixed. $X = X_\Gamma \hookrightarrow X^{min} \,/\, \mathbb{C}$

quasi-proj. alg. var.

normal, projective,

[AMRT].

$$\begin{tikzcd}
 & X^{tor}_\Sigma \arrow[d] \\
X \arrow[ur, hook] \arrow[r, hook] & X^{min}
\end{tikzcd}$$

X^{tor}_Σ ← smooth.

$$\begin{tikzcd}
 & X^{tor}_\Sigma(\mathbb{C}) \arrow[d] \\
\Gamma\backslash\mathbb{H}_g \arrow[ur, hook] \arrow[r, hook] & \Gamma\backslash\mathbb{H}_g^*
\end{tikzcd}$$

$\rightsquigarrow$

Local coordinates. For $0 \le r \le g-1$.

$$\mathbb{H}_g \ni \Omega = \begin{pmatrix} t & w \\ {}^t w & \tau \end{pmatrix} \begin{matrix} \}r \\ \}g-r \end{matrix}$$

$$D_r = \left\{ \begin{pmatrix} t & w \\ {}^t w & \tau \end{pmatrix} \in M_g(\mathbb{C}) \;\middle|\; \begin{matrix} t \in \mathbb{H}_r \\ \tau \in M_{g-r}(\mathbb{C}) \end{matrix} \right\}$$

$\cup$

$\mathbb{H}_g$ "Siegel domain of third kind";

$U_r \subset Sp_{2g} / \mathbb{Q}$

$$U_r(\mathbb{Q}) = \left\{ \begin{pmatrix} I_r & 0 & 0 & 0 \\ 0 & I_{g-r} & 0 & b \\ & & I_r & 0 \\ & & 0 & I_{g-r} \end{pmatrix} \middle| \; b \in M_{g-r}(\mathbb{Q}). \quad b = {}^t b \right\}$$

$\rightsquigarrow$ vector gp $\to$ $\mathbb{Z}$-gp st $U_r(\mathbb{Z}) = P \cap U_r(\mathbb{Q})$

$U_r(\mathbb{Z})^* = \mathrm{Hom}(U_r(\mathbb{Z}), \mathbb{Z})$.

$U_r(\mathbb{R})$

$\cup$

$\overline{C_r}$

$\cup$

C_r a positive cone.

let $\sigma \in \bar{C}_r$ be a top dim cone gen'd by

$\mathbb{Z}$-basis $\{\xi_1, \dots, \xi_n\}$ of $U_r(\mathbb{Z})$

$n = \binom{g-r+1}{2}$.

let $l_1, \dots, l_n \in U_r(\mathbb{Z})^*$ dual basis.

$(U_r(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C})^\times \cong (\mathbb{C}^\times)^n$ torus

$\exp : U_r(\mathbb{C}) \to (U_r(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C})^\times$.

$\Rightarrow D_r \cong U_r(\mathbb{C}) \times \mathbb{C}^k \times F_r \quad k = r(g-r)$.

$$\begin{tikzcd}
\mathbb{H}_g \arrow[r, hook] \arrow[d] & U_r(\mathbb{C}) \times \mathbb{C}^k \times F_r \arrow[d, "\exp \times id \times id."] & \ni (\tau, w, t) \arrow[d, mapsto] \\
U_r(\mathbb{Z}) \backslash \mathbb{H}_g \arrow[r, phantom, "\subset"] \arrow[d] \arrow[dr, hook] & (U_r(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C})^\times \times \mathbb{C}^k \times F_r \arrow[d, equal, "\star"] & \ni \left(\exp(2\pi i\, l_j(\tau)), w, t \right) \\
P \backslash \mathbb{H}_g & (U_r(\mathbb{Z}) \backslash \mathbb{H}_g) &
\end{tikzcd}$$

[illegible] $)_\sigma$ = interior of closure of $U_r(\mathbb{Z})\backslash\mathbb{H}_g$ in $(*)_\sigma$.

$$\begin{tikzcd} \cdots \arrow[dr, bend right] & \cdots)_\sigma \arrow[d] \\ & \widetilde{(P\backslash\mathbb{H}_g)} \end{tikzcd}$$

theory of toric embedding

$\Sigma_{F_r} = \{\sigma_\alpha\}$ rat'l polyhedral cone decomp. of $\overline{C_r}$.

$\rotatebox{90}{$\in$}$

$\sigma \rightsquigarrow X_{F_r,\sigma} = (U_r(\mathbb{Z})\backslash\mathbb{H}_g)_\sigma$.

$G_\ell(F_r) \simeq GL_{g-r} \subset Sp_{2g}/\mathbb{Q}$

acts trivially on F_r, and acts on U_r, $\overline{C_r}$ by conj'n.

$\rightsquigarrow$ can take $\Gamma \cap G_\ell(F_r)(\mathbb{Q})$-invariant cone decomp Σ_{F_r}, modulo

$P \cap G_\ell(F_r)(\mathbb{Q})$ only finitely ma [illegible] only finitely many

“Γ_{Fr}-adm. cone decomp”. cones.

then $(X_{Fr,\sigma})_\sigma$ can be glued into a space

$$\begin{tikzcd}
X'_{\Sigma_{Fr}} \arrow[r] \arrow[dr] & \mathbb{H}_g^{*}. \\
 & P
\end{tikzcd}$$

$$G_\ell(Fr)$$

$$\cap$$

$$U_r \subset N_r \subset Sp_{2g}$$

$$\Gamma_{Fr} = \frac{P \cap N_r(\mathbb{Q})}{U_r(\mathbb{Z})}$$ acts properly

discont. on $X'_{\Sigma_{Fr}}$.

X_Σ = the smot.

F_r

then

$$\begin{tikzcd}
\Gamma_{F_r} \backslash \mathbb{H}_g \arrow[r] \arrow[d, phantom, "\cap"] & \Gamma \backslash \mathbb{H}_g \arrow[d, phantom, "\cap"] \\
X_{\Sigma_{F_r}} \arrow[r] & \Gamma \backslash \mathbb{H}_g^*
\end{tikzcd}$$

F any rat'l bdry comp.

$\Sigma_F = \{ \sigma_\alpha \}$ $\overline{\Gamma}_F$-adm cone decomp. of $\overline{C(F)}$.

$\rightsquigarrow X_{\Sigma_F}$ $\quad W = \coprod_F X_{\Sigma_F} \xrightarrow{\text{quot.}} X^{tor}_{\Sigma}(\mathbb{C})$

$\circlearrowright \Gamma$ $\quad = \tilde{X}_{\Sigma}$.

if $F \subset F'$,

$X_{\Sigma_{F'}} \to X_{\Sigma_F}$ étale

$\Sigma = (\Sigma_F)_F$ "Γ-adm".

Thm: 1) $\hat{X}_\Sigma$ is unique Hausd. anal. sp. containing $\Gamma \backslash \mathbb{H}_g = X_\Gamma$ as an open dense subset. s.t.

a) ∀ rat bdry F, ∃ op morphism π_F s.t.

$$\begin{tikzcd}
U_F(\mathbb{Z}) \backslash \mathbb{H}_g \arrow[r] \arrow[d] & X_{\Sigma_F} \arrow[d, "\pi_F"] \\
\Gamma \backslash \mathbb{H}_g \arrow[r, hook] & \hat{X}_\Sigma
\end{tikzcd}$$

b) ∀ pt of $\hat{X}_\Sigma$ is in the image of π_F for some F.

2) $\hat{X}_\Sigma$ compact normal alg. sp., and ∃ Σ s.t. $\hat{X}_\Sigma$ is smooth. and

projective .

3)
$$\begin{tikzcd}
\Gamma\backslash\mathbb{H}_g \arrow[r, hook] \arrow[dr, hook] & \widetilde{X}_\Sigma \arrow[d] \\
& \Gamma\backslash\mathbb{H}_g^*
\end{tikzcd}$$

- can describe bdry of $\hat{X}_\Sigma$ using theory of semi-abelian varieties.

- Arithmetic compactifications also exist.

[Faltings-Cha