§1: Néron models.

R: Dedekind domain. $K = \mathrm{Frac}(R)$. A/K: abelian var.

Def: A Néron model of A over R is a smooth gp scheme $N(A)$ over R s.t.

- $N(A) \otimes_R K \cong A$.
- If X is a smooth scheme over R, then every map $X_K \to A$ extends uniquely to $X \to N(A)$

Theorem (Néron) The Néron model of an abelian gp exists, and is of finite type over R

Rem: (1) If $\mathcal{A}$ is an abelian scheme over R with $\mathcal{A}_K = A$ then $N(A) = \mathcal{A}$. More generally, if $\mathcal{A}$ is a semi-abelian variety, then $\mathcal{A} = N(A)^\circ$

(2) Let $R \to R'$ be an extension of Dedekind domains. $K' = \mathrm{Frac}(R')$. Then $\exists$ a canonical map $\phi: N(A)_{R'} \longrightarrow N(A_{K'})$.

Moreover, • if $R \to R'$ is a smooth ext. of dvr. then $N(A)_{R'} \xrightarrow{\sim} N(A_{K'})$

- $N(A)^\circ$ is semi-abelian, then ϕ induces an isom $N(A)^\circ \otimes_R R' \xrightarrow{\sim} N(A_{K'})^\circ$

Theorem (Grothendieck) For an abelian variety A/K, there exists a finite separable extension K'/K s.t

$A_{K'}$ has semi-stable reduction over the integral closure R' of R in K'.

§2, Faltings Height:

K number field. A/K abelian variety of dim g
$\mathcal{A}$: Néron model of A over O_K.
$\omega_{\mathcal{A}} := \Lambda^g e^* \Omega^1_{\mathcal{A}/O_K}$. line bundle on $\mathrm{Spec}(O_K)$
$\omega_{\mathcal{A}} \otimes_{O_K} K = \omega_A = H^0(A, \Omega^g_A)$.

• $\forall \tau: K \hookrightarrow \mathbb{C}$ $\exists$ a hermitian struct on $\omega_{A,\tau} = \omega_A \otimes_\tau \mathbb{C}$

$$\langle \alpha, \beta \rangle_\tau := \frac{1}{2^g} \int_{A \otimes_\tau \mathbb{C}} |\alpha \wedge \bar{\beta}|$$

$\leadsto$ $(\omega_{\mathcal{A}}, \langle , \rangle_\tau)$ Hermitian line bundle.

Def: Faltings' height of A is

$$h(A) = \frac{1}{[K:\mathbb{Q}]} \deg(\omega_{\mathcal{A}}, \langle , \rangle_\tau) \qquad \sum_{v \nmid \infty} \varepsilon_v \log \|s\|_v$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad //$$
$$= \frac{1}{[K:\mathbb{Q}]} \log |\omega_{\mathcal{A}}/s \cdot O_K| - \sum_{\tau: K \hookrightarrow \mathbb{C}} \log \|s\|_\tau.$$

$\forall s \in \omega_{\mathcal{A}} \setminus \{0\}$.

Lemma: If K'/K is a finite extension, then $h(A_{K'}) \leq h(A)$, and "=" holds if A has semi-stable reduction over O_K.

Pf: Note that the contributions from arch. places to $h(A)$ and $h(A_{K'})$.
Let $\mathcal{A}'$ be the Néron model of $A_{K'}$ over $O_{K'}$.

Then $\exists$ a canonical map of gp schemes:

$$\mathcal{A} \otimes_{O_K} O_{K'} \longrightarrow \mathcal{A}'$$

which is an isom on generic fibre.

$$\Rightarrow \quad \omega_{\mathcal{A}'} \hookrightarrow \omega_{\mathcal{A}} \otimes_{O_K} O_{K'} \quad \text{injective}$$

and it is an isom if $\mathcal{A}$ is semi-stable

(because $\mathcal{A}'^{0} = \mathcal{A}^{0} \otimes_{O_K} O_{K'}$.)

$$\Rightarrow \quad h(A) - h(A_{K'}) = \frac{1}{[K':\mathbb{Q}]} \log \left| \omega_{\mathcal{A}} \otimes_{O_K} O_{K'} / \omega_{\mathcal{A}'} \right| \quad \square$$

<u>Def</u>: Define $h_{geom}(A) := h(A_{K'})$ for any finite ext. K'/K s.t $A_{K'}$ has semi-stable reduction.

<u>Thm</u> (Northcott thm for A.V).

(1) $\forall g \in \mathbb{N}$, $\exists$ a constant C s.t. $h(A) \geq h_{geom}(A) \geq C$ for any PPAV A of dim g defined over a number field K.

(2) $\forall g \in \mathbb{N}$, $t \in \mathbb{R}$, $d \in \mathbb{N}$. The set of isom. classes of PPAV. A of dim. g defined over a number field K with $[K:\mathbb{Q}] \leq d$ and $h(A) \leq t$. is finite.

<u>Cor</u>: Statement (2) holds also for A.V. without polarization.

Zarhin's trick: $\forall A/K$, with $[K:\mathbb{Q}] < d$, $h(A) \leq t$.

$$B := A^4 \times (A^t)^4$$

has principal polarization.

$$h(B) = 4h(A) + 4h(A^t) = 8h(A)$$

By Thm(2), there exists only finitely many possibilities for B.

For a fixed B, direct summands of B correspond to idempotents in $\mathrm{End}(B)$.

If $e_1, e_2 \in \mathrm{End}(B)$ which are conjugate by elts in $\mathrm{Aut}(B)$, correspond to isomorphic $A \subseteq B$.

The statement follows from the fact that $\mathrm{End}(B)$, which is an order in the semi-simple $\mathbb{Q}$-alg., has only finitely many such conjugacy classes.

$\square$

Strategy for the pf of Faltings' finiteness Thm:

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(1) Compare $h(A)$ with the height on moduli space of PPAV.

(good theory of moduli spaces of PPAV over $\mathbb{Z}$.)

(2) Show that the modular height comes from some Hermitian line bundle on A_g with log. singularity.

Review of Arithmetic theory of moduli space of P.P.A.V.

S_g: moduli stack of PPAV of dim. g.

(algebraic stack over $\mathbb{Z}$, which you can pretend to be a scheme).

$\exists$ universal family of P.P.A.V. A^{univ}/S_g.
s.t. $\forall$ PP.AV A over a scheme T $\exists$ a unique map

$$f: T \longrightarrow S_g \quad \text{s.t.} \quad A = f^*(A^{univ}).$$

$\cdot$ $S_g(\mathbb{C}) = \{$ isom classes of P.P.A.V. over $\mathbb{C}\}$

$$= Sp_{2g}(\mathbb{Z}) \backslash \mathfrak{h}_g$$

with $\mathfrak{h}_g := \{ Z \in M_g(\mathbb{C}) \mid {}^tZ = Z,\ \mathrm{Im}(Z) > 0 \}$

<u>Faltings–Chai</u>: Construct compactifications of S_g over $\mathbb{Z}$.

To explain the toroidal compactification of S_g we need to introduce some combinatorial data.

- Let $X \cong \mathbb{Z}^r$

$C = C(X) = \{$ positive semi-definite sym. bilinear form $b: X_{\mathbb{R}} \times X_{\mathbb{R}} \longrightarrow \mathbb{R}$
$\cdot$ $\ker(\phi_b : X_{\mathbb{R}} \to X_{\mathbb{R}}^*)$ is defined over $\mathbb{Q}$ $\}$

$S^2(X) = \mathrm{Sym}^2(X)$

$B(X) := \mathrm{Hom}(S^2(X), \mathbb{Z})$ $\quad \circlearrowleft \ GL_{\mathbb{Z}}(X) = GL_g(\mathbb{Z})$

$C \subseteq B(X)_{\mathbb{R}}$

Def: A $GL(X)$-adm. polyhedral cone decomposition of $\mathbb{C}$ is a collection $\{\sigma_\alpha\}_{\alpha\in J}$ s.t.

(1) For each $\alpha\in J$. $\exists\ v_1,\cdots,v_k\in B(X)_{\mathbb{Q}}$, s.t.
$$\sigma_\alpha=\mathbb{R}_{>0}v_1+\cdots+\mathbb{R}_{>0}v_k$$
and σ_α does not contain any line.

(2) $C=\coprod_\alpha \sigma_\alpha$. $\overline{\sigma_\alpha}=\bigcup$ finitely many σ_β.

(3) $\{\sigma_\alpha\}$ is $GL(X)$-invariant and $\{\sigma_\alpha\}/GL(X)$

Def: $\{\sigma_\alpha\}_{\alpha\in J}$ is smooth if each σ_α is generated by part of a $\mathbb{Z}$-basis of $B(X)$.

Fact: $GL(X)$-admissible polyhedral cone decomposition exists. Any two such decompositions have a common smooth refinement

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Thm (Faltings-Chai) Let $\Sigma=\{\sigma_\alpha\}$ be a $GL(X)$-adm P.C.D. of $C(X)$. Then $\exists$ an alg. stack $S_{g,\Sigma}^{tor}$ s.t.

(1) $S_{g,\Sigma}^{tor}$ is proper over $\mathrm{Spec}(\mathbb{Z})$ and contains S_g as an open dense substack.

(2) $D_\infty=S_{g,\Sigma}^{tor}-S_g$ is a relative Cartier divisor with normal crossing divisor. in $S_{g,\Sigma}^{tor}$.
$$S_{g,\Sigma}^{tor}=\coprod_{\sigma\in\Sigma/GL(X)} Z(\sigma)$$

3) The universal abelian scheme A^{univ} over S_g: extends to a semi-abelian scheme $A_\Sigma^{tor}/S_{g,\Sigma}^{tor}$

Local coordinates at a 0-dim. cusp

Let $\sigma \in \Sigma$ be an open cone. i.e. $\sigma \subseteq C(X)^\circ$.

$Z(\sigma)$ is 0-dim.

The completion of $S_{g,\Sigma}^{tor}$ along $Z(\sigma)$ is

$$\hat{S}(\sigma) := \mathrm{Spf}\left(\mathbb{Z}[[S^2(X)_\sigma^+]]\right)$$

$$S^2(X)_\sigma^+ := \left\{ q \in S^2(X) \;\middle|\; \begin{array}{l} \forall \ell \in \sigma. \\ \ell(q) > 0 \end{array} \right\}$$

$$= \mathbb{Z}_{\geq 0} q_1 + \cdots + \mathbb{Z}_{\geq 0} q_{\frac{g(g+1)}{2}} \quad \text{s.t. } \{q_i\} \text{ form a } \mathbb{Z}\text{-basis of } S^2(X)$$

over $\mathbb{C}$:

$$U_g := \left\{ \begin{pmatrix} I_g & b \\ 0 & I_g \end{pmatrix} \;\middle|\; \begin{array}{l} b \in M_{g\times g} \\ {}^t b = b \end{array} \right\} \subseteq Sp_{2g}$$

$$U_g(\mathbb{Z}) \cong S^2(X)^*$$

$$\begin{array}{ccc}
Z \in \mathfrak{H}_g & \hookrightarrow & U_g(\mathbb{C}) \\
\downarrow & & \downarrow \; Z \mapsto \exp(2\pi i Z) \\
U_g(\mathbb{Z})\backslash\mathfrak{H}_g & \hookrightarrow & U_g(\mathbb{Z})\backslash U_g(\mathbb{C}) = S^2(X)^* \otimes \mathbb{C}^\times \\
\text{local section} \; \downarrow & & \| \\
Sp_{2g}(\mathbb{Z})\backslash\mathfrak{H}_g & & \mathrm{Spec}(\mathbb{C}[S^2(X)]) \\
\| & & \downarrow \\
S_g(\mathbb{C}) & & \mathrm{Spec}(\mathbb{C}[S^2(X)_\sigma^+])
\end{array}$$

The boundary is defined by $q = 0$, $\forall q \in S^2(X)_\sigma^+$.

$$\begin{array}{ccc} \omega := e^* \Omega^g_{A^{univ}/S_g} & & \omega_\Sigma := e^* \Omega^g_{A^{tor}_\Sigma / S^{tor}_{g,\Sigma}} \\ | & & | \\ S_g & \hookrightarrow & S^{tor}_{g,\Sigma} \end{array}$$

Minimal compactification

Thm: $\exists$ a normal scheme S_g^* proper and of f.t. over $\mathbb{Z}$, an integer $m > 0$, and a very ample line bundle $\mathcal{L}$ on S_g^* s.t.

(1) $\forall$ toroidal compactification $S^{tor}_{g,\Sigma}$, $\exists$ a map

$$\pi_\Sigma : S^{tor}_{g,\Sigma} \longrightarrow S_g^*.$$

$$\pi_\Sigma^*(\mathcal{L}) = \omega_\Sigma^{\otimes m}$$

(2) $\exists$ a canonical isom:

$$S_g^* \cong \mathrm{Proj}\Big(\bigoplus_{n \geq 0} \Gamma(S^{tor}_{g,\Sigma}, \omega_\Sigma^{\otimes n})\Big).$$

(3) $S_g^* = \bigsqcup_{0 \leq i \leq g} [S_i] \to$ coarse moduli space of S_i

(4) For a geom. pt $x \in S^{tor}_{g,\Sigma}$, $\pi_\Sigma(x)$ is the classifying pt of the abelian part of $A^{tor}_{\Sigma,x}$.

The line bundle $\mathcal{L}$ on S_g^* has a Hermitian metric on $S_g(\mathbb{C}) \subseteq S_g^*(\mathbb{C})$:

$x \in S_g(\mathbb{C}) \Leftrightarrow A/\mathbb{C}$ abelian

$\mathcal{L}_x \cong \omega_A^{\otimes m}$ The metric on ω_A induces a metric on $\mathcal{L}_x$.

$\Rightarrow$ Hermitian line bundle $(\mathcal{L}, \|\cdot\|)$. on $S_g(\mathbb{C}) \subseteq S_g^*(\mathbb{C})$

<u>Prop</u>: The metric $\|\cdot\|$ on $\mathcal{L}$ has log singularity along the boundary $S_g^*(\mathbb{C}) - S_g(\mathbb{C})$.

<u>pf</u>:
$$\begin{tikzcd} S_g(\mathbb{C}) \arrow[r, hook] \arrow[d, equal] & S_{g,\Sigma}^{tor}(\mathbb{C}) \arrow[d, "\pi_\Sigma"] \\ S_g(\mathbb{C}) \arrow[r, hook] & S_g^* \end{tikzcd}$$

It suffices to check the metric on ω has log. singularity along the boundary $S_{g,\Sigma}^{tor}(\mathbb{C}) - S_g(\mathbb{C})$

We check here the behavior around a 0-dim. cusp.

$Z(\sigma) \subset S_{g,\Sigma}^{tor}$ for some cone $\sigma \in \Sigma$ with $\sigma \subseteq C(X)^\circ$.

$$\begin{tikzcd} z \in \mathfrak{H}_g \arrow[r, hook] \arrow[d] & U_g(\mathbb{C}) \arrow[d] & & z \arrow[d, mapsto] \\ U_g(\mathbb{Z})\backslash\mathfrak{H}_g \arrow[r, hook] \arrow[d] & U_g(\mathbb{Z})\backslash U_g(\mathbb{C}) \arrow[r, equal] \arrow[d, hook] & S^2(X)^* \otimes \mathbb{C}^\times & \exp(2\pi i z) \\ Sp_{2g}(\mathbb{Z})\backslash\mathfrak{H}_g & \mathrm{Spec}\, \mathbb{C}[S^2(X)^+_\sigma]) & & \end{tikzcd}$$

$$S^2(X)^+_\sigma = \mathbb{Z}_{\geq 0}\, q_1 + \cdots + \mathbb{Z}_{\geq 0}\, q_{\frac{g(g+1)}{2}}$$

For a $Z \in \mathfrak{H}_g$, its corresponding abelian varieties

$A_Z := \mathbb{C}^g / \Lambda_Z$. $\Lambda_Z :=$ lattices in $\mathbb{C}^g$ generated by the column vectors of (I_g, Z)

$$\alpha = dw_1 \wedge \cdots \wedge dw_g \in H^0(A_Z, \Omega^g_{A_Z})$$

$$\|\alpha\|^2 = \frac{1}{2^g} \int_{A_Z(\mathbb{C})} |\alpha \wedge \bar{\alpha}| = \det(\operatorname{Im}(Z))$$

as Z approaches the cusp $Z(\sigma)$,

$$\|\alpha\|^2 \sim \log |q_1 \cdots q_{\frac{g(g+1)}{2}}| .$$

$\square$

A criterion for the semi-stable reduction of A.V.

<u>Thm</u>: Let A/K be an abelian variety defined over a number field. $N \geq 3$ be an integer such that $A[N](\bar{K}) = A[N](K)$. Then A has semi-stable reduction at primes $v \nmid N$.

(cf. Brian Conrad's note « semi-stable reduction of A.V ».

<u>Cor</u>: If A is an A.V over a number field K s.t. $A[12](\bar{K}) = A[12](K)$, then A has semi-stable reduction at all primes of $\mathcal{O}_K$.

Pf. of Faltings' finiteness thm:

Let A/K be an AV. defined over a number field. K. Then $\exists$ a finite ext. K'/K of deg $\leq |GL_{2g}(\mathbb{Z}/12\mathbb{Z})|$ s.t. $A_{K'}$ has semi-stable reduction at all primes of $\mathcal{O}_{K'}$

$$h(A_{K'}) = h_{geom}(A) \leq h(A)$$

Up to replacing d by $d\,|GL_{2g}(\mathbb{Z}/12\mathbb{Z})|$, it suffices to show that the set of P.P.AV. A/K of dim g with $[K:\mathbb{Q}] \leq d$ and with semi-stable reduction is finite.

$$A/K \rightsquigarrow x : \operatorname{Spec}(K) \longrightarrow S_g.$$

extends to $\operatorname{Spec}(\mathcal{O}_K) \xrightarrow{\bar{x}} S_{g,\Sigma}^{tor.}$ since $S_{g,\Sigma}^{tor}$ is proper

s.t. $\bar{x}^* A_\Sigma^{tor} \cong N(A)^\circ$.

$$x^* = \pi_\Sigma \circ \bar{x} : \operatorname{Spec}(\mathcal{O}_K) \to S_g^*$$

$$\Rightarrow h_{\mathcal{L}}(x^*) = m\, h_{geom}(A)$$

$$\pi_\Sigma^* \mathcal{L} = (\omega_\Sigma^{tor})^{\otimes m}$$

Now since the metric of $\mathcal{L}$ at arch. place has log singularity along the boundary $S_g^*(\mathbb{C}) - S_g(\mathbb{C})$, we conclude by

Thm: Let $\bar{X}$ be a proj. var. $/\mathcal{O}_K$. $\mathcal{L}$ be an ample line bundle on $\bar{X}$. $X \subseteq \bar{X}$ open. s.t. $\forall\, \sigma : K \hookrightarrow \mathbb{C}$ $\mathcal{L}|_{X_\sigma}$ has a Hermitian metric with log. singularity then:

(1) $\exists$ a constant C s.t. $h_L(P) \geq C$. $\forall$ $P \in X(\bar{K})$

(2) $\forall$ $B \in \mathbb{R}$, integer d. the set of pts $P \in X(\bar{K})$ defined over a number field of deg $\leq d$. with $h_L(P) \leq B$ is finite.