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Finite group schemes & p-divisible gps

§ Finite gp schemes.

S base sch. $(S = \mathrm{Spec}\, R)$

<u>Def</u> $f: G \to S$ a morphism of schemes is a group scheme if

$\exists\ \mu: G \times G \to G,\quad s: S \to G,\quad i: G \to G$ over S

s.t.

1) $\mu \circ (\mu \times id) = \mu \circ (id \times \mu)$

2) $id_G = \mu \circ (id \times s) = \mu \circ (s \times id)$

3) $\mu \circ (id \times i) = \mu \circ (i \times id) = s \circ f : G \to G$

We say G is comm. if

$$\begin{tikzcd} G \times_S G \arrow[rr, "\sigma"] \arrow[dr, "\mu"'] & \arrow[d, phantom, "\circlearrowleft"] & G \times_S G \arrow[dl, "\mu"] \\ & G & \end{tikzcd}$$

$\sigma(x, y) = (y, x)$.

<u>Def</u>. a group scheme $f: G \to S$ is <u>finite of order n</u>, for $n \geq 1$. if

1) $f: G \to S$ is finite (as morph. of schs)

2) $f_* \mathcal{O}_G$ is loc. free rk n $\mathcal{O}_S$-mod.

<u>Set</u> $\mathrm{ord}(G) = n$.

property: $f: G \to S$ finite comm. gp sch of order n.

$\forall\, m \geq 1.\ [m]: G \to G$. add'n m times

$[n]: G \to G$ annihilates G

i.e. $[n] = s \circ f$.

e.g. 1) $G_a = \operatorname{Spec}(R[t])$. $\mu: t \mapsto t \otimes 1 + 1 \otimes t$.

2) $G_m = \operatorname{Spec} R[t, t^{-1}]$. $\mu: t \mapsto t \otimes t$.

3) $\mu_n = \operatorname{Spec} R[t]/(t^n - 1)$. $\mu: t \mapsto t \otimes t$.

$= \ker(G_m \xrightarrow{[n]} G_m)$.

Def. $f: H \to G$ morph. of gp schs

$$\begin{tikzcd} \ker(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & H \arrow[d, "f"] \\ S \arrow[r, "s"] & G \end{tikzcd}$$

Rmk. X sch / Spec R.

$\rightsquigarrow X: \mathrm{Alg}/R \to \mathrm{Sets}$

$S \longmapsto X(S)$

G gp sch / Spec R

$\rightsquigarrow G: \mathrm{Alg}/R \to \mathrm{Groups}$

$S \longmapsto G(S)$

induced by (μ, s, i).

$G_a(S) = (S, +)$ $\quad$ $G_m(S) = (S^\times, \times)$

$\mu_n(S) = \{x \in S \mid x^n = 1\}$

4) Δ constant gp.

$$\underline{\Delta} = \operatorname{Spec} R^{\Delta}, \quad R^{\Delta} = \operatorname{Fun}(\Delta, R).$$

$$\mu(f) \in R^{\Delta \times \Delta} = R^{\Delta} \otimes_R R^{\Delta}$$

$$\mu(f)(g,h) = f(g \cdot h).$$

5) ζ $(p-1)$-th root of unity $\in \mathbb{Z}_p$, primitive

$$\Lambda_p = \mathbb{Z}\left[\zeta, \frac{1}{p(p-1)}\right] \cap \mathbb{Z}_p \subset \mathbb{Q}_p$$

$$B = \Lambda_p[z] / z^p - 1$$

$i = 1, \dots, p-1.$

$$w_i = \frac{\left(\sum_{m=1}^{p-1} \zeta^{-m}(1 - z^m)\right)^i}{\sum_{m=1}^{p-1} \zeta^{-im}(1 - z^m)} \in B.$$

are units in Λ_p.

$\varphi : \Lambda_p \to R$

$G_{a,b}$ for $a, b \in R$ w/ $ab = p$:

$G_{a,b} = \operatorname{Spec} A$.

$$A = R[t] / t^p - at$$

$$\mu : t \mapsto 1 \otimes t + t \otimes 1 - \frac{b}{1-p} \sum_{i=1}^{p-1} \frac{1}{\varphi(w_i w_{p-i})} t^i \otimes t^{p-i}.$$

$\varepsilon : t \mapsto 0$

$i : t \mapsto -t$.

$G_{a,b}$ is a finite comm. gp sch of ord. p

R complete noeth loc. ring res. char $= p$.

$$\begin{array}{c} \Lambda_p \\ \downarrow \\ \mathbb{Z}_p \longrightarrow R. \end{array}$$

p

Thm. (Oort–Tate) Any G finite comm gp sch $/R$ of order p $G \cong G_{a,b}$.

$G_{a,b} \cong G_{c,d}$ iff $\exists u \in R^\times$ s.t. $c = u^{p-1}a$
$d = u^{1-p} b$.

Def. $0 \to G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3 \to 0$ is <u>exact</u>

if 1) g is faithfully flat
(also say g is surj.)

2) f is closed immersion &
$G_1 \cong \ker(g)$.

$\Rightarrow \operatorname{ord}(G_2) = \operatorname{ord}(G_1) \cdot \operatorname{ord}(G_3)$.

G finite comm. gp sch.
$G = \operatorname{Spec}(A) \to S = \operatorname{spec}(R)$.

$\mu: A \to A \otimes_R A$

comultiplication $\rightsquigarrow$ multn on G.

$m: A \otimes_R A \to A$ multn.

Def. Cartier dual of G is $G' = \operatorname{spec} A'$.

$A' = \operatorname{Hom}_R(A, R)$.

$m': A' \otimes_R A' \to A'$

$\varphi, \psi \mapsto (\varphi \otimes \psi)(\mu(-))$.

$m' : A' \to A' \otimes_R A'$

$\varphi \mapsto \varphi(m(-))$.

e.g. $(\mathbb{Z}/n\mathbb{Z})' \simeq \mu_n$.

<u>differentials</u>. $G \to \operatorname{Spec} R$. $\Omega^1_{G/R} = \Omega_G$.

<u>Prop</u>. R cplt noeth loc. ring res char p.

$a, b \in R$ $ab = p$.

$G = G_{a,b} = \operatorname{Spec} R[t]/(t^p - at)$, $\cdots$

1) $\Omega_G \simeq R[t]/(t^p - at,\ pt^{p-1} - a)\, dt$.

2) $s^* \Omega_G \simeq R/aR$.

<u>Pf</u>. $A = R[t]/t^p - at$.

$I = (t^p - at)$

$$I/I^2 \xrightarrow{d} \Omega_{R[t]/R} \otimes_R A \to \Omega_{A/R} \to 0.$$

$\Omega_{R[t]/R} \otimes_R A \simeq A \cdot dt$

$t^p - at \mapsto (pt^{p-1} - a)\, dt$

$\Rightarrow$ 1).

2): $s : A \to R$, $t \mapsto 0$.

$\Rightarrow s^* \Omega_G \simeq R/aR$ $\square$.

If moreover R is DVR

Cor. $\mathrm{ord}(G) = p$. $\mathrm{length}(s^*\Omega^1_{G/R}) + \mathrm{length}(s^*\Omega^1_{G'/R})$

$= \mathrm{length}\ R/pR$.

Pf. $G = G_{a,b}$, $G' = G_{b,a}$ $ab = p$.

$\rightsquigarrow$ $\mathrm{length}(R/a) + \mathrm{length}(R/b) = \mathrm{length}\ R/p$.

$\boxed{}$.

Thm. R DVR res. char $= p$

fraction field char $= 0$.

1) $0 \to G_1 \to G_2 \to G_3 \to 0$ ex. seq. of fin. comm. gp schs.

then $0 \to s^*\Omega_{G_3} \to s^*\Omega_{G_2} \to s^*\Omega_{G_1} \to 0$

2) G fin. ord. n. G' dual.

$\mathrm{length}(s^*\Omega_G) + \mathrm{length}(s^*\Omega_{G'}) = \mathrm{length}(R/nR)$.

Pf. of 1) ① $0 \to G_1 \to G_2 \to G_3$,

$\Rightarrow s^*\Omega_3 \to s^*\Omega_2 \to s^*\Omega_1 \to 0$. (*)

$A_1 \twoheadleftarrow A_2 \hookleftarrow A_3$

$\rightsquigarrow \Omega^1_{A_2/R} \otimes_{A_2} A_1 \twoheadrightarrow \Omega^1_{A_1/R}$.

$R \to A_3 \to A_2$, $\quad \otimes_{A_2} A_1$:

$$\rightsquigarrow \Omega_{A_3/R} \otimes_{A_3} A_1 \to \Omega_{A_2/R} \otimes_{A_2} A_1 \to \Omega_{A_2/A_3} \otimes_{A_2} A_1 \to 0$$

$$\begin{tikzcd} & \Omega_{A_2/R} \otimes_{A_2} A_1 \arrow[d, "\wr"] & \Omega_{A_2/A_3} \otimes_{A_2} A_1 \arrow[d, "\wr\ \varphi."] \\ & \Omega^1_{(A_2 \otimes_{A_3} R)/R} \arrow[u, "\text{b.c.}"'] \arrow[r, "\theta", "\sim"'] & \Omega^1_{A_1/R} \end{tikzcd}$$

$\otimes_{A_1,s} R \rightsquigarrow (\ast)$.

② If $0 \to G_1 \to G_2 \to G_3 \to 0$

then $(\ast)$ is inj at left.

We need : Thm (Raynaud) $\forall$ fin. gp sch G/R, $\exists$ ab sch A/R.

$$0 \to G \to A \xrightarrow{\varphi} B \to 0$$

B ab sch.

$$\begin{array}{ccccccc} \rightsquigarrow & s^*\Omega_B & \xrightarrow{d\varphi} & s^*\Omega_A & \to & s^*\Omega_G & \to 0. \\ & \wr\wr & & \wr\wr & & & \\ & R^{\oplus g} & \longrightarrow & R^{\oplus g} & & & \end{array}$$

$g = \dim A = \dim B$.

$\det d\varphi = \deg \varphi$, $\operatorname{char}(k) = 0$.

$\Rightarrow d\varphi$ is inj.

$G_1 \hookrightarrow G_2 \hookrightarrow A$. $\quad$ A ab sch $/R$.

$B = A/G_1$

$\rightsquigarrow G_3 = G_2/G_1 \hookrightarrow B$.

$C = B/G_3 = A/G_2$.

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & G_1 \arrow[r] \arrow[d] & A \arrow[r] & B \arrow[r] & 0 \\
0 \arrow[r] & G_2 \arrow[r] \arrow[d] & A \arrow[r] & C \arrow[r] & 0 \\
0 \arrow[r] & G_3 \arrow[r] \arrow[d] & B \arrow[r] & C \arrow[r] & 0. \\
 & 0 & & &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & S^*\Omega_C \arrow[r] & S^*\Omega_B \arrow[r] & S^*\Omega_{G_3} \arrow[r] \arrow[d, "①"] & 0 \\
0 \arrow[r] & S^*\Omega_C \arrow[r] & S^*\Omega_A \arrow[r] & S^*\Omega_{G_2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S^*\Omega_B \arrow[r] & S^*\Omega_A \arrow[r] & S^*\Omega_{G_1} \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

diagram chase $\Rightarrow$ ① inj. !

☒.

Pf of 2)

$0 \to G \to A \xrightarrow{\varphi} B \to 0$, $\quad A, B$ ab schs

$\Rightarrow \quad 0 \to G' \to B' \to A' \to 0$. $\quad B', A'$ dual ab sch.

$$\Rightarrow \quad s^*\Omega_G = \mathrm{cok}\left(s^*\Omega_B \xrightarrow{d\varphi} s^*\Omega_A\right)$$

$$s^*\Omega_{G'} = \mathrm{cok}\left(s^*\Omega_{A'} \to s^*\Omega_{B'}\right)$$

$$= \mathrm{cok}\left(H^1(B, \mathcal{O}_B) \to H^1(A, \mathcal{O}_A)\right).$$

$$\ell(s^*\Omega_G) = \ell\left(\mathrm{cok}\left(\wedge d\varphi : \Gamma(B, \Omega_B^g) \to \Gamma(A, \Omega_A^g)\right)\right)$$

$$\ell(s^*\Omega_{G'}) = \ell\left(\mathrm{cok}\left(\wedge\psi : H^g(B, \mathcal{O}_B) \to H^g(A, \mathcal{O}_A)\right)\right).$$

Serre duality:

$$\begin{tikzcd}
H^g(A, \mathcal{O}_A) \times \Gamma(A, \Omega_A^g) \arrow[r] & R \\
H^g(B, \mathcal{O}_B) \times \Gamma(B, \Omega_B^g) \arrow[r] \arrow[u] & R \arrow[u, "\deg\varphi = \mathrm{ord}(G) = n"']
\end{tikzcd}$$

$\square$.

S p-divisible gps

R cplt noeth loc. ring res char $= p$

Def. a p-div-gp $\underline{G}/R$ of height h is

an inductive system $\underline{G} = (G_k, i_k)$, $k \geq 1$.

$i_k: G_k \hookrightarrow G_{k+1}$. s.t.

1) G_k is fin. comm. of ord. p^{kh}.

2) $\forall\, k \geq 1$. $0 \to G_k \xrightarrow{i_k} G_{k+1} \xrightarrow{[p^k]} G_{k+1}$ is exact.

i.e. $(G_k, i_k) = \ker([p^k])$.

$i_{k,l}: G_k \xhookrightarrow{i_k} G_{k+1} \hookrightarrow \cdots \hookrightarrow G_{k+l}$.

Prop 1) $0 \to G_k \xrightarrow{i_{k,l}} G_{k+l} \xrightarrow{[p^k]} G_{k+l}$ exact.

2) G_k is annihilated by p^k.

3)
$$\begin{tikzcd} 0 \arrow[r] & G_k \arrow[r, hook, "i_{k,l}"] & G_{k+l} \arrow[rr, "{[p^k]}"] \arrow[dr, "j_{k,l}"'] & & G_{k+l} \\ & & & G_l \arrow[ur, "i_{l,k}"'] & \end{tikzcd}$$

4) $0 \to G_k \xrightarrow{i_{k,l}} G_{k+l} \xrightarrow{j_{k,l}} G_l \to 0$ ex.

Pf. 1) Induction on l

2) $0 \to G_k \xrightarrow{i_k} G_{k+1} \xrightarrow{[p^k]} G_{k+1}$

$$\begin{tikzcd} A_k & A_{k+1} \arrow[l, "i_k"'] \\ {} \arrow[u, "p^k"] & {} \arrow[u, "p^k"] \end{tikzcd}$$

$$A_k \overset{i_k}{\hookleftarrow} A_{k+1} \to R.$$

$p^k \circ i_k = i_k \circ p^k$ factors thru zero section

$s: A_{k+1} \to R$

$\Rightarrow [p^k]$ ann. G_k.

3) by 2)
$$\begin{tikzcd} G_{k+l} \arrow[r, "{[p^k]}"] \arrow[dr, dashed, "j_{k,l}"'] & G_{k+l} \arrow[r, "{[p^l]}"] & G_{k+l} \\ & \ker [p^l] = G_l \arrow[u, "i_{l,k}"'] & \end{tikzcd}$$

4) compare $rk_R\, A_l, A_k, A_{l,k}$.

eg. 1) G_m/R. $\mu_{p^k} = \ker(G_m \xrightarrow{[p^k]} G_m)$

$i_k : \mu_{p^k} \hookrightarrow \mu_{p^{k+1}}$.

$(\mu_{p^k}, i_k) := G_m(p)$.

2) $\underline{\mathbb{Q}_p/\mathbb{Z}_p} = (\underline{\frac{1}{p^k}\mathbb{Z}/\mathbb{Z}},\ i_k: \underline{\frac{1}{p^k}\mathbb{Z}/\mathbb{Z}} \hookrightarrow \underline{\frac{1}{p^{k+1}}\mathbb{Z}/\mathbb{Z}})$.

3) A ab sch. $/R$.

$A[p^k] = \ker([p^k] : A \to A)$

$(i_k : A[p^k] \hookrightarrow A[p^{k+1}]) := A(p)$

Def. $0 \to \underline{F} \to \underline{G} \to \underline{H} \to 0$

a seq of p-div. gps.

is exact iff $\forall k$

$0 \to F_k \to G_k \to H_k \to 0$ is exact.