2024 Mordell 猜想讨论班

求真书院

4.25

§. Absolute values

k field; $|\cdot| : k \to [0, +\infty[$ is an <u>abs. value</u> if

(i) $|x| = 0 \iff x = 0$;

(ii) $|xy| = |x||y|$;

(iii) $|x+y| \leq |x| + |y|$,

$|\cdot|$ is <u>non-archimedean</u> if

$|x+y| \leq \max(|x|, |y|)$.

<u>Example</u>: ① $k = \mathbb{Q}$. $M_{\mathbb{Q}} = \{p \text{ prime}\} \cup \{\infty\}$

$|x|_\infty := \max\{x, -x\}$,

$|x|_p := p^{-\mathrm{ord}_p(x)}$

Ostrowski's thm: $|\cdot|_\infty$, $|\cdot|_p$ p prime represent all non-trivial absolute values up to equivalence.

<u>Product formula</u>: $\prod_{v \in M_{\mathbb{Q}}} |x|_v = 1$, $\forall x \in \mathbb{Q}^\times$

② k = number field; i.e. $d := [k : \mathbb{Q}] < +\infty$.

$\forall v \in M_{\mathbb{Q}}$ extends to some $w \in M_k$, write $w \mid v$

Recall: $d_w = [k_w : \mathbb{Q}_v]$ (local degree)

$$|x|_w = \left| N_{k_w/\mathbb{Q}_v}(x) \right|_v^{\frac{1}{d_w}}.$$

$$k \otimes \mathbb{Q}_v = \prod_{w \mid v} k_w \qquad (*)$$

$$d = \sum_{w \mid v} d_w \qquad \forall v \in M_{\mathbb{Q}}$$

<u>Definition</u> (Normalised norm with respect to $\mathbb{Q}$)

$$\|x\|_w := |x|_w^{\frac{d_w}{d}}$$

$$= \left| N_{k_w/\mathbb{Q}_v}(x) \right|_v^{1/d}$$

<u>Proposition</u> (Product formula)

$$\prod_{w \in M_k} \|x\|_w = 1. \quad \forall x \in k^\times$$

proof: $\circledast \Rightarrow$

$$\prod_{w|v} N_{k_w/\mathbb{Q}_v}(x) = N_{k/\mathbb{Q}}(x). \quad \forall x \in k.$$

So
$$\prod_{w \in M_k} \|x\|_w = \prod_{v \in M_\mathbb{Q}} \prod_{w|v} \|x\|_w$$
$$= \prod_{v \in M_\mathbb{Q}} \prod_{w|v} |N_{k_w/\mathbb{Q}_v}(x)|_v^{1/d}$$
$$= \prod_{v \in M_\mathbb{Q}} |N_{k/\mathbb{Q}}(x)|_v^{1/d} = 1. \quad \square$$

§ : Heights on projective spaces

$\mathbb{P}^n_k$ = the projective n-space over k

$\underline{\underline{\text{as a set}}}$ $k^{n+1} - \{\underline{0}\} \Big/ \{(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n),\ \lambda \in k^\times\}$.

Homogeneous coordinates: $[x_0 : \dots : x_n]$.

Definition (Absolute naive height on $\mathbb{P}^n$)

For $P = [x_0 : \dots : x_n] \in \mathbb{P}^n(\overline{\mathbb{Q}})$, choose k number field such that $x_i \in k$ for all i. (write $P \in \mathbb{P}^n(k)$.)

Define
$$H(P) := \prod_{v \in M_k} \max_{0 \le i \le n} \|x_i\|_v \quad \text{(exponential)}$$
$$h(P) := \log H(P) \quad \text{(logarithmic)}$$

Basic properties:

① $H(P) \ge 1$, $h(P) \ge 0$ $\forall P \in \mathbb{P}^n(\overline{\mathbb{Q}})$.

② "Well-defined."

2.1: $\displaystyle\prod_{v \in M_k} \max_{0 \le i \le n} \|\lambda x_i\|_v = \prod_{v \in M_k} \max \|x_i\|_v$

(product formula)

2.2: k'/k finite extension. $P \in \mathbb{P}^n(k) \subseteq \mathbb{P}^n(k')$

$e = \dfrac{d'}{d} = [k' : k]$

$\forall v | w$, $\|x\|_w = \|x\|_v^{[k'_w : k_v]/e}$

$$\prod_{w \in M_{k'}} \max \|x_i\|_w = \prod_{v \in M_k} \prod_{w | v} \max \|x_i\|_w$$
$$= \prod_{v \in M_k} \left(\max \|x\|_v\right)^{\sum_{w|v} [k_w : k_v]/e} = \prod_{v \in M_k} \max \|x\|_v.$$

<u>Remark</u>: H is $G_{\mathbb{Q}}$-invariant, where $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

i.e. $\forall \sigma \in G_{\mathbb{Q}}, \forall P \in \mathbb{P}^n(\overline{\mathbb{Q}}), \quad H(\sigma(P)) = H(P)$.

§ Northcott property.

<u>Elementary case</u> $\quad P = [x_0 : \dots : x_n] \in \mathbb{P}^n(\mathbb{Q})$

can choose a representative $(x_0, \dots, x_n) \in \mathbb{Z}^{n+1} - \underline{0}$

such that $\gcd(x_0, \cdots, x_n) = 1$.

Then $\quad H(P) = \prod_{v \in M_{\mathbb{Q}}} \max_{0 \le i \le n} |x_i|_v = \max_{0 \le i \le n} |x_i|$.

<u>Lemma</u> ("Northcott" over $\mathbb{Q}$)

$$\# \{ P \in \mathbb{P}^n(\mathbb{Q}) : H(P) \le B \} < +\infty.$$

General case

<u>Theorem</u> (Northcott)

$\forall B, D > 0$

$$\# \{ P \in \mathbb{P}^n(\overline{\mathbb{Q}}), \ [\mathbb{Q}(P) : \mathbb{Q}] \le D, \ H(P) \le B \} < +\infty$$

(If $P = [x_0 : \cdots : x_n]$ with $x_0 \neq 0$, let $\mathbb{Q}(P) := \mathbb{Q}\left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right)$)

<u>Proposition</u>: $\forall B > 0, \ d \in \mathbb{N}$

$$\# \{ x \in \overline{\mathbb{Q}} : H(x) \le B, \ [\mathbb{Q}(x) : \mathbb{Q}] = d \} < +\infty$$

$= \prod_{v \in M_{\mathbb{Q}(x)}} \max(\|x\|_v, 1)$ $\quad$ we regard $x = [1 : x] \in \mathbb{P}^1(\overline{\mathbb{Q}})$

Proof of Northcott's theorem.

For $P=[x_0:\dots:x_n]$, WLOG assume $x_0=1$.

Take $k/\mathbb{Q}$ finite s.t. $P\in\mathbb{P}^n(k)$. Then

$\forall v\in M_k$, $\forall 1\leq i\leq n$, $\max(\|x_i\|_v,1)\leq\max(1,\|x_1\|_v,\dots,\|x_n\|_v)$.

Hence $H(P)\leq B\Rightarrow H(x_i)\leq B$.

Proposition $\Rightarrow$ finitely many x_i $1\leq i\leq n$ of height $\leq B$

$\Rightarrow$ finitely many such P. $\square$

Lemma: $\exists$ an absolute constant $C_d>0$ (depending only on d) such that, $\forall x\in\overline{\mathbb{Q}}$, $d=[\mathbb{Q}(x):\mathbb{Q}]$, let $X^d+s_1(x)X^{d-1}+\dots+s_d(x)$ be the minimal polynomial of x over $\mathbb{Q}$; we have

$$H(1,s_1(x),\dots,s_d(x))\leq C_d\,H(x)^{d^2}.$$

Proof of proposition:

Let $N_1(d;B)=\#\{x\in\overline{\mathbb{Q}}: H(x)\leq B,\ [\mathbb{Q}(x):\mathbb{Q}]=d\}$

$$N_2(d;B)=\#\left\{\begin{array}{l}\text{irreducible } X^d+a_1X^{d-1}+\dots+a_d\in\mathbb{Q}[X],\\ H(1,a_1,\dots,a_d)\leq B\end{array}\right\}$$

Every irreducible polynomial of degree d has d roots,

Lemma $\Rightarrow N_1(d;B)\leq d\,N_2(d;CB^d)<+\infty$. $\square$

"Northcott" over $\mathbb{Q}$

Proof of Lemma: Notation: M_k^∞ = archimedean absolute values
M_k^f = non-archimedean abs. values.

Let $G_{\mathbb{Q}} \cdot x = \{x_1, \dots, x_d\}$

$k = \mathbb{Q}(x_1, \dots, x_d)$ splitting field of x

Then $\forall\, 1 \leq r \leq d$, $\forall v \in M_k^\infty$, by triangle inequality,

$$|s_r(x)|_v = \Big|\sum_{(i_1,\dots,i_r)} \pm x_{i_1} \cdots x_{i_r}\Big|_v \leq C_{r,v} \max_{1\leq i\leq d} |x_i|_v^r$$

Note that $C_{r,v} \leq \binom{d}{r} \leq 2^d$. $\forall r, v \in M_k^\infty$

While $\forall v \in M_k^f$, by STRONG triangle inequality,

$$|s_r(x)| \leq \max_{1\leq i\leq d} |x_i|_v^r \quad 1 \leq r \leq d.$$

Hence

$$\max\{1, |s_r(x)|_v, 1\leq r\leq d\} \leq \begin{cases} C_d \max\{1, |x_i|_v, 1\leq i\leq d\}^d, & v \in M_k^\infty \\ \max\{1, |x_i|_v, 1\leq i\leq d\}^d, & v \in M_k^f. \end{cases}$$

We can take $C_d = 2^d$.

$$\Rightarrow H(1, s_1(x), \dots, s_d(x)) \leq 2^d \prod_{i=1}^d H(x_i)^d$$

Since H is $G_{\mathbb{Q}}$-invariant, $H(x_1) = \dots = H(x_d) = H(x)$

$$\Rightarrow H(1, s_1(x), \dots, s_d(x)) \leq 2^d H(x)^{d^2}. \qquad \square$$

§ Functoriality in $\mathbb{P}^n$

Let $f_0, \cdots, f_m$ be homogeneous forms of degree d in $(n+1)$-variables, with coefficients in $\overline{\mathbb{Q}}$.

$$\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m \qquad \text{rational map.}$$
$$\underline{x} \longmapsto (f_0(\underline{x}) : \cdots : f_m(\underline{x}))$$

$Z = V(f_0, \cdots, f_m)$ "base locus of ϕ".

$\phi|_{\mathbb{P}^n \setminus Z}$ is a regular map of degree d.

Theorem Let $X \subseteq \mathbb{P}^n$ be closed & $X \cap Z = \emptyset$ then

$$\forall P \in X(\overline{\mathbb{Q}}) \quad h(\phi(P)) = d\, h(P) + O(1)$$

(i.e. $O(1)$ means that $h \circ \phi - dh : \mathbb{P}^n(\overline{\mathbb{Q}}) \to \mathbb{R} \cup \{\infty\}$ is a bounded function on $X(\overline{\mathbb{Q}})$.)

proof (Sketch)

(1) $\forall P \in \mathbb{P}^n(\overline{\mathbb{Q}}) \setminus Z$, we have
$$h(\phi(P)) \leq d\, h(P) + O(1)$$

(2) Use Hilbert's "Nullstellensatz" to get the reverse inequality on $X(\overline{\mathbb{Q}})$

§ Heights on projective varieties

V projective variety $/\bar{\mathbb{Q}}$.

$L \to V$ line bundle.

- L is <u>globally generated</u> (or the complete linear system $|L|$ is <u>base point free</u>)

if $\exists$ basis $(s_0, \dots, s_n)$ of $H^0(V, L)$ which defines a morphism

$$\phi_L : V \xrightarrow{|L|} \mathbb{P}^n$$

- L is very ample if $\phi_L : V \xrightarrow{|L|} \mathbb{P}^n$ is an immersion;

is ample if $\exists\, m > 0$, $L^{\otimes m}$ is very ample

- $\forall\, \phi : V \to \mathbb{P}^n$ morphism, $c_\phi := [\phi^* \mathcal{O}(1)] \in \mathrm{Pic}(V)$

define $H_\phi(p) := H(\phi(p))$;

$$h_\phi(p) := \log H_\phi(p).$$

- Any $h_1\ h_2 : V(\bar{\mathbb{Q}}) \to \mathbb{R}$ <u>equivalent</u> if $h_1 = h_2 + O(1)$

Theorem (Weil's height machine)

V projective variety $/\bar{\mathbb{Q}}$.

There exists a unique functorial morphism

$$\mathrm{Pic}(V) \to \{\text{functions } V(\bar{\mathbb{Q}}) \to \mathbb{R}\}/\sim \quad \text{s.t.}$$

(1) $\forall c, c'$, $\quad h_{c+c'} = h_c + h_{c'} + O(1)$

(2) For c globally generated $\to \phi_c : V \to \mathbb{P}^n$, we have

$$h_c = h_{\phi_c} + O(1).$$

(3) V, V' projective, $f : V \to V'$ morphism.

$c' \in \mathrm{Pic}(V')$, $c := f^* c' \in \mathrm{Pic}(V)$

Then $h_c = h_{c'} \circ f + O(1)$.

Lemma 1: Let $\phi_1 : V \to \mathbb{P}^{n_1}$, $\phi_2 : V \to \mathbb{P}^{n_2}$ be morphisms.

Assume that $c_{\phi_1} = c_{\phi_2} \in \mathrm{Pic}(V)$.

Then $h_{\phi_1} = h_{\phi_2} + O(1)$.

Proof: Let $L_1 = \phi_1^* \mathcal{O}(1)$, $L_2 = \phi_2^* \mathcal{O}(1)$.

$\Gamma_i = H^0(V, L_i)$ the vector space of global sections of L_i.

Let $N = \dim \Gamma_1 - 1 = \dim \Gamma_2 - 1$

Then there exist an automorphism $\mathbb{P}^N \xrightarrow{\sim} \mathbb{P}^N$,

and linear projections $p_1 : \mathbb{P}^N \dashrightarrow \mathbb{P}^{n_1}$, $p_2 : \mathbb{P}^N \dashrightarrow \mathbb{P}^{n_2}$
such that the following diagram commutes:

$$\begin{tikzcd}
& \mathbb{P}^N \arrow[rr, "\sigma"] \arrow[d, dashed, "p_1"'] & & \mathbb{P}^N \arrow[d, dashed, "p_2"] & \\
& & V \arrow[ul, "|L_1|"] \arrow[ur, "|L_2|"'] \arrow[dl, "\phi_1"] \arrow[dr, "\phi_2"'] & & \\
\mathbb{R} & \mathbb{P}^{n_1} \arrow[l, "h_2"'] & & \mathbb{P}^{n_2} \arrow[r, "h_1"] & \mathbb{R}
\end{tikzcd}$$

The lemma follows from the functoriality for $\mathbb{P}^n$'s. $\square$

<u>Lemma 2</u>: $\forall c \in \mathrm{Pic}(V)$, $\exists$ c_1, c_2 very ample such that
$$c = c_1 - c_2 .$$

<u>proof</u>: Take very ample (V projective) c_0.
$\rightsquigarrow$ $\phi_{c_0} : V \hookrightarrow \mathbb{P}^n$ immersion.
$\forall m \gg 1$, $c_1 = c + mc_0$ is globally generated
$\longrightarrow \phi_{c_1} : V \to \mathbb{P}^m$ morphism.
Consider the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+m+n = N}$
Then $V \to \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ is an immersion
defined by $c_1 + c_0 = c + (m+1) c_0$
Then $c = \underbrace{c + (m+1)c_0}_{\text{very ample}} - \underbrace{(m+1)c_0}_{\text{very ample}}$ $\square$

Proof of Weil's height machine:

① Segre embedding $\Rightarrow c_1, c_2$ very ample. $h_{c_1+c_2} = h_{c_1} + h_{c_2} + O(1)$.

② By Lemma 2 $\forall c \in \mathrm{Pic}(V)$, we have

$$c = c_1 - c_2 \qquad c_i \text{ very ample}$$

let $h_c = h_{c_1} - h_{c_2}$.

if $c = c_1' - c_2'$ $\quad c_i'$ very ample.

Then $c_1' + c_2 = c_1 + c_2'$ both sides being (very) ample

The lemma 1 $\Rightarrow h_{c_1'} + h_{c_2} = h_{c_1} + h_{c_2'} + O(1)$

$\Rightarrow h_c - h_{c_2} = h_{c_1'} - h_{c_2'} + O(1)$. $\square$

§. Heights on abelian varieties & Mordell-Weil thm

Theorem (Néron–Tate)

A/k abelian variety, L ample & symmetric line bundle.

Then $\exists$ a unique $\hat{h}_L^{NT} : A(\bar{k}) \to \mathbb{R}$ "Néron-Tate" height

such that $\hat{h}_L^{NT}$ defines a positive definite quadratic form on

$A(k) \otimes_{\mathbb{Z}} \mathbb{R}$.

Theorem (weak M-W) $\forall m \in \mathbb{N}$ $\quad A(k)/mA(k)$ finite.

Cor (M-W) : $A(k)$ is finitely generated.

§. Metrics, Arakelov heights

k number field. $M_k = M_k^\infty \sqcup M_k^f$
archimedean, non-arch

V variety (not necessarily projective) $/k$, $L \to V$ line bundle.

<u>Definition</u>: $\forall v \in M_k$, a v-adic metric $\|\cdot\|_v$ on L, is a map which, at every $x \in V(k_v)$, associates a norm $\|\cdot\|_v(x)$ on x^*L such that $\forall U \subseteq V$ open, $\forall$ section s of L on U,

$$\begin{tikzcd} x^*L \arrow[r] \arrow[d] & L \arrow[d] \\ \mathrm{Spec}(k_v) \arrow[r, "x"] & V \end{tikzcd}$$

$$x \longmapsto \|s\|_v(x)$$

is continuous in v-adic topology.

— $\ast\ast\ast$ —

From now on assume V proper.

— $\ast\ast\ast$ —

<u>Definition</u>: An adelic metric $(\|\cdot\|_v)_{v \in M_k}$ on L is a family of metrics satisfying: there exists an integral model (flat & generic fibre $\simeq L \to V$) $\mathcal{L} \to \mathcal{V}$ over $\mathcal{O}_{k,S}$ ($M_k^\infty \subseteq S \subseteq M_k$ finite)

s.t. $\forall v \in M_k \setminus S$ $\forall x \in V(k_v) \rightsquigarrow \tilde{x} \in \mathcal{V}(\mathcal{O}_v)$, the norm $\|\cdot\|_v(x)$ on x^*L is defined in such a way that

$$\tilde{x}^*\mathcal{L} = \{ y \in x^*L : \|y\|_v(x) \le 1 \}.$$

("$(\|\cdot\|_v)$ is induced by an integral model for almost all v")

Definition: Given an adelic metrized line bundle $\overline{L} = \underbrace{(L, (\|\cdot\|_v)_{v\in M_k})}$, the **Arakelov height** $\hat{h}_{\overline{L}} : V(\bar{k}) \longrightarrow \mathbb{R}$ is defined by

$$\hat{h}_{\overline{L}}(x) = -\sum_{v\in M_k} \log \|s_x\|_v(x)$$

where s_x is a non-zero local section of L at $x \in V(\bar{k})$.
(Product formula $\Rightarrow$ well-defined)

Lemma. $\forall v \in M_k$, $\forall \|\cdot\|_v, \|\cdot\|_v'$ v-adic metrics on L, $\|\cdot\|_v$ is equivalent to $\|\cdot\|_v'$ (i.e. $\exists\, 0<c<1$ such that $c\|\cdot\|_v^{(x)} \leq \|\cdot\|_v'^{(x)} \leq c^{-1}\|\cdot\|_v^{(x)}$ on x^*L, $\forall x \in V(\mathbb{C}_v)$.)

Proof: $\forall x \in V(\mathbb{C}_v)$, we fix s_x a non-zero local section of L.
Then the map $x \longmapsto \dfrac{\|s_x\|_v(x)}{\|s_x\|_v'(x)}$ — well-defined, continuous.
V proper $\Rightarrow V(\mathbb{C}_v)$ compact $\Rightarrow$ map is bounded $\square$

Proposition: For any adelic metric $(\|\cdot\|_v)$ on $\mathcal{O}(1) \to \mathbb{P}^n$, we have

$$\hat{h}_{\mathcal{O}(1)}(P) = h_{\mathcal{O}(1)}(P) + O(1).$$

Proof: $h_{\mathcal{O}(1)} \Leftrightarrow$ "naive metric" on $\mathcal{O}(1)$.

Direct computation & Lemma on equivalence of metrics. $\square$

§ Distances, logarithmic singularities

V projective. $\forall X \subseteq V$ Zariski closed, $U := V \setminus X$.

<u>Definition</u>: $v \in M_k$. A v-adic <u>logarithmic distance</u> is a function

$$d_{X,v} : U(\bar{k}_v) \longrightarrow [0, +\infty)$$

Satisfying: if X is defined by local equations $f_1 = \dots = f_r = 0$ then the function $\left| d_{X,v} - \log^+\left(\min_{1 \leq j \leq r} |f_j(\cdot)|_v^{-1}\right)\right|$ extends to a bounded function on any open subset of V on which (f_j) are all regular.

$\left(\log^+(\cdot) := \max(\log(\cdot), 0) \right)$

<u>Remarks</u> 1) $d_{X,v} \approx \log\left(\dfrac{1}{v\text{-adic distance to } X}\right)$

2) $\forall Y \subseteq X$ closed, $d_X \geq d_Y + O(1)$

<u>Proposition</u>: Let $L \to V$ be ample. Then $\exists c > 0$ such that
$$h_L(P) \geq c\, d_{X,v}(P) + O(1) \qquad \forall P \in U(k).$$

<u>Proof</u>: ① $\forall L'$ very ample, take $m \in \mathbb{N}$ large so that $L'' := mL - L'$ is globally generated

$\Rightarrow h_{L''} \geq O(1) \Rightarrow m h_L - h_{L'} \geq O(1)$

It suffices to deal with any fixed very ample L'.

② Take D divisor containing X such that $L' := \mathcal{O}_V(D)$ is very ample. By Remark (2), we may assume that $V = \mathbb{P}^N$, $L = \mathcal{O}(1)$, $X = (x_0 = 0)$. $(U = (x_0 \neq 0))$.

③ We let for $P = [x_0 : \cdots : x_n] \in U(k)$

$$d_{X,v}(p) = \log^+\left(\min_i \left\|\frac{x_0}{x_i}\right\|_v^{-1}\right) = \max_i \log\left\|\frac{x_i}{x_0}\right\|_v$$

Then $h_{\mathcal{O}(1)}(p) = \sum_{w \in M_k} \max_i \left(0, \log\left\|\frac{x_i}{x_0}\right\|_w\right)$

$$\geq \max_i \log\left\|\frac{x_i}{x_0}\right\|_v . \qquad \square$$

<u>Remark</u>: In the reduction step ②, $d_{X,v}$ = "local height $h_{X,v}$"
$\left(h_X = \sum_{v \in M_k} h_{X,v}\right)$

<u>Definition</u>: $v \in M_k$, • Let $\|\cdot\|_v'$ be a v-adic metric on $L_{|U} \to U$

We say that $\|\cdot\|_v'$ <u>has logarithmic singularities along X</u> if there exists a v-adic metric $\|\cdot\|_v$ on $L \to V$, a logarithmic distance function $d_{X,v}$ on $U(k_v)$ and constants $c_1, c_2 > 0$ such that

$$\max\left\{\frac{\|\cdot\|_v}{\|\cdot\|_v'}, \frac{\|\cdot\|_v'}{\|\cdot\|_v}\right\} \leq c_1 \left(d_{X,v}(\cdot) + 1\right)^{c_2} \text{ on } U(k_v)$$

• A metrized line bundle $(L_{|U}, (\|\cdot\|_v')_{v \in M_k})$ has log singularities <u>along X</u> if $\|\cdot\|_v'$ does for all $v \in M_k^0$

<u>Remark</u>: log sing w.r.t. a single $\|\cdot\|_v$ on $L \to V$ & single $d_{X,v}$

$\Leftrightarrow$ log sing w.r.t. every $\|\cdot\|_v$ on $L \to V$ & every $d_{X,v}$.

<u>Theorem</u> (Faltings, "Northcott with log-sing's")

Assume $L \to V$ ample. Let $(\|\cdot\|_v')$ be an adelic metric on $L|U$.

- Induced by integal models over $\mathcal{O}_k$: $\begin{matrix} \mathcal{L}|_{\mathcal{U}} \subseteq \mathcal{L} \\ \downarrow \quad\; \downarrow \\ \mathcal{U} \subseteq \mathcal{V} \end{matrix}$

(So we define <u>the Arakelov height</u> $\hat{h}_{\overline{L|U}} : U(k) \to \mathbb{R}$ as before)

- with logarithmic singularities along X.

Then $\forall B > 0 \quad \#\{P \in U(k) : h_{\overline{L|U}}(P) < B\} < +\infty$

<u>Proof</u>: <u>Goal</u>: compare $h_{\overline{L}}(P)$ and $h_{\overline{L|U}}(P)$ for $P \in U(k)$

Let $(\|\cdot\|_v)$ be an adelic metric on L induced by $\mathcal{L} \to \mathcal{V}$

Then $\forall v \in M_k^f$, $\|\cdot\|_v' = \|\cdot\|_v|_U$.

Fix a log-distance $d_{X,v}$ on $U(k_v)$ for all $v \in M_k^\infty$.

$\forall P \in U(k)$, take non-zero local section s_P of L. Then

$$\left| h_{\overline{L}}(P) - h_{\overline{L|U}}(P) \right| \leq \sum_{v \in M_k^\infty} \max\left(\log \frac{\|s_P\|_v(P)}{\|s_P\|_v'(P)}, \log \frac{\|s_P\|_v'(P)}{\|s_P\|_v(P)} \right)$$

$$\leq \sum_{v \in M_k^\infty} \log c_{1,v} (d_{X,v}(P) + 1)^{c_{2,v}}$$

$$\leq C_1 \log(h_{\overline{L}}(P) + 1) + C_2$$

$\Rightarrow h_{\overline{L}} \leq B_1 h_{\overline{L|U}} + B_2$

Northcott for $\overline{L}$ $\Rightarrow$ Northcott for $\overline{L|U}$. $\square$