Seminar on Faltings' Proof of the Mordell Conjecture

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Contents

This is a plan of a seminar run by Ruochuan Liu and Xinyi Yuan in BICMR in Fall 2018. The basic references are:

- Faltings' original paper, whose English translation is in [CS, Chap. II].
- The book [CS], which is a collection of independent papers surveying on theories and results related to Faltings' proof.
- The online note [Mil]. Especially the last chapter has a sketch of Faltings' proof.
- The online note [BS]. This is an outline of a seminar in Michigan. We will roughly follow this note, but with many modifications.
- The online notes in [Sta]. This is a collection of notes for a seminar at Stanford.

1 Lecture 1: Summary of the proof

In this section, we give an overview of the proof. The goal is to state all important theorems involved in the proof, and mention the precise implication relations among the theorems. We will give ideas and references of all implications. This section can serve as a guide to read Faltings' proof, though it is highly not self-contained.

For simplicity, by a *curve* over a field K, we mean a projective, smooth and geometrically integral scheme X over K of dimension 1. Denote by $g(X)$ the genus of X.

Shafarevich conjectures

The goal is to prove:

Theorem A (Mordell conjecture). *Let K be a number field, and C be a curve of genus $g(C) > 1$ over K. Then $C(K)$ is finite.*

By Parshin's construction (before Faltings), the conjecture is to reduced to the following conjecture. The process is amazing as it reduces "counting points" to "counting curves".

Theorem B (Shafarevich conjecture: curves)**.** *Fix an integer $g > 1$, a number field K, and a finite set S of finite places of K. Then up to isomorphism, there are only finitely many curves over K of genus g and with good reduction outside S.*

IDEA (Thm B $\Rightarrow$ Thm A) The details are given in [CS, §VII.9], [Mil, §III.9], or [Mil, §IV.5]. The key is Parshin's construction, which constructs from each $P \in C(K)$ a finite morphism $\pi_P : C_P \to C_{K'}$ over K', where:

(1) K' is a finite extension of K;

(2) C_P is a curve over K' with good reduction outside a finite set S' of finite places of K';

(3) π_P is ramified exactly over P;

(4) $(K', S', g(C_P))$ is *independent* of the choice of P in $C(K)$.

By Theorem B, there are only finitely many such C_P. By a classical result of de Franchis, for each C_P, there are only finitely many finite K'-morphisms $C_P \to C$. Thus there finitely many morphisms $\pi_P : C_P \to C_{K'}$. As π_P determines P by considering ramification, there are only finitely many $P \in C(K)$. □

Remark 1.1. The theorem of de Franchis requires $g(C) > 1$, which is essentially the only part of the proof using the assumption $g(C) > 1$. The field K in Theorem B is actually a finite extension of the field K in Theorem A in the application.

By taking Jacobian varieties, one can further reduce "counting curves" to "counting abelian varieties".

Theorem C (Shafarevich conjecture: abelian varieties)**.** *Fix an integer $g > 0$, a number field K, and a finite set S of finite places of K. Then up to isomorphism, there are only finitely abelian varieties over K of dimension g and with good reduction outside S.*

IDEA (Thm C $\Rightarrow$ Thm B) The details are given in [CS, §VII.12-13] or [Mil, §III.12-13]. By Torelli's theorem, the Jacobian map

$$\{\text{curves of genus} > 1 \text{ over } K\} \longrightarrow \{\text{principally polarized abelian varieties over } K\}$$

is injective. The Jacobian map also keeps places of good reductions. Therefore, it remains to check that for any abelian variety A over any field, there are only finitely many polarizations $A \to A^\vee$ of a fixed degree. See [Mil, Theorem I.15.1]. □

Finiteness Theorems of abelian varieties

It is reduced to prove Theorem C, which is naturally equivalent to the following two theorems.

Theorem D (finiteness I)**.** *Fix a number field K, and an abelian variety A over K. Then up to isomorphism, there are only finitely abelian varieties over K which are isogenous to A.*

Theorem E (finiteness II)**.** *Fix an integer $g > 0$, a number field K, and a finite set S of finite places of K. Then up to isogeny, there are only finitely abelian varieties over K of dimension g and with good reduction outside S.*

For the statement to be reasonable, we need the fact that if A and B are isogenous abelian varieties over K, and v is a finite place of K, then A has good reduction at v if and only if B has good reduction at v. In fact, if A has good reduction, let $\mathcal{A}$ be the projective and smooth model of A over O_{K_v}. Let $A \to B$ be an isogeny, and denote the kernel by G. Denote by $\mathcal{G}$ the Zariski closure of G in $\mathcal{A}$. Then $B = A/G$, and $\mathcal{B} = \mathcal{A}/\mathcal{G}$ is a projective and smooth model of B over O_{K_v}.

PROOF (Thm D+E $\Longleftrightarrow$ Thm C) Trivial. □

In Faltings' proof, he first proves a weaker version of Theorem D, and eventually proves Theorem D and Theorem E by many other pieces of arguments. The weaker version is as follows.

Theorem D' (weak finiteness I)**.** *Fix a number field K, and an abelian variety A over K with everywhere semistable reduction. Let G be an ℓ-divisible subgroup of $A[\ell^\infty]$ over K for a prime ℓ. Denote $G_n = G[\ell^n]$. Let $A_n = A/G_n$ be the quotient abelian variety over K. Then the sequence $\{A_1, A_2, \cdots\}$ has only finitely many isomorphism classes of abelian varieties over K.*

Faltings height

The biggest innovation of Faltings' proof may be the introduction of the Faltings height. For any abelian variety A over a number field K, let $\mathcal{A}$ be the Neron model of A over O_K. The *Hodge bundle* of A is defined to be

$$\omega_{\mathcal{A}} = \epsilon^* \Omega^g_{\mathcal{A}/O_F} = \pi_* \Omega^g_{\mathcal{A}/O_F},$$

where $\pi : \mathcal{A} \to \mathrm{Spec}(O_F)$ denotes the structure morphism and $\epsilon : \mathrm{Spec}(O_F) \to \mathcal{A}$ denotes the identity section. Then $\omega_{\mathcal{A}}$ is a line bundle over $\mathrm{Spec}(O_F)$.

There is a *canonical hermitian metric* on $\omega_{\mathcal{A}}$, and thus one has the Arakelov degree $\widehat{\deg}(\omega_{\mathcal{A}}) \in \mathbb{R}$. Define the *Faltings height* of A to be

$$h(A) = \frac{1}{[K:\mathbb{Q}]} \widehat{\deg}(\omega_{\mathcal{A}}).$$

The most important property of the height is the following finiteness theorem.

Theorem H (Northcott property)**.** *Fix a number field K, an integer $g > 0$, and a real number C. Then up to isomorphism, there are only finitely many abelian varieties A over K of dimension g with $h(A) < C$.*

IDEA (Proof of Thm H) We can assume that A is principally polarized by Zarhin's trick ([Mil, Thm I.13.12]). Denote by S_g the moduli stack of principally polarized abelian varieties of dimension g over $\mathbb{Q}$. Putting a level structure, we can assume that S_g is a (quasi-projective) variety over $\mathbb{Q}$. Let S_g^* be the minimal compactification of S_g. The Hodge bundle ω of S_g extends to an ample line bundle ω^* on S_g^* by construction. The classical Northcott property is about the Weil height $h_{\omega^*} : S_g^*(\overline{\mathbb{Q}}) \to \mathbb{R}$ associated to ω^*. The goal is to convert this statement to the setting we need. Then it amounts to compare $h(A)$ and $h_{\omega^*}(x)$, where A is an abelian variety over $\overline{\mathbb{Q}}$, with a polarization and a level structure, and $x \in S_g(\overline{\mathbb{Q}})$ is the point associated to A. Two major difficulties are the logarithmic singularity of the metric involved and the compactification issue of S_g over $\mathbb{Z}$. These are best treated by the Faltings–Chai theory (cf. [FC, §V.4]). One can also use Gabber's lemma (cf. [BS, §4]). □

IDEA (Thm H ⇒ Thm D') It suffices to bound the Faltings height $h(A_n)$. The change $H(A, A_n) = \exp([K:\mathbb{Q}](h(A) - h(A_n)))$ of the Faltings height is an integer. It amounts to bound $\mathrm{ord}_\ell H(A, A_n)$ for every ℓ. The multiplicity $\mathrm{ord}_\ell H(A, A_n)$ can be expressed in terms of invariants of $\mathcal{G}_n$, the Zariski closure of G_n in the Neron model $\mathcal{A}$ of A over O_{K_v}, for primes v of K above ℓ. If $\mathcal{A}$ is proper, $\mathcal{G}_\infty = \{\mathcal{G}_n\}_n$ is an ℓ-divisible group. To get $\mathrm{ord}_\ell H(A, A_n) = 0$, one needs to compare the height of $\mathcal{G}_\infty$ with the dimension of $\mathcal{G}_\infty$. These two are related by the representation ρ of $\mathrm{Gal}(\bar{K}/K)$ on $T_\ell(\mathcal{G}_\infty)$. The height of $\mathcal{G}_\infty$ is given by the weight of ρ away from ℓ, and the dimension of $\mathcal{G}_\infty$ is given by the Hodge–Tate weight of ρ at v. See [BS, §7]. If $\mathcal{A}$ is not proper, the maximal proper subgroup of $\mathcal{G}_\infty = \{\mathcal{G}_n\}_n$ has a finite subgroup scheme such that the quotient is an ℓ-divisible group. □

Tate modules

By Tate's work (originally over finite fields), Theorem D' actually implies two fundamental results about Tate modules of abelian varieties. These two results also lie in the passage from Theorem D' to Theorem D+E.

Theorem F (semisimplicity)**.** *Let A be an abelian variety over a number field K, and ℓ be any prime. Then the $\mathrm{Gal}(\bar{K}/K)$-module $V_\ell(A)$ is semi-simple.*

Theorem G (Tate conjecture)**.** *Let A and B be abelian varieties over a number field K, and ℓ be any prime. Then the canonical map*

$$\mathrm{Hom}_K(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \longrightarrow \mathrm{Hom}_{\mathrm{Gal}(\bar{K}/K)}(T_\ell A, T_\ell B)$$

is an isomorphism.

IDEA (Thm D' ⇒ Thm F+G) These are in [Mil, §IV.2] or [BS, §2]. The proof of the two theorems are similar, as one needs to construct endomorphism from information on Tate modules in each theorem. The key is the following result:

If $W \subset V_\ell(A)$ is a sub-representation of $\mathrm{Gal}(\bar{K}/K)$, then there exists $f \in \mathrm{End}_K(A)_{\mathbb{Q}_\ell}$ such that $f(V_\ell(A)) = W$.

To prove this result, denote

$$M = W \cap T_\ell(A), \quad M_n = M + \ell^n T_\ell(A), \quad G_n = M_n/\ell^n T_\ell(A).$$

Note that $G_n \simeq M/\ell^n M$ and $G_n \subset T_\ell(A)/\ell^n T_\ell(A) = A(\bar{K})[\ell^n]$ is Galois invariant. Then $\{G_n\}_n$ forms a p-divisible subgroup of $A[\ell^\infty]$. Denote $B_n =$

A/G_n. There is a unique morphism $f_n : B_n \to A$ whose composition with the quotient map $A \to B_n$ is just $[\ell^n] : A \to A$. We further have $f_n(T_\ell(B_n)) = M_n$ in $T_\ell(A)$. By Theorem D', there is an abelian variety B over K, which is isomorphic to B_n for infinitely many n. Thus, for every such n, there is $g_n \in \mathrm{Hom}(B, A)$ such that $g_n(T_\ell(B)) = M_n$ in $T_\ell(A)$. By compactness, there is a limit point $g \in \mathrm{Hom}(B, A) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ of $\{g_n\}_n$, which satisfies $g(T_\ell(B)) = M$ in $T_\ell(A)$. This proves the theorem.

□

IDEA (Thm F+G ⇒ Thm E) This is in [Mil, §IV.3]. By Theorem F+G, the $\mathrm{Gal}(\bar{K}/K)$-module $V_\ell(A)$ determines the isogeny class of an abelian variety A over K. Then it suffices to prove that there are finitely many $\mathrm{Gal}(\bar{K}/K)$-modules $V_\ell(A)$ up to isomorphism. By Theorem F, the $\mathrm{Gal}(\bar{K}/K)$-module $V_\ell(A)$ is semi-simple, so its isomorphism class is determined by its character (trace) $\chi_A : \mathrm{Gal}(\bar{K}/K) \to \mathbb{Q}_\ell$. Note that A is of dimension g and good reduction outside S. By an argument using Chebotarev's density theorem, there is a finite set T of finite places of K, disjoint from S, such that the $\mathrm{Gal}(\bar{K}/K)$-module $V_\ell(A)$ is determined by the restriction of χ_A to $\{\mathrm{Frob}(v) : v \in T\}$. Then the results follows as the eigenvalues of $\mathrm{Frob}(v)$ are Weil numbers.

□

IDEA (Thm H+F+G ⇒ Thm D) This is in [Lip]. The proof is similar to that of Theorem D'. Let B be isogenous to A. We need to bound $h(B)$. The change $H(A, B) = \exp([K : \mathbb{Q}](h(A) - h(B)))$ of the Faltings height is still an integer, and we need to bound the multiplicity $\mathrm{ord}_\ell H(A, B)$ for every ℓ. For large ℓ, apply a Theorem of Raynaud on group schemes to prove that the multiplicity is just 0. For small ℓ, note that $T_\ell(B)$ is a $\mathrm{Gal}(\bar{K}/K)$-invariant lattice of $V_\ell(A)$. First, all $\mathrm{Gal}(\bar{K}/K)$-invariant lattices of $V_\ell(A)$ lie in finitely many isomorphism classes of $\mathrm{Gal}(\bar{K}/K)$-modules over $\mathbb{Z}_\ell$. Second, if $T_\ell(B)$ and $T_\ell(B')$ are isomorphic as $\mathrm{Gal}(\bar{K}/K)$-modules, by Theorem G, an isomorphism $T_\ell(B) \to T_\ell(B')$ can be approximated by an isogeny $B \to B'$ with degree prime to ℓ. Then $\mathrm{ord}_\ell H(B, B') = 0$.

□

Table of implications

The theorems involved are Theorems A,B, $\cdots$, H, D'. The implications mentioned above are:

- Prove Thm H.
- Thm H $\Rightarrow$ Thm D' $\Rightarrow$ Thm F+G $\Rightarrow$ Thm E.
- Thm H+F+G $\Rightarrow$ Thm D.
- Thm D+E $\Rightarrow$ Thm C is trivial.
- Thm C $\Rightarrow$ Thm B $\Rightarrow$ Thm A.

In the following, by postponing the proof involving Thm H as much as possible, we introduce the proof by the following order:

- Thm D' $\Rightarrow$ Thm F+G $\Rightarrow$ Thm E.
- Prove Thm H.
- (Thm H $\Rightarrow$ Thm D') and (Thm H+F+G $\Rightarrow$ Thm D).
- Thm C $\Rightarrow$ Thm B $\Rightarrow$ Thm A.

2 Lecture 2: Thm D' $\Rightarrow$ Thm F+G

Follow [BS, §2] or [Mil, §IV.2]. This is Tate's proof, which treats the situation of finite fields. The proof does not use any special property of number fields. In fact, it proves that if a field K satisfies Thm D', then it satisfies Thm F and Thm G.

3 Lecture 3: Abelian varieties over finite fields

There are three goals for this talk:

(1) Prove that up to isomorphism, there are only finitely many abelian varieties of a fixed dimension over a fixed finite field. Follow [Mil, Cor I.13.13]. Prove Zarhin's trick ([Mil, Thm I.13.12]), and explain the other ingredients.

(2) Prove the Riemann hypothesis for abelian varieties over finite fields. Follow [Mil, Thm II.1.1].

(3) State the Honda–Tate theorem (cf. [Mil, Thm II.2.1]). Relate the injectivity of the map to Theorem F+G over finite fields.

4 Lecture 4: Thm F+G ⇒ Thm E

Follow [Mil, §IV.3].

5 Lecture 5: Basics of group schemes

Give a general introduction following [CS, §III.1-4].

6 Lecture 6: Basics of Neron models and semi-abelian schemes

Cover [BS, §3]. Add some of [CS, VIII] depending on time.

7 Lecture 7: Proof of Thm H

Follow [BS, §4]. May also modify the proof by [FC, §V.4].

8 Lecture 8: Tate–Raynaud Theorem

Cover [BS, §5].

9 Lecture 9: Basics of p-divisible groups

Cover [BS, §6]. Add some of [CS, §III.5-7] depending on time.

10 Lecture 10: Raynaud's extension theorem on group schemes

Cover [BS, §9].

11 Lecture 11: (Thm H ⇒ Thm D') & (Thm H+F+G ⇒ Thm D)

Follow [BS, §7] and [Lip].

12 Lecture 12: Thm C ⇒ Thm B ⇒ Thm A.

Follow [Mil, §IV.4-5, §III.9, §III.12]. Note that the [Mil, §IV.5] and [Mil, §III.9] give different ways to construct branched covers, so choose one of them.

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