SPRING 2025 TSINGHUA NUMBER THEORY LEARNING SEMINAR: CANONICAL INTEGRAL MODELS OF SHIMURA VARIETIES

Last updated March 9, 2025.

Goal: Understand the construction of canonical integral models for Shimura varieties of exceptional type given by [BST24].

Time: Thursdays 1:15-2:45 PM

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Overview

Recall that a Shimura datum is a pair (G, X) of a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class of homomorphisms $h\colon \mathbb{S} = \mathbb{C}^\times \to G(\mathbb{R})$ satisfying certain assumptions. Let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup. The double quotient

$$G(\mathbb{Q}) \setminus (X \times G(\mathbb{A}_f)/K) =: \mathrm{Sh}_K(G, X)(\mathbb{C})$$

has the structure of a complex algebraic variety, which we denote by $\mathrm{Sh}_K(G, X)$. The datum (G, X) has a naturally associated reflex field $E \subset \mathbb{C}$, which is a number field, and which acts a natural field of definition for $\mathrm{Sh}_K(G, X)$. Along these lines, there exists a canonical model, which we will also denote $\mathrm{Sh}_K(G, X)$, which is an algebraic variety defined over E whose $\mathbb{C}$ points are isomorphic to the double quotient. These models are canonically characterized by certain Galois compatibility conditions concerning "special points" on the Shimura variety, formulated in terms of the Artin reciprocity map in global class field theory.

The simplest examples of Shimura varieties correspond to $G = \mathrm{GL}_2$ and $X = \mathbb{H}^\pm$, the upper and lower half planes. Then the Shimura variety $\mathrm{Sh}_K(\mathrm{GL}_2, \mathbb{H}^\pm)$ is nothing more than the modular curve with the level structure given by the choice of compact open subgroup K. In higher dimensions, taking $G = \mathrm{GSp}_{2g}$ gives an algebraic variety which is the moduli space of g-dimensional (principally) polarized abelian varieties. One would like to study the $\mathbb{F}_q$ points of these Shimura varieties for various reasons. For instance, the Hasse–Weil zeta function for Sh_K is conjecturally a product of automorphic L-functions. In the case of the modular curve, the connection is given by the Eichler–Shimura congruence relation. However, in order to study the $\mathbb{F}_q$ points, one must first have a model of the Shimura variety over local ring completion $\mathcal{O}_{E,(v)}$ for some place v of E and then consider the mod p points.

Given a variety over a number field E, there are many ways to construct smooth varieties over $\mathcal{O}_{E,(v)}$ whose generic fibers agree. Thus, we need to impose some conditions on this integral model in order for it to be unique. One candidate condition turns out to be an extension property. The property roughly says that for a certain class of admissible $\mathcal{O}_{E,(v)}$-test schemes, any map of generic fibers extends uniquely to the whole scheme. It was first formulated by Milne [Mil92] and Moonen [Moo98]; see also [Kis10]. In [BST24], a different but related definition of this extension property is used.

Langlands conjectured that integral canonical models of Shimura varieties exist whenever the level structure K_p at p is chosen to be hyperspecial. These integral canonical models were first studied by Kottwitz ([Kot92]) in the setting of Shimura varieties of PEL-type. These types of Shimura varieties are moduli spaces for abelian varieties together with extra data about their endomorphism rings. By framing the Shimura variety as a moduli problem, he was able to show that the moduli functor was representable over an integral base and that the resulting scheme is the canonical integral model, when K_p is hyperspecial. Later, Kisin in [Kis10] was able to prove the existence of canonical integral models when (G, X) is of Hodge type (parametrizing abelian varieties along with some Hodge cycles) and of abelian type (ones whose data is closely related to Hodge type). All other pairs of Shimura datum are called exceptional type. For exceptional Shimura varieties, there is no moduli interpretation in terms of abelian varieties, making the construction of integral models much more difficult.

We will cover the main result of [BST24]: Over all finite places v sufficiently large, there exists an integral canonical model for $\mathrm{Sh}_K(G, X)$.

The semester will be divided into two halves. The first half will be introductory and will introduce Shimura varieties as well as cover their canonical models over their reflex fields. We presume familiarity with the language of schemes and algebraic groups. We will primarily follow Deligne ([Del71]) and Milne ([Mil05]). During the second half of the semester, we will cover [BST24, §2-5].

Talk schedule

(1) **Hermitian symmetric domains and locally symmetric varieties** Give the definition of Hodge structures as well as the definition for a family of Hodge structures over a variety. Discuss how variations of Hodge structures can be mapped into Grassmannians. State the relationship between hermitian symmetric domains and the moduli spaces of variation of Hodge structures. Then define arithmetic subgroups and state the theorem that arithmetic quotients of Hermitian symmetric domains are algebraic varieties. This should cover [Mil05, §1–§3].

(2) **Definition of Shimura Varieties** Cover [Del71, §1] and [Mil05, §5] up to "Structure of Shimura Varieties". Start with [Mil05, Def. 5.5] or [Del71, 1.5] defining the notion of Shimura datum. Give a review of the the notion of Cartan involution. Then discuss the examples [Mil05, Ex. 5.6] and [Del71, 1.6]. Give details in the latter example as this will be used frequently in the future. Prove [Mil05, Cor. 5.8 and Prop. 5.9] simultaneously, for which you need to also sketch a proof of [Mil05, Prop. 4.8]. (Do not spend too much time on the notion of connected Shimura datum.) Pay special attention to the relation between the conditions [Mil05, Thm. 1.21 (a)], [Mil05, Def. 4.4, SU1], and [Mil05, Def. 5.5, SV1]. Cover [Mil05, 5.11-5.14]. Then prove [Mil05, Thm. 5.16] by giving a detailed proof of [Del71, 1.15]. If you still have time, discuss [Del71, 1.10-1.12]

(3) **Structure of Shimura Varieties** Cover [Del71, §2] and the rest of [Mil05, §5]. State and prove [Mil05, Thm. 5.17] and equivalently [Del71, 2.7]. Then state, as an immediate consequence, that the Shimura variety is an algebraic variety over $\mathbb{C}$. After this, feel free to omit the parts "zero-dimensional Shimura varieties", "additional axioms", "arithmetic subgroups of tori" in [Mil05, §5]. State and give a complete proof of (at least) the first assertion in [Mil05, Thm. 5.28]. Explain Remarks 5.29, 5.30.

(4) **Definition of Canonical Models** First review the main theorem of complex multiplication following [Mil05, §11]. Make necessary preparations and state Theorem 11.2 without proof. Then follow [Mil05, §12] ([Del71, §3]). Do include Example 12.4 (b), Example 12.7. After giving Definition 12.8, explain that the definition is motivated by the main theorem of CM by briefly talking about the final subsubsection "CM-tori" in §12, but do not get too involved into technical details. After this, cover [Mil05, §13], and give a complete proof of Key Lemma 13.4, which is [Del71, 5.1]. (If you do not have time, you can just prove Key Lemma 13.4 and omit everything else in §13; we will cover §13 in a later talk.)

(5) **Siegel Modular Variety** Cover [Mil05, §6] covering in depth the example of the moduli space of abelian varieties $\mathcal{A}_g$. Give the proof that this represents the moduli functor of principally polarized abelian varieties.

(6) **Existence of Canonical Models of Hodge (and Abelian) Type** Cover [Del71, §4] and [Mil05, §14] which prove the existence of a canonical model for $G = \mathrm{GSp}$ and Sh_K is the moduli space of abelian varieties. If time you may mention how PEL-type, Hodge type, and abelian type are covered in [Del79].

(7) **Integral Canonical Models of Hodge (and Abelian) Type** Discuss [Kis10].

(8) **Extension Property of Integral Models** Cover [BST24, §2.1-2.3]. Give the definition of admissible test schemes and cover [BST24, Ex. 2.12] about why we must allow for finite type flat schemes as models instead of requiring smooth models. Cover [BST24, Cor. 2.5, 2.9] and their proofs on the crucial induction step on how to reduce to a codimension 2 subset.

(9) **Extension Property of Integral Models** Cover [BST24, §2.5-2.6] by proving [BST24, Prop. 2.16] about how the existence of integral models is equivalent to the extension of a certain local system.

(10) **Inverse Cartier Transform** Cover [BST24, §3.1-3.2.1] by giving the definition and properties of flat sheaves, Higgs sheaves, the nonabelian Hodge correspondence, and inverse Cartier transform of pullbacks of Higgs sheaves.

(11) **Inverse Cartier Transform on Nodal Curves** Cover the rest of [BST24, §3.2] culminating in the proof of Proposition 3.7.

(12) **Application to Fontaine–Laffaille Modules** Cover the rest of [BST24, §3]. Give the definition of Fontaine–Laffaille modules and then prove the main result of Proposition 3.9 which is a degree bound.

(13) **Reduction to Fontaine–Laffaille Modules** Cover [BST24, §4.1] by proving Theorem 4.1 and Theorem 4.3.

(14) **Alternate Proof** Cover [BST24, §4.2] and give another proof of Theorem 4.1 and 4.3 by proving Proposition 4.7.

(15) **Proof of Integral Models** Cover [BST24, §5.1] proving the existence of integral canonical models.

References

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