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THE STABILIZATION OF THE FROBENIUS–HECKE TRACES ON THE INTERSECTION COHOMOLOGY OF ORTHOGONAL SHIMURA VARIETIES

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THE STABILIZATION OF THE FROBENIUS–HECKE TRACES ON THE INTERSECTION COHOMOLOGY OF ORTHOGONAL SHIMURA VARIETIES

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Abstract. — We study Shimura varieties associated with special orthogonal groups over the field of rational numbers. We prove a version of Morel's formula for the Frobenius–Hecke traces on the intersection cohomology of the Baily–Borel compactification. Our main result is the stabilization of this formula. As an application, we compute the Hasse–Weil zeta function of the intersection cohomology in some special cases, using the recent work of Arthur and Taïbi on the endoscopic classification of automorphic representations of special orthogonal groups.

Résumé. — Nous étudions les variétés de Shimura associées à des groupes spéciaux orthogonaux sur le corps des nombres rationnels. Nous prouvons une version de la formule de Morel pour les traces de Frobenius–Hecke sur la cohomologie d'intersection de la compactification de Baily–Borel. Notre résultat principal est la stabilisation de cette formule. Comme application, nous calculons la fonction zêta de Hasse–Weil de la cohomologie d'intersection dans certains cas particuliers, en utilisant les travaux récents d'Arthur et Taïbi sur la classification endoscopique des représentations automorphes des groupes spéciaux orthogonaux.

其始也，皆收視反聽，耽思傍訊，精騖八極，心遊萬仞。其致也，情曈曨而彌鮮，物昭晰而互進。

陸機《文賦》

In the beginning,
All external vision and sound are suspended,
Perpetual thought itself gropes in time and space;
Then, the spirit at full gallop reaches the eight limits of the cosmos,
And the mind, self-buoyant, will ever soar to new insurmountable heights.
When the search succeeds,
Feeling, at first but a glimmer, will gradually gather into full luminosity,
Whence all objects thus lit up glow as if each the other's light reflects.(1)

Excerpt from *Essay on Literature*
by LU Ji (261–303 AD)

(1) Translated from Chinese by CHEN Shixiang.

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INTRODUCTION

Inspired by the early works of Eichler, Shimura, Kuga, Sato, and Ihara, the ongoing study of expressing Hasse–Weil zeta functions of Shimura varieties through automorphic L-functions remains a focal point within the Langlands program. Langlands approached this problem by proposing a comparison of the Frobenius–Hecke traces on the cohomology of Shimura varieties with the stable Arthur–Selberg trace formulas, as detailed in [**Lan77, Lan79a, Lan79b**]. Kottwitz further formalized these ideas into precise conjectures [**Kot90, Kot92b**]. In this paper, we confirm a version of Kottwitz's conjecture specifically for the intersection cohomology of orthogonal Shimura varieties.

The conjectures

Let $(G, \mathcal{X})$ be a Shimura datum with reflex field E. For each sufficiently small compact open subgroup $K \subset G(\mathbb{A}_f)$, we have the Shimura variety

$$\mathrm{Sh}_K = \mathrm{Sh}_K(G, \mathcal{X}),$$

which is a smooth quasi-projective algebraic variety over E. Let $\overline{\mathrm{Sh}_K}$ be the Baily–Borel compactification of Sh_K. Let $\mathbf{IH}^*$ be the intersection cohomology of $\overline{\mathrm{Sh}_K} \otimes_E \overline{E}$ with $\overline{\mathbb{Q}}_\ell$-coefficients. (More generally, a non-trivial "automorphic" coefficient system is allowed, which we ignore in the introduction.) Let p be a hyperspecial prime for K, i.e., $K = K_p K^p$ with $K_p \subset G(\mathbb{Q}_p)$ a hyperspecial subgroup and $K^p \subset G(\mathbb{A}_f^p)$ a compact open subgroup. (Here $\mathbb{A}_f^p$ denotes the finite adeles away from p.) Assume that $p \neq \ell$. On $\mathbf{IH}^*$, we have commuting actions of $\mathrm{Gal}(\overline{E}/E)$ and the Hecke algebra $\mathcal{H}(G(\mathbb{A}_f^p) \mathbin{/\!\!/} K^p)_{\overline{\mathbb{Q}}_\ell}$ consisting of the $\overline{\mathbb{Q}}_\ell$-valued smooth compactly supported K^p-bi-invariant distributions on $G(\mathbb{A}_f^p)$. Fix $f^{p,\infty} \in \mathcal{H}(G(\mathbb{A}_f^p) \mathbin{/\!\!/} K^p)_{\overline{\mathbb{Q}}_\ell}$, and let $\Phi = \Phi_\mathfrak{p}$ be a geometric Frobenius at a place $\mathfrak{p}$ of E above p. Let $a \in \mathbb{Z}_{\geq 1}$.

Conjecture 1 **(Kottwitz, see [Kot90, §10]).** — *The action of* $\mathrm{Gal}(\overline{E}/E)$ *on* $\mathbf{IH}^*$ *is unramified at* $\mathfrak{p}$*, and under simplifying assumptions of a group-theoretic nature, we have*

$$\sum_k (-1)^k \operatorname{Tr}(f^{p,\infty} \times \Phi^a \mid \mathbf{IH}^k) = \sum_H \iota(G,H) ST^H(f^H). \tag{0.1}$$

On the right, H runs through the isomorphism classes of elliptic endoscopic data of G. For each H, $ST^H(\cdot)$ is the geometric side of the stable trace formula for H, and f^H is a function on $H(\mathbb{A})$ determined by the Shimura datum, $f^{p,\infty}$, and a.

In addition, Kottwitz also formulated the following conjecture for the compact support cohomology $\mathbf{H}_c^*$ of $\mathrm{Sh}_K \otimes_E \overline{E}$.

Conjecture 2 **(Kottwitz, see [Kot90, §7]).** — *The action of* $\mathrm{Gal}(\overline{E}/E)$ *on* $\mathbf{H}_c^*$ *is unramified at* $\mathfrak{p}$*, and under simplifying assumptions we have*

$$\sum_k (-1)^k \operatorname{Tr}(f^{p,\infty} \times \Phi^a \mid \mathbf{H}_c^k) = \sum_H \iota(G,H) ST_e^H(f^H). \tag{0.2}$$

Here H and f^H are the same as in Conjecture 1, while $ST_e^H(\cdot)$ is the elliptic part of the geometric side of the stable trace formula for H.

The main result

Let (V,q) be a quadratic space over $\mathbb{Q}$ of signature $(n,2)$, where $n \geq 3$. We assume that V has a 2-dimensional totally isotropic subspace, which is automatic if $n \geq 5$. Let $G = \mathrm{SO}(V,q)$. We have a natural Shimura datum $(G,\mathcal{X})$, where $\mathcal{X}$ can be identified with the set of oriented negative definite planes in $V_{\mathbb{R}}$. This Shimura datum is of abelian type (but not of Hodge type). The associated Shimura varieties are called *orthogonal Shimura varieties*. They are n-dimensional varieties over the reflex field $\mathbb{Q}$.

Theorem 1 **(Corollary 8.17.5).** — *Conjecture 1 is true for the orthogonal Shimura varieties associate to (V,q), for almost all primes p and for all sufficiently large a.*

We refer the reader to the statements of Theorem 1.8.4 and Corollary 8.17.5 for the precise meaning of "almost all primes p". Here we just mention that the set of primes to be excluded should depend on a fixed element f^∞ of the "full" Hecke algebra $\mathcal{H}(G(\mathbb{A}_f) /\!\!/ K)_{\overline{\mathbb{Q}}_\ell}$, whereas $f^{p,\infty}$ in (0.1) should be the component of f^∞ away from p, after p has been chosen.

Some remarks

From a group-theoretic point of view, both sides of (0.2) are less complicated compared to (0.1). In fact, the RHS of (0.2) has an elementary definition in terms of stable orbital integrals. For the LHS of (0.2), Kottwitz computed it for PEL Shimura varieties of type A or C in [**Kot92b**] by counting (virtual) abelian varieties with additional structures over finite fields and using the Grothendieck–Lefschetz–Verdier trace formula. He obtained:

$$(0.3)\quad \sum_k (-1)^k \operatorname{Tr}(f^{p,\infty} \times \Phi^a \mid \mathbf{H}_c^k) = \sum_{(\gamma_0,\gamma,\delta)} c(\gamma_0,\gamma,\delta) O_\gamma(f^{p,\infty}) TO_\delta(\phi) \operatorname{Tr}(\gamma_0 \mid \mathbb{V}).$$

We do not explain the terms on the RHS in detail here, but only mention that they are group-theoretic in nature and include orbital integrals $O_\gamma(\cdot)$ and twisted orbital integrals $TO_\delta(\cdot)$. In [**Kot90**], Kottwitz conjectured that (0.3) should hold for general Shimura varieties (at least under some simplifying assumptions of a group-theoretic nature). In the same paper Kottwitz *stabilized* the RHS of (0.3), namely he found(2) the functions f^H such that the RHS of (0.3) is equal to the RHS of (0.2). In [**KSZ**], both the formula (0.3) and the stabilization step are generalized to arbitrary Shimura varieties of abelian type, and Conjecture 2 is proved for these varieties.

One should view Conjecture 1 as one step forward from Conjecture 2. From a spectral perspective, it is ST^H rather than ST^H_e that sees the "whole picture". More specifically, ST^H has a spectral expansion, from which one can eventually make a link to automorphic representations. By contrast it is unclear how ST^H_e can be directly related to spectral information in general.

We also mention that the expectation that the intersection cohomology is the correct cohomology to insert in (0.1) is motivated by Zucker's conjecture and Arthur's work on L^2-cohomology, among other things. We refer the reader to [**Mor10a**] for a more detailed discussion on these motivations.

Application: the Hasse–Weil zeta functions

In [**Kot90**], Kottwitz showed that one can combine Conjecture 1 with the conjectural framework of Arthur parameters and Arthur's multiplicity conjectures to infer a description of the Galois–Hecke module $\mathbf{IH}^*$, and in particular a formula for the Hasse–Weil zeta function associated to $\mathbf{IH}^*$.

Currently some of these premises related to Arthur's conjectures have been established in special cases. Most notably, Arthur [**Art13**] has established the multiplicity conjectures for quasi-split classical groups.(3) In fact, our interest in delving into special orthogonal groups within this paper is driven by a desire to connect with Arthur's work. This intentional decision distinguishes our focus from similar groups such as

(2)The construction of f^H relies on the Langlands–Shelstad Transfer Conjecture and the Fundamental Lemma, which were unproven at the time of [**Kot90**]. They are now theorems thanks to the work of numerous mathematicians, most notably Ngô and Waldspurger.

(3)The results in [**Art13**] are contingent on the release of several upcoming papers, including the reference [A25], which have not appeared as of the time of writing.

GSpin, whose Shimura varieties display a relative simplicity in various aspects, for instance, being of Hodge type.

Unfortunately, when the rank is large the special orthogonal groups that have Shimura varieties cannot be quasi-split even over $\mathbb{R}$, because of the signature $(n,2)$ condition. Arthur's work has been generalized to limited cases of inner forms by Taïbi [**Taï19**] (building on earlier work of Kaletha [**Kal18, Kal16**] and Arancibia–Moeglin–Renard [**AMR18**], among others). We combine Theorem 1 with Arthur's and Taïbi's work to obtain the following theorem. Here we state it only for odd n for simplicity.

***Theorem 2* (Theorem 9.7.5, Remark 9.7.6).** — *Assume that n is odd, and that $G = \mathrm{SO}(V,q)$ is quasi-split at all finite places. For any finite set S of prime numbers, let $\zeta^S(\mathbf{IH}^*, s)$ be the S-partial Hasse–Weil zeta function associated to $\mathbf{IH}^*$. When S is sufficiently large, we have*

$$\log \zeta^S(\mathbf{IH}^*, s) \\ = \sum_{\psi} \sum_{\pi^\infty} \sum_{\nu} \dim(\pi^\infty)^K m(\pi^\infty, \psi, \nu)(-1)^n \nu(s_\psi) \log L^S(\mathcal{M}(\psi,\nu), s).$$

Here ψ runs through a certain set of Arthur's substitutes of global Arthur parameters, π^∞ runs through the away-from-∞ global packet of ψ, and ν runs through characters of the centralizer group of ψ (which is finite abelian). The three-fold summation is over a finite range. The numbers $m(\pi^\infty, \psi, \nu) \in \{0,1\}$ and $\nu(s_\psi) \in \{\pm 1\}$ are defined in terms of constructions in [**Art13**] *and* [**Taï19**]*. The term $L^S(\mathcal{M}(\psi,\nu), s)$ is a finite product of S-partial standard automorphic L-functions for general linear groups (with some shifting in the variable s), and hence has meromorphic continuation to $\mathbb{C}$. In particular, the above formula implies that $\zeta^S(\mathbf{IH}^*, s)$ has meromorphic continuation to $\mathbb{C}$.*

In the proof of Theorem 2, one crucial ingredient is a relatively simple formula for $ST^H(f^H)$ when the test function f^H is *stable cuspidal* at the real place; see Hypothesis 9.1.2. This formula follows from Kottwitz's stabilization of the L^2 Lefschetz number formula in his unpublished notes, and is also used in Morel's work [**Mor10b, Mor11**]. A self-contained proof of this formula for $ST^H(f^H)$, from a different point of view, is given in a recent paper by Z. Peng [**Pen19**].

We also prove a refinement of Theorem 2 concerning the decomposition of $\mathbf{IH}^*$ in the Grothendieck group of Galois–Hecke modules, under the same assumption on G. When n is odd (as well as in some cases when n is even), we express $\mathbf{IH}^*$ in terms of the known Galois representations associated to regular algebraic cuspidal automorphic representations of general linear groups, with multiplicities given in a similar way as the multiplicities in Theorem 2. See Theorem 9.8.5, Corollary 9.8.8, and Corollary 9.8.10. When n is even, both the computation of the partial Hasse–Weil zeta function and the decomposition of $\mathbf{IH}^*$ proved in this paper are weaker than the conjectures in [**Kot90**], in that a certain ambiguity up to outer automorphism is constantly present. This is due to the extra ambiguity in the endoscopic classification of representations for even special orthogonal groups in [**Art13**] and [**Taï19**], which seems intrinsic to the methods therein.

As a byproduct of our refinement of Theorem 2, we prove that if an Arthur parameter ψ contributes to $\mathbf{IH}^*$, then the cuspidal automorphic representations of general linear groups that constitute ψ all satisfy the Ramanujan–Petersson conjecture at almost all primes. These representations need not be regular algebraic, in which case the conjecture was previously known. See Theorem 9.8.5 (3) and Remark 9.8.6.

Reduction to the stabilization of the boundary terms

We now discuss the structure of the proof of Theorem 1. For some period of time, the study of the LHS of (0.1) had been restricted to sporadic low dimensional cases; see for instance [**LR92**]. The essential tools for treating arbitrary dimensions were developed by Morel [**Mor06, Mor08**] (cf. [**Mor10a**]), who went on to prove Conjecture 1 for some unitary similitude Shimura varieties and the Siegel modular varieties of arbitrary dimensions in [**Mor10b**] and [**Mor11**] respectively. We use Morel's work to obtain the following result for the orthogonal Shimura varieties associated to (V, q). We fix a minimal parabolic subgroup of $G = \mathrm{SO}(V, q)$ and fix a Levi component of it. Thus we get a notion of standard parabolic subgroups and standard Levi subgroups of G.

Theorem 3 **(Theorem 1.8.4).** — *For almost all primes p, we have*

$$\sum_k (-1)^k \operatorname{Tr}(f^{p,\infty} \times \Phi^j \mid \mathbf{IH}^k) = \sum_M \operatorname{Tr}_M, \tag{0.4}$$

where M runs through the standard Levi subgroups of G.

Let us roughly describe the terms Tr_M. For $M = G$, we have

$$\operatorname{Tr}_G = \sum_k (-1)^k \operatorname{Tr}(f^{p,\infty} \times \Phi^j \mid \mathbf{H}_c^k),$$

where $\mathbf{H}_c^k$ is the compact support cohomology of $\mathrm{Sh}_{K,\overline{\mathbb{Q}}}$. For a proper M, the term Tr_M is a more complicated mixture of the following ingredients.

– The analogue of $\sum_k (-1)^k \operatorname{Tr}(f^{p,\infty} \times \Phi^j \mid \mathbf{H}_c^k)$ for a boundary stratum in $\overline{\mathrm{Sh}_K}$. In another words, an enumeration of points on the stratum fixed under certain Frobenius–Hecke operators.

– The topological fixed point formula of Goresky–Kottwitz–MacPherson as in [**GKM97**], for the trace of a Hecke operator on the compact support cohomology of a certain locally symmetric space.

– *Kostant–Weyl terms.* By this we mean characters for certain algebraic sub-representations of M_P inside

$$\mathbf{H}^*(\operatorname{Lie} N_P, \mathbb{V}),$$

where P is a standard parabolic subgroup of G containing M, and $P = M_P N_P$ is the standard Levi decomposition. These sub-representations are defined by certain truncations of weights, and can be understood in terms Kostant's theorem [**Kos61**] describing $\mathbf{H}^*(\operatorname{Lie} N_P, \mathbb{V})$.

As we have already mentioned, in [**KSZ**] the term Tr_G is computed and stabilized for all Shimura varieties of abelian type. Thus Tr_G is known to be equal to the RHS of (0.2). In view of this, Theorem 1 follows from Theorem 3 and the following result, which may be viewed as the "stabilization of the boundary terms".

***Theorem 4* (Theorem 8.17.2).** — *We have*

$$\sum_{M \subsetneq G} \mathrm{Tr}_M = \sum_H \iota(G,H)[ST^H(f^H) - ST^H_e(f^H)]. \tag{0.5}$$

Stabilization of the boundary terms

The method for proving Theorem 4 is by calculating the two sides of (0.5) and matching the explicit expressions. To calculate the RHS, we use Kottwitz's formula in his unpublished notes, as mentioned below Theorem 2. According to this formula (to be recalled in §8.3), we have an expansion of the form

$$ST^H(f^H) - ST^H_e(f^H) = \sum_{M' \neq H} ST^H_{M'}(f^H),$$

where M' runs through standard proper Levi subgroups of H, and each term $ST^H_{M'}(\cdot)$ has a relatively simple expression.

Roughly speaking, we label the pairs (H, M') appearing in the above summation by either a standard proper Levi subgroup M of G or the symbol $\emptyset$. We write $(H, M') \sim M$, or $(H, M') \sim \emptyset$. In order to prove Theorem 4, we need to show

$$\mathrm{Tr}_M = \sum_{(H,M') \sim M} ST^H_{M'}(f^H), \tag{0.6}$$

where M is either a standard proper Levi subgroup of G or the symbol $\emptyset$, and we define $\mathrm{Tr}_\emptyset$ to be 0. The proof of (0.6) involves the following ingredients.

(i) Fixed point formula for a boundary stratum. — We need a formula that enumerates points on a boundary stratum fixed under a Frobenius–Hecke operator, of a form similar to (0.3). The boundary stratum in question is (a finite quotient of) either a modular curve or a zero-dimensional Shimura variety, so such a formula is essentially a classical result. However, the zero-dimensional case causes some extra complication. We will come back to this technical point later in the introduction.

(ii) Archimedean comparison. — We need a series of identities between the archimedean contributions to the two sides of (0.6). These are identities between terms of two different natures, namely discrete series character values (which appear on the RHS of (0.6)) and Kostant–Weyl terms (which appear on the LHS of (0.6); see the discussion below Theorem 3). We establish such identities by explicit computation. On the discrete series side, we use formulas due to Harish-Chandra [**HC65**] and Herb [**Her79**]. On the Kostant–Weyl side, we use Kostant's theorem [**Kos61**] and the Weyl character formula.

We point out that *a priori* it is not clear which identities between the archimedean contributions would eventually lead to the proof of (0.6). Finding the correct forms of the archimedean identities seems to be a harder task than proving them. It would be desirable to have a more conceptual understanding of how the archimedean comparison should be woven into the proof of (0.6) in general.

(iii) Computation at p. — We need to compute the p-adic contributions to the two sides of (0.6) explicitly. *A priori* there are more p-adic terms on the RHS than the LHS. We will need to prove, among other things, that the extra terms eventually cancel each other.

This finishes our discussion on the structure of the proof of Theorem 1. Next we highlight three new features in the proof which did not show up in Morel's work [**Mor11, Mor10b**] for symplectic similitude and unitary similitude groups.

Arithmetic feature: Shimura varieties of abelian type

The orthogonal Shimura varieties are of abelian type and not of PEL type. In this paper we take as a black box the main result of [**KSZ**] that proves Conjecture 2 for these Shimura varieties. In Morel's work, the Shimura varieties are of PEL type, and for them Conjecture 2 was already proved by Kottwitz.

The reason that Theorem 1 is proved only for primes outside an unspecified finite set is also due to a certain lack of understanding of Shimura varieties of abelian type. Ideally one would like to prove the theorem for all hyperspecial primes p, but a prerequisite for that would be a robust theory of integral models of the Baily–Borel and toroidal compactifications. Such a theory has been established by Madapusi Pera [**MP19**] in the case of Hodge type. For the Baily–Borel compactifications alone, a "crude" construction of the integral models in the case of abelian type has been given by Lan–Stroh [**LS18**]. However, for the above-mentioned purpose the integral models of toroidal compactifications are equally important, and this is currently unavailable beyond the case of Hodge type.

All the difficulty about integral models of compactifications can be circumvented at the cost of excluding an unspecified finite set of primes, and this is the point of view taken in this paper. We refer the reader to §3.1 for a more detailed discussion.

Geometric feature: zero-dimensional boundary strata as quotients of Shimura varieties

In general, the boundary strata of the Baily–Borel compactification are naturally isomorphic to finite quotients of Shimura varieties at certain natural levels. Often these quotients are isomorphic to genuine Shimura varieties. However this is not true for the zero-dimensional boundary strata in the present case. From a group-theoretic point of view, this issue corresponds to the fact that the orthogonal Shimura datum does not satisfy Morel's axioms in [**Mor10b**, Chap. 1]. As a result, in the proof of Theorem 3 we need to modify the axiomatic approach in *loc. cit.*, and the terms Tr_M in (0.4) are also given by formulas that are slightly different from those in [**Mor10b, Mor11**].

Endoscopic-theoretic feature: normalizing transfer factors

In the proof of (0.6), signs are utterly important. One source of signs is the difference between the normalizations of transfer factors at the real place. The necessity of computing these signs was not emphasized in [**Mor10b, Mor11**]. For the orthogonal Shimura varieties, these signs form a delicate pattern.

To understand these signs we need to compare the normalization $\Delta_{j,B}$ introduced in [**Kot90**, §7], and the Whittaker normalization. Here we explicitly fix $G_{\mathbb{R}}$ as a pure inner form of its quasi-split inner form $G^*_{\mathbb{R}}$ and fix a Whittaker datum for $G^*_{\mathbb{R}}$, so the Whittaker normalizations for the transfer factors between $G_{\mathbb{R}}$ and its endoscopic groups can be defined. The normalization $\Delta_{j,B}$ naturally shows up in the description of the archimedean component of f^H. To compare these two normalizations, we compare the corresponding spectral transfer factors that appear in the endoscopic character relations and compute the sign between them.

Extra complication arises when $G^*_{\mathbb{R}}$ has more than one equivalence class of Whittaker data. This happens if and only if $\dim V$ is divisible by 4, when there are precisely two equivalence classes. In this case, we need to study how the two (different) Whittaker normalizations relate to the explicit formulas of Waldspurger [**Wal10**], the latter having the merit of being easier to keep track of when passing to Levi subgroups. In this direction we prove Theorem 6.3.11, which may be of independent interest in representation theory.

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LEITFADEN

In §1, we recall the setting of orthogonal Shimura varieties and state Morel's formula in Theorem 1.8.4. The terms in this formula are defined in §2, and the proof is given in §3. For a more detailed introduction to the structure of the proof see §3.1.

In §4, we carry out the archimedean comparison between the Kostant–Weyl terms and the stable discrete series characters. The results proved in this chapter to be used later are Propositions 4.4.2, 4.5.2, 4.6.12, 4.6.13, and 4.6.14.

In §5, we review the endoscopic data for special orthogonal groups and give explicit presentations which are important for the later computations.

In §6, we compare different normalizations of archimedean transfer factors for special orthogonal groups. The goal is to explicitly determine certain signs.

In §7, we calculate some Satake transforms at p that will be needed later in the stabilization.

In §8, we prove the stabilization of the boundary terms by assembling the preceding ingredients and explicit manipulation. We deduce the main result Theorem 1 of this paper in Corollary 8.17.5.

In §9, we apply our main result to the actual computation of Hasse–Weil zeta functions in some special cases, after reviewing results of Arthur and Taïbi on the endoscopic classification of automorphic representations. The main results in this chapter are Theorems 9.6.4, 9.7.5, and 9.8.5.

CONVENTIONS AND NOTATIONS

– For $x \in \mathbb{R}$, we denote by $\lfloor x \rfloor$ the largest integer $\le x$ and denote by $\lceil x \rceil$ the smallest integer $\ge x$. If $x \ge 0$, we denote by $x^{1/2}$ the non-negative square root of x.

– We denote $i \in \mathbb{C}$ alternatively by $\sqrt{-1}$.

– For any $n \in \mathbb{Z}_{\ge 1}$, we denote by $[n]$ the set $\{1, 2, \cdots, n\}$. We denote by $\mathfrak{S}_n$ the symmetric group of the set $[n]$.

– Let A be a subset of $\mathbb{Z}_{\ge 1}$. For each $i \in \mathbb{Z}_{\ge 1}$, we set $\nabla_i(A) = 1$ if $i \in A$, and $\nabla_i(A) = -1$ if $i \notin A$.

– When the symbol $\pm$ appears for multiple times in a single expression, it is understood that all possible combinations of the signs are considered. For example, we shall write $\{\pm x \pm y\}$ for the set $\{x + y, x - y, -x + y, -x - y\}$.

– A *basis* of a finite-dimensional vector space is always understood as an ordered basis. We often just use the notation for a set such as $\{e_1, \cdots, e_d\}$ to denote a basis, but the ordering is understood.

– For $x_1, \cdots, x_n \in \mathbb{C}^\times$, we write $\mathrm{symdiag}(x_1, \cdots, x_n)$ for the $2n \times 2n$ diagonal matrix $\mathrm{diag}(x_1, \cdots, x_n, x_n^{-1}, \cdots, x_1^{-1})$.

– For any square matrix A, we write $A^\intercal$ for the transpose.

– If a group G acts on a set X, we write $\mathrm{Cent}_G X$ for the action kernel, namely the largest subgroup of G acting trivially on X.

– When x is an element of a group, we write $\mathrm{Int}(x)$ for the automorphism $y \mapsto xyx^{-1}$.

– If Σ is a finite set of prime numbers, we denote by $\mathbb{Z}[1/\Sigma]$ the ring $\mathbb{Z}[1/p, p \in \Sigma]$.

– For $a \in \mathbb{Z}_{\ge 1}$ and p a prime number, we denote by $\mathbb{Q}_{p^a}$ the degree a unramified extension of $\mathbb{Q}_p$, and by $\mathbb{Z}_{p^a}$ the valuation ring of $\mathbb{Q}_{p^a}$. We denote by σ the arithmetic p-Frobenius acting on $\mathbb{Q}_{p^a}$.

– If H is either a locally profinite group or a real Lie group, we write $C_c^\infty(H)$ for the set of compactly supported smooth $\mathbb{C}$-valued functions on H.

– We use the following abbreviations:

$$\Gamma_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \qquad \Gamma_p = \Gamma_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p), \qquad \Gamma_\infty = \Gamma_{\mathbb{R}} = \mathrm{Gal}(\mathbb{C}/\mathbb{R}).$$

More generally, if F is a field, we write Γ_F for the absolute Galois group of F.

– We say that a profinite Galois étale covering $Y \to X$ of schemes is a *G-torsor*, where G is a profinite group, if G is the limit $\varprojlim_{i\in I} G_i$ of finite groups G_i, and $Y \to X$ is the limit of finite Galois étale coverings $Y_i \to X$ that is a G_i-torsor.

– By a *reductive group*, we always mean a connected reductive group.

– For a reductive group G over $\mathbb{R}$ and a maximal torus T in G defined over $\mathbb{R}$, we write $\Omega_{\mathbb{C}}(G,T)$ for the complex Weyl group $\mathrm{Nor}_{G(\mathbb{C})}(T)/T(\mathbb{C})$, and write $\Omega_{\mathbb{R}}(G,T)$ for the real Weyl group $\mathrm{Nor}_{G(\mathbb{R})}(T)/T(\mathbb{R})$.

– For a reductive group G over $\mathbb{R}$, we denote by $G(\mathbb{R})^0$ the identity connected component of the real Lie group $G(\mathbb{R})$.

– In the structural theory of reductive groups, the words "pinning", "splitting", and "épinglage" are synonyms. We use the word "splitting".

– If P is a parabolic subgroup of a reductive group over a field, we write N_P for the unipotent radical of P. We reserve the notation U_P for a different purpose. We write M_P for P/N_P. When it is clear from the context, M_P also denotes a fixed Levi component of P.

– We freely use the language of abelianized Galois cohomology as developed in [**Bor98**] and [**Lab99**]. For an overview, cf. [**KSZ**, §1]. We also use Kottwitz's more classical formulation [**Kot86**] in terms of centers of Langlands dual groups.

– Let G be a reductive group over $\mathbb{Q}$. We denote by $\ker^1(\mathbb{Q},G)$ the kernel set

$$\ker(\mathbf{H}^1(\mathbb{Q},G) \to \mathbf{H}^1(\mathbb{A},G)).$$

It is well known that $\ker^1(\mathbb{Q},G)$ has the canonical structure of an abelian group; see for instance [**Bor98**].

– When normalizing transfer factors, we use the classical normalization of local class field theory as opposed to Deligne's normalization, cf. [**KS12**, §§4.1–4.2].

– Names of Dynkin types are denoted by sans serif letters, e.g., $\mathsf{A}_n, \mathsf{B}_n$, etc.

– We sometimes use the abbreviations "LHS" and "RHS" for "left hand side" and "right hand side".

CHAPTER 1

THE ORTHOGONAL SHIMURA VARIETIES

1.1. General definitions concerning reductive groups

We collect some definitions that will appear repeatedly in the paper.

Definition 1.1.1. — Let G be a reductive group over a field F. Let P be a parabolic subgroup of G, with unipotent radical N_P. Let M be a Levi component of P.

(1) We denote by A_M the *split component* of M, namely the maximal F-split torus in the center of M.

(2) Let $\mathrm{Nor}_G(M)$ be the normalizer of M in G. We denote by $\mathcal{W}_M^G$ the quotient group $\mathrm{Nor}_G(M)(F)/M(F)$, and denote by n_M^G the cardinality of $\mathcal{W}_M^G$.

(3) For any $\gamma \in M(F)$, we define

$$D_M^G(\gamma) := \det\big(1 - \mathrm{Ad}(\gamma) \mid \mathrm{Lie}\, G/\,\mathrm{Lie}\, M\big) \in F.$$

(4) Assume that $F = \mathbb{Q}_v$ for a place v of $\mathbb{Q}$. For any $\gamma \in P(\mathbb{Q}_v)$, we define

$$\delta_{P(\mathbb{Q}_v)}(\gamma) := \big|\det\big(\mathrm{Ad}(\gamma) \mid \mathrm{Lie}\, N_P \otimes \mathbb{Q}_v\big)\big|_v \in \mathbb{R}_{>0},$$

where $|\cdot|_v$ denotes the usual absolute value on $\mathbb{Q}_v$.

Remark 1.1.2. — In Definition 1.1.1 (2), we in fact have $\mathrm{Nor}_G(M)(F) = \mathrm{Nor}_G(A_M)(F)$, and $M(F) = \mathrm{Cent}_G(A_M)(F)$. Hence $\mathcal{W}_M^G$ is isomorphic to the image of $\mathrm{Nor}_G(A_M)(F)$ in $\mathrm{Aut}(A_M)$.

Definition 1.1.3. — Let G be a quasi-split reductive group over a field F. By a *Borel pair* in G, we mean a pair (T, B) consisting of a maximal torus T in G and a Borel subgroup B of G containing T. Given a Borel pair (T, B), we denote the sets of roots, coroots, positive roots, positive coroots by $\Phi(G,T)$, $\Phi(G,T)^\vee$, $\Phi(G,T)^+$, $\Phi(G,T)^{\vee,+}$ respectively. We write $\mathrm{BRD}(T, B)$ for the based root datum

$(X^*(T), \Phi(G,T), \Phi(G,T)^+, X_*(T), \Phi(G,T)^\vee, \Phi(G,T)^{\vee,+})$. We define the *Weyl denominator*

$$\Delta_B(\gamma) := \prod_{\alpha\in\Phi(G,T)^+} (1-\alpha^{-1}(\gamma)) \in \overline{F}, \qquad \forall \gamma \in T(\overline{F}).$$

Definition 1.1.4. — Let G be a reductive group over $\mathbb{R}$. We denote by X_G the symmetric space associated to G, namely $X_G = G(\mathbb{R})/KA_G(\mathbb{R})^0$, where K is a maximal compact subgroup of $G(\mathbb{R})$. Thus X_G is a smooth manifold. We let $q(G)$ be the half of the dimension of X_G.

Remark 1.1.5. — In Definition 1.1.4, K meets every connected component of $G(\mathbb{R})$ by Matsumoto's theorem (see [**BT65**, 14.4]). Hence X_G is connected.

Definition 1.1.6. — We call a reductive group G over $\mathbb{Q}$ *cuspidal* if $G_{\mathbb{R}}$ contains elliptic maximal tori and Z_G^0 has equal $\mathbb{Q}$-split and $\mathbb{R}$-split rank. Equivalently, $(G/A_G)_{\mathbb{R}}$ contains $\mathbb{R}$-anisotropic maximal tori, where A_G is the split component of G over $\mathbb{Q}$.

Remark 1.1.7. — In this paper, every reductive group over $\mathbb{Q}$ that appears will be a direct product of special orthogonal groups and general linear groups. Thus the only case where the center can have different $\mathbb{Q}$-split and $\mathbb{R}$-split ranks is when we have a direct factor SO_2 which is non-split over $\mathbb{Q}$ but split over $\mathbb{R}$.

Definition 1.1.8. — Let G be a reductive group over $\mathbb{Q}$. We say that an element $\gamma \in G(\mathbb{Q})$ is $\mathbb{R}$-*elliptic*, if there is an elliptic maximal torus T in $G_{\mathbb{R}}$ such that $\gamma \in T(\mathbb{R})$.

1.2. Generalities on quadratic spaces

1.2.1. — Let F be a field of characteristic zero, with a fixed algebraic closure $\overline{F}$. In this paper, all quadratic spaces over F are assumed to be finite-dimensional and non-degenerate. Let (V,q) be a quadratic space over F. We denote by $[\cdot,\cdot]_q : V\otimes V \to F$ the associated bilinear pairing, defined as $[x,y]_q = q(x,y) - q(x) - q(y)$. When no confusion can arise we simply write V for (V,q), and write $[\cdot,\cdot]$ for $[\cdot,\cdot]_q$. Recall that the *determinant* of q, denoted by $\det q$, is the image in $F^\times/F^{\times,2}$ of the determinant of the matrix of q under any basis of V. We define the *discriminant* of (V,q) to be

$$\delta := (-1)^{\lfloor \dim V/2 \rfloor} \det q \in F^\times/F^{\times,2}.$$

For $m \in \mathbb{Z}_{\geq 1}$, we write J_m for the $m\times m$ matrix

$$J_m = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Definition 1.2.2. — Let (V,q) be a quadratic space over F of dimension d and discriminant δ. Let $m = \lfloor d/2 \rfloor$. We define the following notions.

(1) A basis $\{e_1, \cdots, e_d\}$ of V is called *hyperbolic*, if the matrix $([e_i, e_j]_q)$ is of the form

$$\begin{pmatrix} & & J_m \\ & x & \\ J_m & & \end{pmatrix}$$

for some $x \in F^\times$ when d is odd, and is equal to

$$\begin{pmatrix} & J_m \\ J_m & \end{pmatrix}$$

when d is even. Note that when d is even, a hyperbolic basis exists only when δ is trivial.

(2) Assume that d is even, and that δ is non-trivial. In this case a basis $\{e_1, \cdots, e_d\}$ of V is called *near-hyperbolic* , if the matrix $([e_i, e_j]_q)$ is equal to

$$\begin{pmatrix} & & & J_{m-1} \\ & 1 & & \\ & & -x & \\ J_{m-1} & & & \end{pmatrix}$$

for some $x \in F^\times$. Note that x is a lift of $\delta \in F^\times/F^{\times,2}$. We say that x is the *discriminant* of $\{e_1, \cdots, e_d\}$.

Definition 1.2.3. — We call (V, q) *quasi-split*, if there exists a hyperbolic basis or a near-hyperbolic basis of V. If there exists a hyperbolic basis we also say that V is *split*; this is equivalent to requiring that V contains a totally isotropic subspace of dimension $\lfloor \dim V/2 \rfloor$.

Example 1.2.4. — Let $F = \mathbb{R}$. Then a quadratic space over $\mathbb{R}$ of signature (p, q) is quasi-split if and only $p - q \in \{1, -1, 2\}$. For any $p \in \mathbb{Z}_{\geq 1}$, the quadratic spaces of signature (p, p) and $(p + 1, p - 1)$ are both quasi-split, and their discriminants are 1 and $-1 \in \mathbb{R}^\times/\mathbb{R}^{\times,2}$ respectively.

1.2.5. — Let $m \in \mathbb{Z}_{\geq 1}$. We denote by $\mathrm{RD}(\mathsf{B}_m)$ the *standard type* B_m *root datum*, given by

$$(\mathbb{Z}^m = \mathrm{span}_{\mathbb{Z}} \{\epsilon_1, \cdots, \epsilon_m\}, R, \mathbb{Z}^m = \mathrm{span}_{\mathbb{Z}} \{\epsilon_1^\vee, \cdots, \epsilon_m^\vee\}, R^\vee),$$

where $\langle \epsilon_i, \epsilon_j^\vee \rangle = \delta_{i,j}$, and

$$R = \{\pm\epsilon_i \mid 1 \leq i \leq m\} \cup \{\pm\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq m\},$$
$$R^\vee = \{\pm 2\epsilon_i^\vee \mid 1 \leq i \leq m\} \cup \left\{\pm\epsilon_i^\vee \pm \epsilon_j^\vee \mid 1 \leq i < j \leq m\right\}.$$

(If $m = 1$, then $R = \{\epsilon_1\}, R^\vee = \{2\epsilon_1^\vee\}$.) By the *standard simple roots* we mean the following choice of simple roots:

$$\epsilon_1 - \epsilon_2, \cdots, \epsilon_{m-1} - \epsilon_m, \epsilon_m.$$

We denote by $\mathrm{BRD}(\mathsf{B}_m)$ the based root datum corresponding to the above choice of simple roots, called the *standard based root datum*. The Weyl group of $\mathrm{RD}(\mathsf{B}_m)$ is naturally identified with $\{\pm 1\}^m \rtimes \mathfrak{S}_m$.

Similarly, for $m \in \mathbb{Z}_{\geq 1}$ we denote by $\mathrm{RD}(\mathsf{D}_m)$ the *standard type* D_m *root datum*, given by

$$(\mathbb{Z}^m, R, \mathbb{Z}^m, R^\vee),$$

where

$$R = \{\pm\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq m\},$$
$$R^\vee = \left\{\pm\epsilon_i^\vee \pm \epsilon_j^\vee \mid 1 \leq i < j \leq m\right\}.$$

(If $m = 1$, then $R = R^\vee = \emptyset$.) By the *standard simple roots* we mean the following choice of simple roots:

$$\epsilon_1 - \epsilon_2, \cdots, \epsilon_{m-1} - \epsilon_m, \epsilon_{m-1} + \epsilon_m.$$

We denote by $\mathrm{BRD}(\mathsf{D}_m)$ the corresponding based root datum. The Weyl group of $\mathrm{RD}(\mathsf{D}_m)$ is naturally identified with $(\{\pm 1\}^m)' \rtimes \mathfrak{S}_m$, where $(\{\pm 1\}^m)'$ denotes the kernel of the homomorphism $\{\pm 1\}^m \to \{\pm 1\}$ taking the product of the coordinates.

Definition 1.2.6. — Let $\alpha \in \overline{F}$ be an element such that $\alpha^2 \in F^\times$ and $\alpha \notin F$. Let $\mathrm{U}(1)_\alpha$ be the norm-one subtorus of $\mathrm{Res}_{F(\alpha)/F}\,\mathbb{G}_m$. We have a canonical isomorphism $\mathrm{U}(1)_{\alpha,\overline{F}} \cong \mathbb{G}_{m,\overline{F}}$ corresponding to the inclusion $F(\alpha) \hookrightarrow \overline{F}$. In particular, we canonically identify $X^*(\mathrm{U}(1)_\alpha)$ and $X_*(\mathrm{U}(1)_\alpha)$ with $\mathbb{Z}$. We also have a canonical injective F-homomorphism $\iota_\alpha : \mathrm{U}(1)_\alpha \to \mathrm{GL}_2$, which represents the multiplication action of $\mathrm{U}(1)_\alpha$ on $F(\alpha)$ under the F-basis $\{1, \alpha\}$ of $F(\alpha)$. If $F = \mathbb{R}, \overline{F} = \mathbb{C}, \alpha = \sqrt{-1}$, we simply write $\mathrm{U}(1)$ for $\mathrm{U}(1)_\alpha$.

1.2.7. — Let $V = (V, q)$ be a quadratic space over F of dimension d and discriminant δ. Let $G = \mathrm{SO}(V)$. Then G is a reductive algebraic group over F, and semi-simple if $d \neq 2$. The absolute rank of G is $m = \lfloor d/2 \rfloor$.

Assuming that (V, q) is quasi-split, we shall obtain an explicit description of a Borel pair in G and the associated based root datum as follows. There are two cases to consider.

The first case is when V has a hyperbolic basis $\mathbb{B} = \{e_1, \cdots, e_d\}$. We then identify G with a subgroup of GL_d using the basis $\mathbb{B}$. When d is odd, we obtain an F-embedding

$$\iota_{\mathbb{B}} : \mathbb{G}_m^m \longrightarrow G, \qquad (z_1, \cdots, z_m) \longmapsto \mathrm{diag}(z_1, \cdots, z_m, 1, z_m^{-1}, \cdots, z_1^{-1}).$$

When d is even, we obtain an F-embedding

$$\iota_{\mathbb{B}} : \mathbb{G}_m^m \longrightarrow G, \qquad (z_1, \cdots, z_m) \longmapsto \mathrm{diag}(z_1, \cdots, z_m, z_m^{-1}, \cdots, z_1^{-1}).$$

For both parities of d, the image T of $\iota_{\mathbb{B}}$ is a split maximal torus in G. Also, the intersection of G with the upper triangular Borel subgroup of GL_d is a Borel subgroup

B of G containing T. Under $\iota_{\mathbb{B}}$, the based root datum $\mathrm{BRD}(T, B)$ is identified with the standard based root datum $\mathrm{BRD}(\mathsf{B}_m)$ (resp. $\mathrm{BRD}(\mathsf{D}_m)$) when d is odd (resp. even).

The second case is when d is even, δ is non-trivial, and V has a near-hyperbolic basis $\mathbb{B} = \{e_1, \cdots, e_d\}$. Let $x \in F^\times$ be the discriminant of $\mathbb{B}$ (see Definition 1.2.2), and fix a square root $\alpha \in \overline{F}$ of x. We identify G with a subgroup of GL_d using the basis $\mathbb{B}$, and obtain an F-embedding

$$\begin{aligned} \iota_{\alpha,\mathbb{B}} : \mathbb{G}_m^{m-1} \times \mathrm{U}(1)_\alpha &\longrightarrow G \\ (z_1, \cdots, z_{m-1}, z_m) &\longmapsto \mathrm{diag}(z_1, \cdots, z_{m-1}, \iota_\alpha(z_m), z_{m-1}^{-1}, \cdots, z_1^{-1}). \end{aligned}$$

Here $\mathrm{U}(1)_\alpha$ and $\iota_\alpha : \mathrm{U}(1)_\alpha \to \mathrm{GL}_2$ are as in Definition 1.2.6. The image T of $\iota_{\alpha,\mathbb{B}}$ is a maximal torus in G. Recall from Definition 1.2.6 that $X^*(\mathrm{U}(1)_\alpha)$ and $X_*(\mathrm{U}(1)_\alpha)$ are canonically identified with $\mathbb{Z}$, so $X^*(\mathbb{G}_m^{m-1} \times \mathrm{U}(1)_\alpha)$ and $X_*(\mathbb{G}_m^{m-1} \times \mathrm{U}(1)_\alpha)$ are canonically identified with $\mathbb{Z}^m$. Under $\iota_{\alpha,\mathbb{B}}$, the root datum of $(T_{\overline{F}}, G_{\overline{F}})$ is identified with $\mathrm{RD}(\mathsf{D}_m)$. The standard based root datum $\mathrm{BRD}(\mathsf{D}_m)$ thus gives rise to a Borel subgroup $B_{\overline{F}}$ of $G_{\overline{F}}$ containing $T_{\overline{F}}$. The Γ_F-action on $X^*(\mathbb{G}_m^{m-1} \times \mathrm{U}(1)_\alpha) \cong \mathbb{Z}^m$ factors through $\mathrm{Gal}(F(\alpha)/F)$, and the non-trivial element of $\mathrm{Gal}(F(\alpha)/F)$ acts by $\mathbb{Z}^m \to \mathbb{Z}^m, (a_1, \cdots, a_m) \mapsto (a_1, \cdots, a_{m-1}, -a_m)$. Hence the Γ_F-action preserves the set of standard simple roots. It follows that $B_{\overline{F}}$ comes from a Borel subgroup B of G. Thus (T, B) is a Borel pair in G, and $\iota_{\alpha,\mathbb{B}}$ induces an isomorphism between $\mathrm{BRD}(\mathsf{D}_m)$ and $\mathrm{BRD}(T, B)$.

Proposition 1.2.8. — *Let (V, q) be a quadratic space over F of dimension d and discriminant δ. Let $G = \mathrm{SO}(V)$. Assume that $d \geq 3$. The following statements hold.*

(1) *The quadratic space V is split if and only if G is split.*

(2) *If d is odd, then G is split if and only if G is quasi-split.*

(3) *If d is even, then G is split if and only if G is quasi-split and δ is trivial.*

(4) *Assume that d is even, δ is non-trivial, and V is quasi-split. Then G is quasi-split.*

(5) *Assume that d is even, δ is non-trivial, and G is quasi-split over F. Then G is split over $F(\alpha)$, for any $\alpha \in \overline{F}$ whose square is a lift of δ.*

(6) *Keep the assumptions in (5), and assume that F is a non-archimedean local field of characteristic zero. Then G is unramified if and only if $F(\alpha)$ is unramified over F, if and only if $\delta \in F^\times/F^{\times,2}$ has a representative in $\mathcal{O}_F^\times/\mathcal{O}_F^{\times,2}$.*

(7) *Suppose $F = \mathbb{Q}_p$ for an odd prime p. Then (V, q) is quasi-split if and only if the Hasse invariant is $(-1)^{\frac{p-1}{2} v_p(\delta) \lfloor \frac{d-1}{2} \rfloor}$. Here $v_p(\delta)$ is well defined in $\mathbb{Z}/2\mathbb{Z}$.*

(8) *Suppose $F = \mathbb{Q}$. Then $(V, q) \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is quasi-split for almost all primes p.*

Proof. — **(1)** This is well known; see for instance [**PR94**, Prop. 2.14].

(2) This follows from the fact that the Dynkin diagram of type $\mathsf{B}_{(d-1)/2}$ does not have non-trivial automorphisms.

(3) If G is split, then V is split by part (1), and so δ is trivial. Conversely, assume that G is quasi-split and δ is trivial. By the abstract classification of quasi-split semi-simple groups of type D_m (where $m = \frac{d}{2} \geq 2$), we know that the split rank of G is at least $\frac{d}{2} - 1$. This implies that V is an orthogonal direct sum of $\frac{d}{2} - 1$ hyperbolic planes and a 2-dimensional quadratic space V_0, by [**PR94**, Prop. 2.14]. The discriminant of V_0 is the same as that of V, which is trivial. Therefore there is a basis of V_0 under which the matrix of the quadratic form on V_0 is $\mathrm{diag}(a, -ab^2)$ for some $a, b \in F^\times$. Clearly this implies that V_0 is a hyperbolic plane. Hence V is split, and therefore G is split by (1).

(4) By §1.2.7, G admits a Borel subgroup over F.

(5) This follows from (3) by base changing both V and G from F to $F(\alpha)$.

(6) Since δ is non-trivial, by (1) we know that G is non-split. By the abstract classification of quasi-split non-split semi-simple groups of type D_m (with $m \geq 2$), we know that G splits over a unique quadratic extension E/F inside $\overline{F}$, and that any splitting field of G inside $\overline{F}$ must contain E. Thus G is unramified if and only if E/F is unramified. By (5), we know that $E = F(\alpha)$. Thus G is unramified if and only if $F(\alpha)$ is unramified over F, which is also equivalent to that δ has a representative in $\mathcal{O}_F^\times / \mathcal{O}_F^{\times,2}$.

(7) If (V, q) is quasi-split, then it has matrix representation

$$\begin{pmatrix} I_{\frac{d-1}{2}} & & \\ & x & \\ & & -I_{\frac{d-1}{2}} \end{pmatrix}$$

when d is odd and

$$\begin{pmatrix} I_{\frac{d}{2}} & & \\ & -x & \\ & & -I_{\frac{d}{2}-1} \end{pmatrix}$$

when d is even, for some $x \in F^\times$ representing δ. Hence the Hasse invariant is $(x, -1)_p^{\lfloor \frac{d-1}{2} \rfloor} = (-1)^{\frac{p-1}{2} v_p(x) \lfloor \frac{d-1}{2} \rfloor}$. This proves the "only if" direction. The "if" direction follows since two quadratic spaces over $\mathbb{Q}_p$ with the same dimension, discriminant, and Hasse invariant are isomorphic.

(8) For almost all p, $v_p(\delta) = 0 \in \mathbb{Z}/2\mathbb{Z}$ and the Hasse invariant of (V, q) at p is trivial. By (7) we know that $(V, q) \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is quasi-split for such p. □

Remark 1.2.9. — From the assumptions that d is even, δ is non-trivial, and $G = \mathrm{SO}(V)$ is quasi-split over F, it does not follow that V is quasi-split. For example, the quadratic spaces over $\mathbb{R}$ of signatures $(n+2, n)$ and $(n, n+2)$ define isomorphic special orthogonal groups, but only the former quadratic space is quasi-split; cf. Example 1.2.4.

1.3. Generalities on Shimura data and rational boundary components

In this section we collect some general facts concerning the formalism of mixed Shimura data and rational boundary components in [**Pin90**].

1.3.1. — According to the definition of Pink [**Pin90**, Chap. 2], a *mixed Shimura datum* is a tuple $(P, U, \mathcal{Y}, h)$, where P is a connected linear algebraic group over $\mathbb{Q}$, U is a subgroup of the unipotent radical of P that is normal in P, $\mathcal{Y}$ is a left homogeneous space under the subgroup $P(\mathbb{R})U(\mathbb{C})$ of $P(\mathbb{C})$, and h is a $P(\mathbb{R})U(\mathbb{C})$-equivariant map $\mathcal{Y} \to \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P)$, satisfying the axioms in [**Pin90**, 2.1]. (Here $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$.) If h is clear from the context, we omit it from the notation. If U is trivial, we also omit it from the notation. The mixed Shimura datum is called *pure* if P is reductive. Note that the notion of a pure Shimura datum according to Pink's definition is less restrictive than Deligne's definition in [**Del79**, 2.1], in that h is allowed to be non-injective, cf. [**Pin90**, 2.2 (d)]. In the sequel all pure Shimura data are understood in the sense of Pink.

Some comments on the homogeneous space $\mathcal{Y}$ are in order. First, note that $P(\mathbb{R})U(\mathbb{C})$ is the preimage of $(P/U)(\mathbb{R})$ along the map $P(\mathbb{C}) \to (P/U)(\mathbb{C})$, since $\mathbf{H}^1(\mathbb{R}, U)$ is trivial. It follows that $P(\mathbb{R})U(\mathbb{C})$ is a closed Lie subgroup of the real Lie group $P(\mathbb{C})$. Recall that for any real Lie group $\mathcal{G}$, a left homogeneous space under $\mathcal{G}$ is a set S equipped with a transitive left action of $\mathcal{G}$ such that the stabilizers are closed Lie subgroups of $\mathcal{G}$. Then S has the unique structure of a smooth manifold such that the $\mathcal{G}$-action is smooth. In the definition of a mixed Shimura datum, $\mathcal{Y}$ is required to be a left homogeneous space under the real Lie group $P(\mathbb{R})U(\mathbb{C})$, and so $\mathcal{Y}$ is canonically a smooth manifold. As explained in [**Pin90**, 2.2], $\mathcal{Y}$ has finitely many connected components, and the smooth structure on $\mathcal{Y}$ can be upgraded to a canonical complex structure, which is invariant under $P(\mathbb{R})U(\mathbb{C})$.

1.3.2. — By definition ([**Pin90**, 2.3]), a *morphism* between two mixed Shimura data $(P, U, \mathcal{Y}, h)$ and $(P', U', \mathcal{Y}', h')$ is a pair (π, F), where $\pi : P \to P'$ is a homomorphism of $\mathbb{Q}$-algebraic groups, and $F : \mathcal{Y} \to \mathcal{Y}'$ is a map, required to satisfy the following conditions:

– π maps U into U'.

– F is equivariant with respect to the homomorphism $P(\mathbb{R})U(\mathbb{C}) \to P'(\mathbb{R})U'(\mathbb{C})$ induced by π.

– For any $y \in \mathcal{Y}$, the homomorphism $h'(F(y)) : \mathbb{S}_{\mathbb{C}} \to P'_{\mathbb{C}}$ is equal to the composite homomorphism

$$\mathbb{S}_{\mathbb{C}} \xrightarrow{h(y)} P_{\mathbb{C}} \xrightarrow{\pi} P'_{\mathbb{C}}.$$

As shown in [**Pin90**, 2.4], if (π, F) is a morphism as above, then F is automatically holomorphic with respect to the canonical complex structures on $\mathcal{Y}$ and $\mathcal{Y}'$.

1.3.3. — Let $(P, U, \mathcal{Y}, h)$ be a mixed Shimura datum. In [**Pin90**, 2.9], Pink constructs the *quotient* of $(P, U, \mathcal{Y}, h)$ by a normal subgroup P_0 of P. This is a mixed Shimura datum for the group P/P_0 equipped with a morphism from $(P, U, \mathcal{Y}, h)$ satisfying a universal property. In the following, we give an alternative construction of the quotient in the special case where P_0 is the unipotent radical of P.

Let W be the unipotent radical of P. We write G for P/W, and write π for the projection $P \to G$. Since $\mathbf{H}^1(\mathbb{R}, W)$ is trivial, and since $W(\mathbb{R})$ and $U(\mathbb{C})$ are connected, the natural map $P(\mathbb{R})U(\mathbb{C}) \to G(\mathbb{R})$ is surjective and induces an isomorphism $\pi_0(P(\mathbb{R})U(\mathbb{C})) \xrightarrow{\sim} \pi_0(G(\mathbb{R}))$. In particular, $\pi_0(G(\mathbb{R}))$ acts on $\pi_0(\mathcal{Y})$.

Suppose we have a left $\pi_0(G(\mathbb{R}))$-set Γ and a $\pi_0(G(\mathbb{R}))$-equivariant map $\lambda : \pi_0(\mathcal{Y}) \to \Gamma$. We define the map

$$\begin{aligned} \mathbb{H}_\lambda : \mathcal{Y} &\longrightarrow \Gamma \times \mathrm{Hom}(\mathbb{S}_\mathbb{C}, G_\mathbb{C}). \\ y &\longmapsto (\lambda([y]), \pi \circ h(y)). \end{aligned}$$

We have a diagonal $G(\mathbb{R})$-action on $\Gamma \times \mathrm{Hom}(\mathbb{S}_\mathbb{C}, G_\mathbb{C})$, where the action on the second factor is by conjugation. The map $\mathbb{H}_\lambda$ is equivariant with respect to the natural homomorphism $P(\mathbb{R})U(\mathbb{C}) \to G(\mathbb{R})$. Let $\mathcal{X}_\lambda := \mathrm{im}(\mathbb{H}_\lambda)$. Let $h_\lambda : \mathcal{X} \to \mathrm{Hom}(\mathbb{S}_\mathbb{C}, G_\mathbb{C})$ be the projection map to the second factor. It is easy to check that $(G, \mathcal{X}_\lambda, h_\lambda)$ is a pure Shimura datum, and that the pair $(\pi : P \to G,\ \mathbb{H}_\lambda : \mathcal{Y} \to \mathcal{X}_\lambda)$ is a morphism $(P, U, \mathcal{Y}, h) \to (G, \mathcal{X}_\lambda, h_\lambda)$ between mixed Shimura data. Since $\mathbb{H}_\lambda : \mathcal{Y} \to \mathcal{X}_\lambda$ is surjective by the definition of $\mathcal{X}_\lambda$, it induces a surjection $\pi_0(\mathbb{H}_\lambda) : \pi_0(\mathcal{Y}) \to \pi_0(\mathcal{X}_\lambda)$.

Lemma 1.3.4. — *Let Γ and λ be as above. The following statements hold.*

(1) *The map $\mathcal{X}_\lambda \to \Gamma$ given by the projection to the first factor induces an injection $\pi_0(\mathcal{X}_\lambda) \to \Gamma$.*

(2) *The surjection $\pi_0(\mathbb{H}_\lambda) : \pi_0(\mathcal{Y}) \to \pi_0(\mathcal{X}_\lambda)$ is a bijection if and only if λ is injective.*

(3) *If λ is injective, then the morphism $(\pi, \mathbb{H}_\lambda) : (P, U, \mathcal{Y}, h) \to (G, \mathcal{X}_\lambda, h_\lambda)$ identifies $(G, \mathcal{X}_\lambda, h_\lambda)$ with the quotient of $(P, U, \mathcal{Y}, h)$ by W.*

Proof. — **(1)** A connected component of $\mathcal{X}_\lambda$ is the same thing as a $G(\mathbb{R})^0$-orbit in $\mathcal{X}_\lambda$, but $G(\mathbb{R})^0$ acts trivially on Γ.

(2) The composition of $\pi_0(\mathbb{H}_\lambda)$ followed by the injection $\pi_0(\mathcal{X}_\lambda) \to \Gamma$ in part (1) is equal to λ.

(3) Let $(\pi, F) : (P, U, \mathcal{Y}, h) \to (G, \mathcal{X}_{\mathrm{abs}}, h_{\mathrm{abs}})$ be the abstract quotient by W, which is characterized by a universal property and constructed in [**Pin90**, 2.9]. By the universal property, there is a unique $G(\mathbb{R})$-equivariant map $j : \mathcal{X}_{\mathrm{abs}} \to \mathcal{X}_\lambda$ such that $h_\lambda \circ j = h_{\mathrm{abs}}$ and $j \circ F = \mathbb{H}_\lambda$. We only need to show that j is a bijection. Since $\mathbb{H}_\lambda : \mathcal{Y} \to \mathcal{X}_\lambda$ is surjective, so is j. By part (2), j induces an injection $\pi_0(\mathcal{X}_{\mathrm{abs}}) \to \pi_0(\mathcal{X}_\lambda)$. It remains to show that the restriction of j to each connected component of $\mathcal{X}_{\mathrm{abs}}$ is injective. For this, it is enough to show that the restriction of $h_\lambda \circ j = h_{\mathrm{abs}}$ to each connected component of $\mathcal{X}_{\mathrm{abs}}$ is injective. But this is [**Pin90**, 2.12]. □

1.3.5. — We recall the formalism of rational boundary components developed in [**Pin90**, Chap. 4]. Let $(G, \mathcal{X}) = (G, \mathcal{X}, h)$ be a pure Shimura datum. For simplicity, we assume that G^{ad} is $\mathbb{Q}$-simple, which will suffice for our applications. We denote by $\mathrm{AdmPar}(G)$ the set of *admissible parabolic subgroups of* G, namely G itself and the maximal proper parabolic subgroups of G (defined over $\mathbb{Q}$). For any $P \in \mathrm{AdmPar}(G)$, Pink [**Pin90**, 4.7, 4.8] defines a canonical normal subgroup P^{Pink} of P, and a canonical normal subgroup U_P of P^{Pink} contained in the unipotent radical of P^{Pink}.(1) As G is reductive, the proof of [**Pin90**, 4.8] shows that the unipotent radical of P^{Pink} is equal to the unipotent radical N_P of P. In particular, the subgroup $P^{\mathrm{Pink}} \subset G$ uniquely determines P. We shall write M_P for P/N_P and write G_P for P^{Pink}/N_P.

We define

$$\mathscr{Y}_P := \pi_0(\mathcal{X}) \times \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P^{\mathrm{Pink}}_{\mathbb{C}}),$$

equipped with the diagonal action of $P(\mathbb{R})U_P(\mathbb{C})$. Here the action on the first factor is via $\pi_0(P(\mathbb{R})U_P(\mathbb{C})) \cong \pi_0(P(\mathbb{R})) \to \pi_0(G(\mathbb{R}))$, and on the second factor via conjugation. We write p_1^P and p_2^P for the projection maps from $\mathscr{Y}_P$ to the two factors. In [**Pin90**, 4.11], Pink defines a canonical $P(\mathbb{R})$-equivariant map

$$\omega_P : \mathcal{X} \longrightarrow \mathscr{Y}_P$$

such that $p_1^P \circ \omega_P$ is the natural projection $\mathcal{X} \to \pi_0(\mathcal{X})$.

By definition ([**Pin90**, 4.11]), a *rational boundary component* of $(G, \mathcal{X})$ is a pair $(P, \mathcal{Y})$, where $P \in \mathrm{AdmPar}(G)$, and $\mathcal{Y}$ is any $P^{\mathrm{Pink}}(\mathbb{R})U_P(\mathbb{C})$-orbit in $\mathscr{Y}_P$ such that $\mathcal{Y} \cap \mathrm{im}(\omega_P) \neq \emptyset$. We denote by $\mathcal{RBC}(G, \mathcal{X})$ or simply $\mathcal{RBC}$ the set of all rational boundary components. For each $P \in \mathrm{AdmPar}(G)$, we denote by $\mathcal{RBC}_P(G, \mathcal{X})$ or simply $\mathcal{RBC}_P$ the set of all rational boundary components whose first coordinate is P. For $(P, \mathcal{Y}) \in \mathcal{RBC}$, we write $\mathcal{X}^{\mathcal{Y}}$ for the subset $\omega_P^{-1}(\mathcal{Y})$ of $\mathcal{X}$. We have the following facts (see [**Pin90**, Chap. 4]):

(I) For $(P, \mathcal{Y}) \in \mathcal{RBC}$, the $P^{\mathrm{Pink}}(\mathbb{R})U_P(\mathbb{C})$-action on $\mathcal{Y}$ and the map $p_2^P|_{\mathcal{Y}} : \mathcal{Y} \to \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P^{\mathrm{Pink}}_{\mathbb{C}})$ make the tuple $(P^{\mathrm{Pink}}, U_P, \mathcal{Y})$ a mixed Shimura datum.

(II) For $(P, \mathcal{Y}) \in \mathcal{RBC}$, the set $\mathcal{X}^{\mathcal{Y}}$ is the union of some connected components of $\mathcal{X}$. The map ω_P maps $\mathcal{X}^{\mathcal{Y}}$ injectively and holomorphically into $\mathcal{Y}$, inducing a bijection

$$\gamma_{\mathcal{Y}} : \pi_0(\mathcal{X}^{\mathcal{Y}}) \xrightarrow{\sim} \pi_0(\mathcal{Y}). \tag{1.3.5.1}$$

Moreover, the map $\pi_0(\mathcal{Y}) \to \pi_0(\mathcal{X})$ induced by $p_1^P|_{\mathcal{Y}} : \mathcal{Y} \to \pi_0(\mathcal{X})$ is the inverse of $\gamma_{\mathcal{Y}}$.

(III) For each fixed $P \in \mathrm{AdmPar}(G)$, $\mathcal{X}$ is the disjoint union

$$\mathcal{X} = \coprod_{(P,\mathcal{Y}) \in \mathcal{RBC}_P} \mathcal{X}^{\mathcal{Y}}. \tag{1.3.5.2}$$

(1) Our P, P^{Pink}, and U_P are denoted respectively by Q, P_1, and U_1 in [**Pin90**, 4.7, 4.8].

1.3.6. — We keep the setting of §1.3.5. Let $P \in \mathrm{AdmPar}(G)$. For each $(P, \mathcal{Y}) \in \mathcal{RBC}_P$, we let $(G_P, \mathcal{X}_\mathcal{Y})$ be the quotient of the mixed Shimura datum $(P^{\mathrm{Pink}}, U_P, \mathcal{Y})$ by the unipotent radical N_P of P (which is also the unipotent radical of P^{Pink}), and let $F_\mathcal{Y} : \mathcal{Y} \to \mathcal{X}_\mathcal{Y}$ be the canonical $P^{\mathrm{Pink}}(\mathbb{R})U_P(\mathbb{C})$-equivariant map. By Lemma 1.3.4, we know that $F_\mathcal{Y}$ induces a bijection between the sets of connected components.

Let $\pi_\mathcal{Y}$ be the composition

$$\pi_\mathcal{Y} : \mathcal{X}^\mathcal{Y} \xrightarrow{\omega_P} \mathcal{Y} \xrightarrow{F_\mathcal{Y}} \mathcal{X}_\mathcal{Y}.$$

Then $\pi_\mathcal{Y}$ is holomorphic and induces a bijection between the sets of connected components, since both $\omega_P|_{\mathcal{X}^\mathcal{Y}}$ and $F_\mathcal{Y}$ have these properties. Moreover, $\pi_\mathcal{Y}$ is equivariant with respect to the surjective Lie group homomorphism $P^{\mathrm{Pink}}(\mathbb{R})U_P(\mathbb{C}) \to G_P(\mathbb{R})$. In particular, $\pi_\mathcal{Y}$ is a surjective submersion, since $\mathcal{X}^\mathcal{Y}$ (resp. $\mathcal{X}_\mathcal{Y}$) is a left homogeneous space under $P^{\mathrm{Pink}}(\mathbb{R})U_P(\mathbb{C})$ (resp. $G_P(\mathbb{R})$).

Let $\mathcal{X}_P$ be the disjoint union

$$\mathcal{X}_P := \coprod_{(P,\mathcal{Y}) \in \mathcal{RBC}_P} \mathcal{X}_\mathcal{Y},$$

as a complex manifold with a $G_P(\mathbb{R})$-action. In view of (1.3.5.2), we have a map

$$\pi_P := \coprod_{(P,\mathcal{Y}) \in \mathcal{RBC}_P} \pi_\mathcal{Y} : \mathcal{X} \longrightarrow \mathcal{X}_P.$$

Then π_P is holomorphic, surjective, submersive, equivariant with respect to $P^{\mathrm{Pink}}(\mathbb{R})U_P(\mathbb{C}) \to G_P(\mathbb{R})$, and induces a bijection between the sets of connected components, since each $\pi_\mathcal{Y}$ has these properties. When $P = G$, the map π_G is an isomorphism.

Consider the set-theoretic disjoint union [(2)]

$$\mathcal{X}^* = \coprod_{P \in \mathrm{AdmPar}(G)} \mathcal{X}_P = \coprod_{(P,\mathcal{Y}) \in \mathcal{RBC}} \mathcal{X}_\mathcal{Y}. \tag{1.3.6.1}$$

There is a natural $G(\mathbb{Q})$-action on $\mathcal{X}^*$, satisfying the following properties (see [**Pin90**, 4.16, 6.2]):

– The action respects the stratification of $\mathcal{X}^*$ by the subsets $\mathcal{X}_\mathcal{Y}$.

– For $g \in G(\mathbb{Q})$ and $P \in \mathrm{AdmPar}(G)$, we have $g(\mathcal{X}_P) = \mathcal{X}_{gPg^{-1}}$. In particular, $\mathrm{Stab}_{G(\mathbb{Q})}\mathcal{X}_P = P(\mathbb{Q})$.

– For $P \in \mathrm{AdmPar}(G)$, the map $\pi_P : \mathcal{X} \to \mathcal{X}_P$ is $P(\mathbb{Q})$-equivariant. Here $P(\mathbb{Q})$ acts on X_P since $\mathrm{Stab}_{G(\mathbb{Q})}\mathcal{X}_P = P(\mathbb{Q})$. Moreover, the $P(\mathbb{Q})$-action on $\mathcal{X}_P$ factors through the quotient map $P(\mathbb{Q}) \to M_P(\mathbb{Q})$.

Let $P \in \mathrm{AdmPar}(G)$. As discussed above we have an $M_P(\mathbb{Q})$-action on $\mathcal{X}_P$. Since $M_P(\mathbb{Q})$ is dense in $M_P(\mathbb{R})$, there is at most one way to extend this action to a

(2) While we shall only consider $\mathcal{X}^*$ as a set, there is a natural *Satake topology* on $\mathcal{X}^*$; see [**Pin90**, 6.2]. Under this topology, $\mathcal{X}^*$ contains $\mathcal{X}$ as a dense open subset.

continuous $M_P(\mathbb{R})$-action. It is shown in [**Pin90**, 3.6] that such an extension indeed exists. Since we need to explicitly describe this $M_P(\mathbb{R})$-action later for the orthogonal Shimura datum, we give its construction in the following proposition.

Proposition 1.3.7. — *Keep the setting of §1.3.5, and let $P \in \mathrm{AdmPar}(G)$. The following statements hold.*

(1) *There is a unique extension of the $G_P(\mathbb{R})$-action on $\mathcal{X}_P$ to an $M_P(\mathbb{R})$-action such that the map $\pi_P : \mathcal{X} \to \mathcal{X}_P$ is equivariant with respect to the homomorphism $P(\mathbb{R}) \to M_P(\mathbb{R})$.*

(2) *The $M_P(\mathbb{R})$-action on $\mathcal{X}_P$ in (1) factors through the natural homomorphism*

$$\begin{aligned} M_P(\mathbb{R}) &\longrightarrow \pi_0(M_P(\mathbb{R})) \times \mathrm{Aut}(G_{P,\mathbb{R}}) \\ m &\longmapsto ([m], \mathrm{Int}\, m). \end{aligned} \tag{1.3.7.1}$$

(3) *The $M_P(\mathbb{R})$-action on $\mathcal{X}_P$ in (1) is transitive and continuous. Its restriction to $M_P(\mathbb{Q})$ coincides with the $M_P(\mathbb{Q})$-action discussed in §1.3.6.*

Proof. — **(1)** The uniqueness immediately follows from the surjectivity of π_P. We prove the existence. Using the canonical isomorphism $\pi_0(P(\mathbb{R})U_P(\mathbb{C})) \cong \pi_0(M_P(\mathbb{R}))$, we view $\pi_0(\mathcal{X})$ as a $\pi_0(M_P(\mathbb{R}))$-set. In particular, $\pi_0(\mathcal{X})$ is a $\pi_0(G_P(\mathbb{R}))$-set. To simplify notation, we write $\mathbb{H}_P$ for the set $\pi_0(\mathcal{X}) \times \mathrm{Hom}(\mathbb{S}_\mathbb{C}, G_{P,\mathbb{C}})$, which is equipped with the diagonal $G_P(\mathbb{R})$-action as in §1.3.3 (where we take Γ to be $\pi_0(\mathcal{X})$). The $G_P(\mathbb{R})$-action on $\mathbb{H}_P$ extends to an $M_P(\mathbb{R})$-action in the obvious way (using the normality of G_P in M_P). We have a natural map

$$\begin{aligned} \mathscr{F}_P : \mathscr{Y}_P = \pi_0(\mathcal{X}) \times \mathrm{Hom}(\mathbb{S}_\mathbb{C}, P_\mathbb{C}^{\mathrm{Pink}}) &\longrightarrow \mathbb{H}_P = \pi_0(\mathcal{X}) \times \mathrm{Hom}(\mathbb{S}_\mathbb{C}, G_{P,\mathbb{C}}) \\ ([x], l) &\longmapsto ([x], (\mathbb{S}_\mathbb{C} \xrightarrow{l} P_\mathbb{C}^{\mathrm{Pink}} \to G_{P,\mathbb{C}})), \end{aligned}$$

which is equivariant with respect to $P(\mathbb{R}) \to M_P(\mathbb{R})$.

Let $(P, \mathcal{Y}) \in \mathcal{RBC}_P$. We denote by $\lambda_\mathcal{Y}$ the injective map

$$\pi_0(\mathcal{Y}) \xrightarrow{\gamma_\mathcal{Y}^{-1}} \pi_0(\mathcal{X}^\mathcal{Y}) \hookrightarrow \pi_0(\mathcal{X}),$$

where $\gamma_\mathcal{Y}$ is as in (1.3.5.1). As in §1.3.3, $\lambda_\mathcal{Y}$ induces a map $\mathbb{H}_{\lambda_\mathcal{Y}} : \mathcal{Y} \to \mathbb{H}_P$, whose image is denoted by $\mathcal{X}_{\lambda_\mathcal{Y}}$. By Lemma 1.3.4, we may assume that $\mathcal{X}_\mathcal{Y}$ is equal to $\mathcal{X}_{\lambda_\mathcal{Y}}$, and that the map $F_\mathcal{Y} : \mathcal{Y} \to \mathcal{X}_\mathcal{Y}$ is equal to the map $\mathbb{H}_{\lambda_\mathcal{Y}}$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{Y} & \hookrightarrow & \mathscr{Y}_P \\ \downarrow{\scriptstyle F_\mathcal{Y}} & & \downarrow{\scriptstyle \mathscr{F}_P} \\ \mathcal{X}_\mathcal{Y} & \hookrightarrow & \mathbb{H}_P \end{array}$$

For different elements $(P, \mathcal{Y}) \neq (P, \mathcal{Y}') \in \mathcal{RBC}_P$, the subsets $\mathcal{X}_\mathcal{Y}$ and $\mathcal{X}_{\mathcal{Y}'}$ of $\mathbb{H}_P$ are disjoint, because their projections in $\pi_0(\mathcal{X})$ are the disjoint subsets $\pi_0(\mathcal{X}^\mathcal{Y})$ and $\pi_0(\mathcal{X}^{\mathcal{Y}'})$. Therefore we may identify $\mathcal{X}_P$ with the union of the $\mathcal{X}_\mathcal{Y}$'s inside $\mathbb{H}_P$. Under this identification, the map $\pi_P : \mathscr{X} \to \mathscr{X}_P$ is given by the composite map

$$\mathscr{X} \xrightarrow{\omega_P} \mathscr{Y}_P \xrightarrow{\mathscr{F}_P} \mathbb{H}_P.$$

Since $\pi_P : \mathscr{X} \to \mathscr{X}_P$ is surjective, and since $\mathscr{F}_P \circ \omega_P : \mathscr{X} \to \mathbb{H}_P$ is equivariant with respect to $P(\mathbb{R}) \to M_P(\mathbb{R})$, we see that $\mathscr{X}_P$ is an $M_P(\mathbb{R})$-stable subset of $\mathbb{H}_P$. We define the desired $M_P(\mathbb{R})$-action on $\mathscr{X}_P$ to be the one inherited from the $M_P(\mathbb{R})$-action on $\mathbb{H}_P$. Then π_P is indeed equivariant with respect to $P(\mathbb{R}) \to M_P(\mathbb{R})$.

(2) It suffices to observe that the $M_P(\mathbb{R})$-action on $\mathbb{H}_P$ factors through (1.3.7.1), which is obvious.

(3) Firstly, by [**Pin90**, 4.7], the $G(\mathbb{R})$-action on $\mathscr{X}$ restricts to a transitive $P(\mathbb{R})$-action on $\mathscr{X}$. Since $\pi_P : \mathscr{X} \to \mathscr{X}_P$ is surjective, the $M_P(\mathbb{R})$-action on $\mathscr{X}_P$ is transitive. Secondly, the continuity of the $M_P(\mathbb{R})$-action on $\mathscr{X}_P$ follows from the continuity of the $P(\mathbb{R})$-action on $\mathscr{X}$, and the fact that the maps $\pi_P : \mathscr{X} \to \mathscr{X}_P$ and $P(\mathbb{R}) \to M_P(\mathbb{R})$ are surjective submersions. Finally, the last statement in (3) follows from the surjectivity and $P(\mathbb{Q})$-equivariance of $\pi_P : \mathscr{X} \to \mathscr{X}_P$, where $P(\mathbb{Q})$ acts on $\mathscr{X}_P$ in the way described in §1.3.6. □

Remark 1.3.8. — In the above exposition, we started with the rational boundary components in the sense of [**Pin90**], and used them to construct the $M_P(\mathbb{R})$-homogeneous space $\mathscr{X}_P$, the $P(\mathbb{R})$-equivariant map $\pi_P : \mathscr{X} \to \mathscr{X}_P$, and the $G(\mathbb{Q})$-set $\mathscr{X}^*$. This is the approach taken in [**Pin90**]. Alternatively, one could apply the classical (i.e. non-adelic) formalism of rational boundary components in [**AMRT10**] to each connected component of the Hermitian symmetric domain $\mathscr{X}$ in order to construct each connected component of $\mathscr{X}_P$ and each connected component of $\mathscr{X}^*$. One could then construct the whole $\mathscr{X}_P$ and $\mathscr{X}^*$ by taking suitable disjoint unions, and reconstruct the subsets $\mathscr{X}_{\mathcal{Y}} \subset \mathscr{X}_P$ as the $G_P(\mathbb{R})$-orbits in $\mathscr{X}_P$. This alternative approach is the point of view taken in [**Pin92a**]. These two approaches are logically equivalent. Our usage of the notations $\mathscr{X}^*$ and $\mathscr{X}_P$ agrees with [**Pin92a**, §3.6] and [**Mor10b**, §1.1].

1.4. The group-theoretic setting

In this section we fix the group-theoretic setting for our discussion of orthogonal Shimura varieties.

1.4.1. — Let (V, q) be a quadratic space over $\mathbb{Q}$, of signature $(n, 2)$. We always assume that $n \geq 3$. Let $d = \dim V = n + 2$, and let $m = \lfloor d/2 \rfloor$. Let $G = \mathrm{SO}(V)$. Throughout the paper, we shall refer to "*the odd case*" and "*the even case*" according to the parity of d.

Since $n \geq 3$, the maximal totally isotropic subspaces of $V_{\mathbb{R}}$ are of dimension 2. Throughout the paper we assume that the maximal totally isotropic subspaces of V

are also of dimension 2. If $n \geq 5$, this assumption is automatic by Meyer's theorem (see [**Ser73**, §IV.3.2 Cor. 2]). We fix a flag

$$0 \subset V_1 \subset V_2 \subset V_2^{\perp} \subset V_1^{\perp} \subset V, \tag{1.4.1.1}$$

where V_i is an i-dimensional totally isotropic $\mathbb{Q}$-subspace of V. We set

$$W_i := V_i^{\perp}/V_i,$$

for $i \in \{1,2\}$. Define

$$\begin{aligned} P_1 &:= \mathrm{Stab}_G(V_2) \subset G, \\ P_2 &:= \mathrm{Stab}_G(V_1) \subset G, \\ P_{12} &:= P_1 \cap P_2 \subset G. \end{aligned}$$

Then P_{12} is a minimal parabolic subgroup of G, and P_1 and P_2 are the only proper parabolic subgroups of G strictly containing P_{12}. If S is a non-empty subset of $\{1,2\}$, we write P_S for the one of P_1, P_2, and P_{12} corresponding to S.

1.4.2. — We fix once and for all a splitting of the flag (1.4.1.1). Then we obtain a Levi component M_S of P_S for each non-empty $S \subset \{1,2\}$. We have

$$\begin{aligned} M_1 &\cong \mathrm{GL}(V_2) \times \mathrm{SO}(W_2), \\ M_2 &\cong \mathrm{GL}(V_1) \times \mathrm{SO}(W_1), \\ M_{12} &\cong \mathrm{GL}(V_1) \times \mathrm{GL}(V_2/V_1) \times \mathrm{SO}(W_2). \end{aligned} \tag{1.4.2.1}$$

In the sequel we call parabolic subgroups of G containing P_{12} *standard*. For each standard parabolic subgroup P, we denote by M_P the Levi component of P containing M_{12}, also called *standard*, and denote by N_P the unipotent radical of P. Thus the standard proper parabolic subgroups are P_1, P_2, P_{12}, and for $P = P_S$ we have $M_P = M_S$. We also write N_S for N_{P_S}.

1.4.3. — We fix a basis $\{e_1\}$ of V_1 and a basis $\{e_2\}$ of V_2/V_1. By the fixed splitting of the flag (1.4.1.1), we can view e_2 as a vector in $V_2 \subset V$. Let $e_1' \in V/V_1^{\perp}$ and $e_2' \in V_1^{\perp}/V_2^{\perp}$ be determined by the conditions $[e_i, e_i'] = 1, i = 1,2$. We view e_1', e_2' as vectors in V as well. Under these choices we have identifications

$$\mathrm{GL}(V_i) \cong \mathrm{GL}_i,\ i \in \{1,2\}, \qquad \text{and} \qquad \mathrm{GL}(V_2/V_1) \cong \mathrm{GL}_1,$$

which we shall use freely in the sequel. In particular, the decomposition (1.4.2.1) becomes

$$M_{12} \cong \mathbb{G}_m \times \mathbb{G}_m \times \mathrm{SO}(W_2).$$

We shall refer to the factor corresponding to $\mathrm{GL}(V_1)$ as *the first* $\mathbb{G}_m$, and refer to the factor corresponding to $\mathrm{GL}(V_2/V_1)$ as *the second* $\mathbb{G}_m$.

1.4.4. — Let M be a standard proper Levi subgroup of G. We set

$$M^{\mathrm{GL}} := \begin{cases} \mathrm{GL}(V_2), \\ \mathrm{GL}(V_1), \\ \mathrm{GL}(V_1) \times \mathrm{GL}(V_2/V_1), \end{cases} \qquad M^{\mathrm{SO}} := \begin{cases} \mathrm{SO}(W_2), \\ \mathrm{SO}(W_1), \\ \mathrm{SO}(W_2), \end{cases}$$

$$M_h := \begin{cases} \mathrm{GL}(V_2), \\ \mathrm{GL}(V_1), \\ \mathrm{GL}(V_1), \end{cases} \qquad M_l := \begin{cases} \mathrm{SO}(W_2), \\ \mathrm{SO}(W_1), \\ \mathrm{GL}(V_2/V_1) \times \mathrm{SO}(W_2), \end{cases}$$

where the three cases are when $M = M_1, M_2$, and M_{12} respectively. (Here h stands for "hermitian" and l stands for "linear".) We have

$$M = M^{\mathrm{GL}} \times M^{\mathrm{SO}} = M_h \times M_l.$$

1.5. The orthogonal Shimura datum

1.5.1. — Let (V, q) and $G = \mathrm{SO}(V)$ be as in §1.4. In this paper we are concerned with the *orthogonal Shimura datum* on G. In the following we recall its definition and some basic facts. More details can be found in [**MP16**].

Consider the set $\mathcal{X}$ of oriented, negative definite, two-dimensional subspaces of $V_{\mathbb{R}}$. Then $\mathcal{X}$ is a left homogeneous space under the natural action of $G(\mathbb{R})$. Moreover, $\mathcal{X}$ has two connected components, and the action of $\pi_0(G(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$ on $\pi_0(\mathcal{X})$ is the non-trivial one.

Let $x \in \mathcal{X}$. For any $re^{i\theta} \in \mathbb{C}^\times$ (with $r \in \mathbb{R}_{>0}, \theta \in \mathbb{R}$), we let

$$\underline{h}(x)(re^{i\theta}) \in G(\mathbb{R})$$

be the element which acts on $V_{\mathbb{R}} = x \oplus x^{\perp}$ as the rotation on x by angle -2θ (according to the given orientation on x) and as the identity on $x^{\perp}$. The map $\underline{h}(x) : \mathbb{C}^\times \to G(\mathbb{R})$ comes from an $\mathbb{R}$-algebraic group homomorphism

$$h(x) : \mathbb{S} \longrightarrow G_{\mathbb{R}}.$$

Moreover, the association $x \mapsto h(x)$ is $G(\mathbb{R})$-equivariant and identifies $\mathcal{X}$ with a $G(\mathbb{R})$-conjugacy class of homomorphisms $\mathbb{S} \to G_{\mathbb{R}}$. The tuple $(G, \mathcal{X}, h)$ is a pure Shimura datum, called the *orthogonal Shimura datum*. From now on we also denote this Shimura datum by $\mathbf{O}(V)$. It is known that $\mathbf{O}(V)$ is of abelian type. In fact, the pair $(\mathrm{GSpin}(V), \mathcal{X})$ can be upgraded to a Shimura datum of Hodge type, and $\mathbf{O}(V)$ is the quotient of that by the central $\mathbb{G}_m$ in $\mathrm{GSpin}(V)$.

The Hodge cocharacter $\mu : \mathbb{G}_m \to G$ of $\mathbf{O}(V)$ (well-defined up to $G(\overline{\mathbb{Q}})$-conjugacy) is given as follows. Choose an arbitrary hyperbolic basis $\mathbb{B}$ of $V_{\overline{\mathbb{Q}}}$, and let $\iota_{\mathbb{B}} : \mathbb{G}_m^m \hookrightarrow G_{\overline{\mathbb{Q}}}$ be the embedding constructed in §1.2.7. Let $\{\epsilon_1^\vee, \cdots, \epsilon_m^\vee\}$ be the standard basis of $X_*(\mathbb{G}_m^m)$. Then μ is conjugate to $\iota_{\mathbb{B}} \circ \epsilon_1^\vee$. Moreover, it is possible to find a representative $\mu : \mathbb{G}_m \to G$ defined over $\mathbb{Q}$. In fact, we may assume that the first and the last vectors in $\mathbb{B}$ are respectively e_1 and e_1'. Then $\iota_{\mathbb{B}} \circ \epsilon_1^\vee$ is defined over $\mathbb{Q}$. Consequently, the reflex field of $\mathbf{O}(V)$ is $\mathbb{Q}$.

Next we determine some explicit information about the rational boundary components of $\mathbf{O}(V)$. We follow the notation in §1.3. In the present case the set $\mathrm{AdmPar}(G)$ consists of G and the $G(\mathbb{Q})$-conjugates of P_1 and P_2.

Proposition 1.5.2. — *The following statements hold.*

(1) *For each $P \in \mathrm{AdmPar}(G)$, the set $\mathcal{RBC}_P(\mathbf{O}(V))$ is a singleton.*

(2) *For $i = 1, 2$, we have $P_i^{\mathrm{Pink}} = M_{i,h}N_i$. In particular, $G_{P_i} = P_i^{\mathrm{Pink}}/N_i$ is naturally identified with $M_{i,h}$.*

(3) *For $i = 1, 2$, under the identification $M_{i,h} \cong \mathrm{GL}_{3-i}$, the Shimura datum $(M_{i,h}, \mathcal{X}_{P_i})$ is identified with the Siegel Shimura datum $(\mathrm{GL}_{3-i}, \mathcal{H}_{2(2-i)})$ (see* [**Pin90**, 2.7, 2.8]*).*

(4) *The action of the subgroup $M_{1,l}(\mathbb{R}) \subset M_1(\mathbb{R})$ on $\mathcal{X}_{P_1}$ is trivial.*

(5) *The groups $\pi_0(M_{2,h}(\mathbb{R}))$, $\pi_0(M_{2,l}(\mathbb{R}))$, and $\pi_0(G(\mathbb{R}))$ are all isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The map*

$$\pi_0(M_2(\mathbb{R})) \cong \pi_0(M_{2,h}(\mathbb{R})) \times \pi_0(M_{2,l}(\mathbb{R})) \longrightarrow \pi_0(G(\mathbb{R}))$$

induced by the inclusion $M_2(\mathbb{R}) \hookrightarrow G(\mathbb{R})$ is given by

$$\begin{aligned} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} &\longrightarrow \mathbb{Z}/2\mathbb{Z} \\ (a, b) &\longmapsto a + b. \end{aligned}$$

The action of $M_2(\mathbb{R})$ on $\mathcal{X}_{P_2}$ as in Proposition 1.3.7 is given by the composite map $M_2(\mathbb{R}) \to \pi_0(M_2(\mathbb{R})) \to \pi_0(G(\mathbb{R}))$ followed by the unique non-trivial action of $\pi_0(G(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$ on the two-element set $\mathcal{X}_{P_2} = \mathcal{H}_0$.

Proof. — Statements (1) (2) (3) follow from [**Hör14**, Prop. 2.4.5]. To show (4), note that $M_{1,l}(\mathbb{R}) \cong \mathrm{SO}(n-2, 0)(\mathbb{R})$ is connected, and that it commutes with $G_{P_1} = M_{1,h}$. The statement then follows from Proposition 1.3.7 (2). We now show (5). We have

$$M_{2,h}(\mathbb{R}) \cong \mathbb{R}^\times, \qquad M_{2,l}(\mathbb{R}) \cong \mathrm{SO}(n-1, 1)(\mathbb{R}), \qquad G(\mathbb{R}) \cong \mathrm{SO}(n, 2)(\mathbb{R}).$$

The first two statements in (5) follow from the standard description of the connected components of special orthogonal groups; see for instance [**Kna02**, I.17]. The third statement follows from the fact that the map $\pi_{P_2} : \mathcal{X} \to \mathcal{X}_{P_2}$ is $P_2(\mathbb{R})$-equivariant and induces a bijection $\pi_0(\mathcal{X}) \xrightarrow{\sim} \pi_0(\mathcal{X}_{P_2}) = \mathcal{X}_{P_2}$; see §1.3.6 and Proposition 1.3.7. □

1.6. Shimura varieties

From now on until the end of §1, we let $\mathbf{O}(V) = (G, \mathcal{X}, h)$ be the orthogonal Shimura datum fixed in §1.5. Let K be a neat compact open subgroup of $G(\mathbb{A}_f)$. (See [**Pin90**, 0.6] for the meaning of "neat".) As usual we define

$$\mathrm{Sh}_K(\mathbf{O}(V))(\mathbb{C}) := G(\mathbb{Q})\backslash\mathcal{X} \times G(\mathbb{A}_f)/K.$$

This is the set of $\mathbb{C}$-points of the *canonical model* $\mathrm{Sh}_K(\mathbf{O}(V))$, which is a smooth quasi-projective variety of dimension $n = d-2$ over the reflex field $\mathbb{Q}$. As $\mathbf{O}(V)$ is of abelian type, the existence of the canonical model follows from [**Del79**]. We write Sh_K for $\mathrm{Sh}_K(\mathbf{O}(V))$.

Let K_1 and K_2 be neat compact open subgroups of $G(\mathbb{A}_f)$, and let g be an element of $G(\mathbb{A}_f)$ such that $K_1 \subset gK_2g^{-1}$. We have a finite étale $\mathbb{Q}$-morphism

$$[\cdot g]_{K_1,K_2} : \mathrm{Sh}_{K_1} \longrightarrow \mathrm{Sh}_{K_2}$$

called a *Hecke operator*. On $\mathbb{C}$-points, it is induced by

$$\begin{aligned} \mathcal{X} \times G(\mathbb{A}_f) &\longrightarrow \mathcal{X} \times G(\mathbb{A}_f) \\ (x,k) &\longmapsto (x,kg). \end{aligned}$$

When the context is clear we will simply write $[\cdot g]$ for $[\cdot g]_{K_1,K_2}$.

We recall the following facts proved in [**Pin90**, 12.3]. For any neat compact open subgroup $K \subset G(\mathbb{A}_f)$, the Shimura variety Sh_K has the canonical Baily–Borel compactification

$$j : \mathrm{Sh}_K \longrightarrow \overline{\mathrm{Sh}_K},$$

where $\overline{\mathrm{Sh}_K}$ is a normal projective variety over $\mathbb{Q}$, and j is a dense open embedding defined over $\mathbb{Q}$. At the level of $\mathbb{C}$-points, we have

$$\overline{\mathrm{Sh}_K}(\mathbb{C}) = G(\mathbb{Q})\backslash\mathcal{X}^* \times G(\mathbb{A}_f)/K,$$

where $\mathcal{X}^*$ is the $G(\mathbb{Q})$-set defined in (1.3.6.1), and j is induced by $\omega_G : \mathcal{X} \xrightarrow{\sim} \mathcal{X}_G \hookrightarrow \mathcal{X}^*$. For K_1, K_2, and g as in the last paragraph, the morphism $[\cdot g] : \mathrm{Sh}_{K_1} \to \mathrm{Sh}_{K_2}$ uniquely extends to a finite $\mathbb{Q}$-morphism $\overline{[\cdot g]} = \overline{[\cdot g]}_{K_1,K_2} : \overline{\mathrm{Sh}_{K_1}} \to \overline{\mathrm{Sh}_{K_2}}$.

1.7. Automorphic λ-adic sheaves

1.7.1. — Let $\mathbb{V}$ be a finite-dimensional vector space over a number field $\mathbb{E}$ equipped with a G-representation, i.e., an $\mathbb{E}$-algebraic group homomorphism $G_{\mathbb{E}} \to \mathrm{GL}(\mathbb{V})$. Let λ be a finite place of $\mathbb{E}$. Then by a well-known construction, for any neat compact open subgroup $K \subset G(\mathbb{A}_f)$ there is an $\mathbb{E}_\lambda$-sheaf on Sh_K associated to $\mathbb{V}$, which we denote by $\mathcal{F}^K\mathbb{V}$. Moreover, for each Hecke operator $[\cdot g] : \mathrm{Sh}_{K_1} \to \mathrm{Sh}_{K_2}$ (with K_1, K_2 neat), there is a canonical isomorphism

$$\mathcal{F}_{[\cdot g]} : \mathcal{F}^{K_1}\mathbb{V} \xrightarrow{\sim} [\cdot g]^*\mathcal{F}^{K_2}\mathbb{V}. \tag{1.7.1.1}$$

We refer the reader to [**Pin92a**, §5.1] and [**KSZ**, §1.5] for more details.

Let ℓ be the rational prime below λ, and fix a $\mathbb{Q}_\ell$-algebra embedding $\mathbb{E}_\lambda \hookrightarrow \overline{\mathbb{Q}}_\ell$. Let K be as above. We view the $\mathbb{E}_\lambda$-sheaf $\mathcal{F}^K\mathbb{V}$ as a $\overline{\mathbb{Q}}_\ell$-sheaf and keep the same notation. We have the intersection complex

$$\mathrm{IC}^K\,\mathbb{V} := \Big(j_{!*}\big((\mathcal{F}^K\mathbb{V})[n]\big)\Big)[-n] \in D^b_c(\overline{\mathrm{Sh}_K}, \overline{\mathbb{Q}}_\ell).$$

Here j is the open embedding $\mathrm{Sh}_K \hookrightarrow \overline{\mathrm{Sh}_K}$, and remember that $n = \dim \mathrm{Sh}_K$.

1.7.2. — We have analogues of the canonical isomorphisms (1.7.1.1) for the intersection complexes, which we now explain. Consider a Hecke operator $[\cdot g] : \mathrm{Sh}_{K_1} \to \mathrm{Sh}_{K_2}$ and its extension $\overline{[\cdot g]} : \overline{\mathrm{Sh}_{K_1}} \to \overline{\mathrm{Sh}_{K_2}}$. To ease notation we write g for $[\cdot g]$ and write $\bar{g}$ for $\overline{[\cdot g]}$. For $i = 1, 2$, we write $\mathcal{F}_i$ and IC_i for $\mathcal{F}^{K_i}\mathbb{V}$ and $\mathrm{IC}^{K_i}\,\mathbb{V}$ respectively, and write j_i for the open embedding $\mathrm{Sh}_{K_i} \to \overline{\mathrm{Sh}_{K_i}}$.

For any $\mathscr{F} \in D^b_c(\mathrm{Sh}_{K_2}, \overline{\mathbb{Q}}_\ell)$, we have the commutative diagram

$$\begin{array}{ccc} \bar{g}^* Rj_{2,!}\mathscr{F} & \longrightarrow & Rj_{1,!}g^*\mathscr{F} \\ \downarrow & & \downarrow \\ \bar{g}^* Rj_{2,*}\mathscr{F} & \longrightarrow & Rj_{1,*}g^*\mathscr{F} \end{array}$$

where the horizontal maps are the base change maps, and the vertical maps are induced by the natural maps $Rj_{i,!}(\cdot) \to Rj_{i,*}(\cdot),\ i = 1, 2$. Since $\bar{g}$ is finite (see §1.6), $\bar{g}^*$ is exact with respect to the (middle-perversity) perverse t-structures. Therefore the above commutative diagram induces a natural map

$$\bar{g}^* j_{2,!*}\mathscr{F} \longrightarrow j_{1,!*}g^*\mathscr{F}. \tag{1.7.2.1}$$

Taking $\mathscr{F}$ to be $\mathcal{F}_2[n]$, we obtain a map

$$\bar{g}^* j_{2,!*}(\mathcal{F}_2[n]) \longrightarrow j_{1,!*}g^*(\mathcal{F}_2[n]).$$

The composition of the above map followed by $j_{1,!*}(\mathcal{F}^{-1}_{[\cdot g]})$ gives a map

$$\bar{g}^* j_{2,!*}(\mathcal{F}_2[n]) \longrightarrow j_{1,!*}(\mathcal{F}_1[n]).$$

Shifting by $[-n]$ we obtain a map

$$\bar{g}^*\,\mathrm{IC}_2 \longrightarrow \mathrm{IC}_1\,. \tag{1.7.2.2}$$

Similarly, using the base co-change maps (see [**SGA73**, XVIII])

$$Rj_{1,!}g^!\mathscr{F} \longrightarrow \bar{g}^! Rj_{2,!}\mathscr{F},$$
$$Rj_{1,*}g^!\mathscr{F}, \longrightarrow \bar{g}^! Rj_{2,*}\mathscr{F},$$

we obtain a map

$$j_{1,!*}g^!\mathscr{F} \longrightarrow \bar{g}^! j_{2,!*}\mathscr{F} \tag{1.7.2.3}$$

as a counterpart of (1.7.2.1). Note that because g is finite étale (see §1.6), we have $g^! = g^*$. Again, taking $\mathscr{F}$ to be $\mathcal{F}_2[n]$ in (1.7.2.3), pre-composing with $j_{1,!*}(\mathcal{F}_{[\cdot g]})$, and shifting by $[-n]$, we obtain a map

$$\mathrm{IC}_1 \longrightarrow \bar{g}^!\,\mathrm{IC}_2\,. \tag{1.7.2.4}$$

Now for Hecke operators $[\cdot g_1]_{K',K_1}$ and $[\cdot g_2]_{K',K_2}$, we obtain a canonical cohomological correspondence

$$\mathscr{H}_{g_1,g_2,K_1,K_2,K'} : \ \bar{g}_1^* \, \mathrm{IC}^{K_1} \, \mathbb{V} \longrightarrow \bar{g}_2^! \, \mathrm{IC}^{K_2} \, \mathbb{V} \tag{1.7.2.5}$$

by composing (1.7.2.2) for $g = g_1$ with (1.7.2.4) for $g = g_2$.

1.8. Intersection cohomology and Morel's formula

1.8.1. — Keep the setting of §1.7.1. Let K be a neat compact open subgroup of $G(\mathbb{A}_f)$. Define

$$\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V}) := \mathbf{H}^*_{\text{ét}}(\overline{\mathrm{Sh}_K} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \ \mathrm{IC}^K \, \mathbb{V}),$$
$$\mathbf{H}^*_c(\mathrm{Sh}_K, \mathbb{V}) := \mathbf{H}^*_{\text{ét},c}(\mathrm{Sh}_K \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \ \mathcal{F}^K \mathbb{V}),$$

which we view as graded $\overline{\mathbb{Q}}_\ell$-vector spaces. We denote by $\mathcal{H}(G(\mathbb{A}_f) \mathbin{/\!\!/} K)_{\mathbb{Q}}$ the Hecke algebra of $\mathbb{Q}$-valued smooth compactly supported K-bi-invariant distributions on $G(\mathbb{A}_f)$. If we choose a Haar measure dg^∞ on $G(\mathbb{A}_f)$ that gives rational volumes to compact open subgroups, then each element of $\mathcal{H}(G(\mathbb{A}_f) \mathbin{/\!\!/} K)_{\mathbb{Q}}$ can be uniquely written as $f^\infty dg^\infty$, where f^∞ is a smooth compactly supported K-bi-invariant function $G(\mathbb{A}_f) \to \mathbb{Q}$. We have commuting actions of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\mathcal{H}(G(\mathbb{A}_f) \mathbin{/\!\!/} K)_{\mathbb{Q}}$ on $\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})$ and $\mathbf{H}^*_c(\mathrm{Sh}_K, \mathbb{V})$. Here the $\mathcal{H}(G(\mathbb{A}_f) \mathbin{/\!\!/} K)_{\mathbb{Q}}$-action on $\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})$ is characterized as follows. For any $g \in G(\mathbb{A}_f)$, the element

$$1_{KgK} \cdot \mathrm{vol}_{dg^\infty}(K)^{-1} dg^\infty \in \mathcal{H}(G(\mathbb{A}_f) \mathbin{/\!\!/} K)_{\mathbb{Q}}$$

depends only on g and not on the choice of dg^∞. We require that this element acts on $\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})$ via the endomorphism induced by the cohomological correspondence

$$\mathscr{H}_{g,1,K,K,gKg^{-1}\cap K} : \ \bar{g}^* \, \mathrm{IC}^K \, \mathbb{V} \longrightarrow \bar{1}^! \, \mathrm{IC}^K \, \mathbb{V},$$

where the notation is as in (1.7.2.5). By linearity, this determines the $\mathcal{H}(G(\mathbb{A}_f) \mathbin{/\!\!/} K)_{\mathbb{Q}}$-action on $\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})$. The $\mathcal{H}(G(\mathbb{A}_f) \mathbin{/\!\!/} K)_{\mathbb{Q}}$-action on $\mathbf{H}^*_c(\mathrm{Sh}_K, \mathbb{V})$ is characterized similarly.

If p is a prime and K^p is a compact open subgroup of $G(\mathbb{A}_f^p)$, we denote by $\mathcal{H}(G(\mathbb{A}_f^p) \mathbin{/\!\!/} K^p)_{\mathbb{Q}}$ the Hecke algebra of $\mathbb{Q}$-valued smooth compactly supported K^p-bi-invariant distributions on $G(\mathbb{A}_f^p)$. Similarly as before, its elements can be written as $f^{p,\infty} dg^{p,\infty}$, where $f^{p,\infty}$ is a function $G(\mathbb{A}_f^p) \to \mathbb{Q}$ and $dg^{p,\infty}$ is a Haar measure on $G(\mathbb{A}_f^p)$ giving rational volumes to compact open subgroups.

Definition 1.8.2. — Let K be a compact open subgroup of $G(\mathbb{A}_f)$ and let p be a prime number.

(1) We say that p is a *hyperspecial prime* for K, if we have $K = K^p K_p$, with K_p a hyperspecial subgroup of $G(\mathbb{Q}_p)$, and K^p a compact open subgroup of $G(\mathbb{A}_f^p)$.

(2) Let $f^\infty dg^\infty \in \mathcal{H}(G(\mathbb{A}_f) \mathbin{/\!\!/} K)_{\mathbb{Q}}$. We say that p is an *unramified prime* for $f^\infty dg^\infty$, if p is hyperspecial for K, and we have $f^\infty dg^\infty = f^{p,\infty} dg^{p,\infty} 1_{K_p} dg_p$, where

$f^{p,\infty}dg^{p,\infty}$ is an element of $\mathcal{H}(G(\mathbb{A}_f^p) /\!\!/ K^p)_{\mathbb{Q}}$, $1_{K_p} : G(\mathbb{Q}_p) \to \mathbb{Q}$ is the characteristic function of K_p, and dg_p is a Haar measure on $G(\mathbb{Q}_p)$ giving rational volumes to compact open subgroups.

1.8.3. — Fix a neat compact open subgroup K of $G(\mathbb{A}_f)$, and fix $f^\infty dg^\infty \in \mathcal{H}(G(\mathbb{A}_f) /\!\!/ K)_{\mathbb{Q}}$. Let Σ_0 be the finite set consisting of the prime ℓ, the primes not hyperspecial for K, and the primes not unramified for $f^\infty dg^\infty$. For each prime $p \notin \Sigma_0$, we write $K = K^p K_p$ and $f^\infty dg^\infty = f^{p,\infty} dg^{p,\infty} 1_{K_p} dg_p$ as in Definition 1.8.2. Without loss of generality, we may and shall assume that $\mathrm{vol}_{dg_p}(K_p) = 1$ by rescaling $f^{p,\infty}dg^{p,\infty}$.

Recall from §1.7.1 that we have fixed an embedding $\mathbb{E}_\lambda \hookrightarrow \overline{\mathbb{Q}}_\ell$. We now also fix a field embedding $\mathbb{E} \hookrightarrow \mathbb{C}$. For any endomorphism u of the graded $\overline{\mathbb{Q}}_\ell$-vector space $\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})$, we write $\mathrm{Tr}(u \mid \mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V}))$ for the alternating sum

$$\sum_k (-1)^k \,\mathrm{Tr}(u \mid \mathbf{IH}^k(\overline{\mathrm{Sh}_K}, \mathbb{V})) \in \overline{\mathbb{Q}}_\ell.$$

(The sum is finite, since the terms are zero unless $0 \le k \le 2\dim \mathrm{Sh}_K$.) The same convention is applied to $\mathbf{H}_c^*(\mathrm{Sh}_K, \mathbb{V})$.

***Theorem 1.8.4* (Morel's formula).** — *In the setting of §1.8.3, there exists a finite set of prime numbers $\Sigma = \Sigma(\mathbf{O}(V), \mathbb{V}, \lambda, K, f^\infty)$ containing Σ_0 such that the following statements hold for all primes $p \notin \Sigma$.*

(1) *The actions of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *on* $\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})$ *and on* $\mathbf{H}_c^*(\mathrm{Sh}_K, \mathbb{V})$ *are both unramified at p.*

(2) *Let* $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *be a geometric Frobenius at p. There exists a positive integer $a_0 = a_0(\mathbf{O}(V), \mathbb{V}, \lambda, K, f^\infty, p)$ such that for all integers $a \ge a_0$ we have*

$$\begin{aligned}(1.8.4.1)\quad & \mathrm{Tr}(\mathrm{Frob}_p^a \times f^\infty dg^\infty \mid \mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})) \\ &= \mathrm{Tr}(\mathrm{Frob}_p^a \times f^\infty dg^\infty \mid \mathbf{H}_c^*(\mathrm{Sh}_K, \mathbb{V})) + \sum_M \mathrm{Tr}_M(f^{p,\infty}dg^{p,\infty}, K, a).\end{aligned}$$

Here in the summation M runs through the standard proper Levi subgroups of G, and $\mathrm{Tr}_M(f^{p,\infty}dg^{p,\infty}, K, a)$ *will be given in Definition 2.4.3 below (which depends on the embedding $\mathbb{E} \hookrightarrow \mathbb{C}$). The two sides of (1.8.4.1) are* a priori *numbers in $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$ respectively, but they actually both lie in $\mathbb{E}$.*

The proof of Theorem 1.8.4 will be given in §3.

Remark 1.8.5. — We expect that Theorem 1.8.4 should in fact hold for $\Sigma = \Sigma_0$. The proof of this would require a robust theory of integral models of the Baily–Borel compactification and the toroidal compactifications of Sh_K at all hyperspecial primes, which is currently unavailable. See §3.1 below for a more detailed discussion.

CHAPTER 2

DEFINITION OF THE TERMS IN MOREL'S FORMULA

In this chapter we define the terms $\mathrm{Tr}_M(f^{p,\infty}dg^{p,\infty}, K, a)$ in Theorem 1.8.4. We keep the setting in §1.4–§1.8. In particular, we fix $\mathbb{E} \hookrightarrow \mathbb{C}$ as in §1.8.3.

2.1. Truncated Lie algebra cohomology

Definition 2.1.1. — For $i \in \{1, 2\}$, let $\varpi_i : \mathbb{G}_m \to M_{i,h}$ be the weight cocharacter of the Shimura datum $(M_{i,h}, \mathcal{X}_{P_i})$, and let $t_i = \dim \mathcal{X}_{P_i} - \dim \mathcal{X}$. (Here dim means the complex dimension.)

Lemma 2.1.2. — *The following statements hold.*

(1) *The cocharacter* ϖ_1 *of* $M_{1,h} = \mathrm{GL}(V_2) \cong \mathrm{GL}_2$ *is given by* $z \mapsto \mathrm{diag}(z, z)$.
(2) *The cocharacter* ϖ_2 *of* $M_{2,h} = \mathrm{GL}(V_1) \cong \mathbb{G}_m$ *is given by* $z \mapsto z^2$.
(3) *We have* $t_1 = 3 - d$, *and* $t_2 = 2 - d$.

Proof. — By Proposition 1.5.2, we have $(M_{i,h}, \mathcal{X}_{P_i}) \cong (\mathrm{GL}_{2-i}, \mathcal{H}_{2(2-i)})$. The statements about ϖ_1 and ϖ_2 are clear. To determine t_1 and t_2, we use that $\dim \mathcal{X} = n = d - 2$, $\dim \mathcal{X}_{P_1} = 1$, and $\dim \mathcal{X}_{P_2} = 0$. □

2.1.3. — Let S be a non-empty subset of $\{1, 2\}$. By Kostant's theorem [**Kos61**] (cf. [**GHM94**, §11] or §4.3 below), the Lie algebra cohomology

$$\mathbf{H}^k(\mathrm{Lie}(N_S)_{\mathbb{C}}, \mathbb{V} \otimes_{\mathbb{E}} \mathbb{C})$$

is a finite-dimensional algebraic representation of $M_S(\mathbb{C})$, and is non-zero only for finitely many non-negative integers k. For $i \in S$, since we have $M_i = M_{i,h} \times M_{i,l}$ and since ϖ_i is a central cocharacter of $M_{i,h}$ defined over $\mathbb{Q}$, we know that ϖ_i is a cocharacter of the split component A_{M_i} of M_i, and *a fortiori* a cocharacter of the split component A_{M_S} of M_S.

Definition 2.1.4. — Let S be a non-empty subset of $\{1,2\}$. We write

$$\mathbf{H}^k(\mathrm{Lie}(N_S)_{\mathbb{C}}, \mathbb{V}\otimes_{\mathbb{E}}\mathbb{C})_{>t_S}$$

for the maximal $M_S(\mathbb{C})$-sub-representation of $\mathbf{H}^k(\mathrm{Lie}(N_S)_{\mathbb{C}}, \mathbb{V}\otimes_{\mathbb{E}}\mathbb{C})$ on which ϖ_i has weights strictly greater than t_i for each $i \in S$. (Here we say that a $\mathbb{G}_m$-representation has weights greater than a number t if all the appearing characters $z \mapsto z^k$ satisfy $k > t$.) We define the virtual $M_S(\mathbb{C})$-representation:

$$R\Gamma(\mathrm{Lie}\, N_S, \mathbb{V})_{>t_S} := \sum_{k\geq 0}(-1)^k\, \mathbf{H}^k(\mathrm{Lie}(N_S)_{\mathbb{C}}, \mathbb{V}\otimes_{\mathbb{E}}\mathbb{C})_{>t_S}.$$

When $P = P_S$ is fixed in the context, we also replace the symbol "$> t_S$" by "$> t_P$".

2.2. The Kostant–Weyl term L_M

In this section, let M be a standard proper Levi subgroup of G, i.e., $M \in \{M_1, M_2, M_{12}\}$.

Definition 2.2.1. — Let $\mathcal{P}(M)$ be the set of pairs (P,g), where P is a standard proper parabolic subgroup of G, and g is an element of $G(\mathbb{Q})$, satisfying the following conditions.

(1) We have $M_h = M_{P,h}$, and M_l is a Levi subgroup of $M_{P,l}$. In particular, $M \subset M_P$.

(2) The element g centralizes $M_h \subset G$, and normalizes $M_l \subset G$. In particular, g normalizes $M \subset G$.

Let $\sim$ be the equivalence relation on $\mathcal{P}(M)$ such that $(P,g)\sim(P',g')$ if and only if $P = P'$ and $g \in M_P(\mathbb{Q})g'M(\mathbb{Q})$. (Here M_P is the standard Levi component of P, which may not be the same as M.) For any standard proper parabolic subgroup Q of G, let

$$\mathcal{P}(M,Q) := \{(P,g)\in\mathcal{P}(M) \mid P = Q\} \subset \mathcal{P}(M).$$

Definition 2.2.2. — Set $\mathtt{m}_M$ to be 1 if $M = M_1$, and 2 if $M \in \{M_2, M_{12}\}$. For $\gamma \in M(\mathbb{R})$ and $(P,g)\in\mathcal{P}(M)$, define the complex number

$$\begin{aligned} L_{M,P,g}(\gamma) := \mathtt{m}_M(-1)^{\dim A_M/A_{M_P}}(n_M^{M_P})^{-1}\left|D_M^{M_P}(g\gamma g^{-1})\right|_{\mathbb{R}}^{1/2} \\ \cdot\,\delta_{P(\mathbb{R})}(g\gamma g^{-1})^{1/2}\,\mathrm{Tr}(g\gamma g^{-1} \mid R\Gamma(\mathrm{Lie}\,N_P,\mathbb{V})_{>t_P}).\end{aligned}$$

Here the terms $n_M^{M_P}$, $D_M^{M_P}(\cdot)$, $\delta_{P(\mathbb{R})}(\cdot)$ are all defined in §1.1, and $R\Gamma(\mathrm{Lie}\,N_P,\mathbb{V})_{>t_P}$ is as in Definition 2.1.4.

It is easy to see that $L_{M,P,g}(\gamma)$ depends on (P,g) only via the $\sim$-equivalence class of (P,g). We use this fact in the next definition.

Definition 2.2.3. — For $\gamma \in M(\mathbb{R})$, define the *Kostant–Weyl term*

$$L_M(\gamma) := \sum_{(P,g)\in\mathcal{P}(M)/\sim} |\mathcal{P}(M,P)/\sim|^{-1} L_{M,P,g}(\gamma) \in \mathbb{C}. \tag{2.2.3.1}$$

Proposition 2.2.4. — *Let $i = 1$ or 2. Then every element of $\mathcal{P}(M_i)$ is $\sim$-equivalent to $(P_i, 1)$. In particular, for $\gamma \in M_i(\mathbb{R})$ we have*

$$L_{M_i}(\gamma) = \mathtt{m}_{M_i}\delta_{P_i(\mathbb{R})}(\gamma)^{1/2}\operatorname{Tr}(\gamma \mid R\Gamma(\operatorname{Lie} N_i, \mathbb{V})_{>t_i}).$$

Proof. — It is clear that $(P_i, 1) \in \mathcal{P}(M_i)$. Let $(P, g) \in \mathcal{P}(M_i)$. By condition (1) in Definition 2.2.1, we have $P = P_i$. Since $M_{i,h}$ contains A_{M_i}, the centralizer of $M_{i,h}$ in G is contained in M_i. Hence by condition (2) in Definition 2.2.1, we have $g \in M_i(\mathbb{Q})$. It then follows that $(P, g) \sim (P_i, 1)$. □

2.2.5. — Next we give an explicit description of the set $\mathcal{P}(M_{12})/\sim$. Recall from §1.4 that we have identified V with the orthogonal direct sum of $\operatorname{span}_{\mathbb{Q}}\{e_1, e_1'\}$ and W_1, and identified W_1 with the orthogonal direct sum of $\operatorname{span}_{\mathbb{Q}}\{e_2, e_2'\}$ and W_2. Also recall that $M_{2,l} = \mathrm{SO}(W_1) \subset M_2 = \mathrm{GL}(V_1) \times \mathrm{SO}(W_1)$.

Definition 2.2.6. — Let $M_{2,l}(\mathbb{Q})^\sharp$ be the set consisting of $g \in M_{2,l}(\mathbb{Q})$ satisfying the following conditions:

(1) $g(e_2) = e_2'$, $g(e_2') = e_2$.

(2) g stabilizes W_2, and $g|_{W_2}$ is an element of $\mathrm{O}(W_2)(\mathbb{Q})$ with determinant -1.

Remark 2.2.7. — Since $\dim W_2 = n - 2 \geq 1$, the group $\mathrm{O}(W_2)(\mathbb{Q})$ indeed contains elements with determinant -1. It is then clear that $M_{2,l}(\mathbb{Q})^\sharp \neq \emptyset$.

Proposition 2.2.8. — *The set $\mathcal{P}(M_{12}, P_1)$ is empty. Every element of $\mathcal{P}(M_{12}, P_2)$ is $\sim$-equivalent to $(P_2, 1)$. The set $\mathcal{P}(M_{12}, P_{12})$ is the union of exactly two $\sim$-equivalence classes, and they are represented by $(P_{12}, 1)$ and (P_{12}, g_0), where g_0 is any element of $M_{2,l}(\mathbb{Q})^\sharp$.*

Proof. — Since $M_{12,l}$ is not contained in $M_{1,l}$, we have $\mathcal{P}(M_{12}, P_1) = \emptyset$. Since $M_{12,h} = \mathrm{GL}(V_1) = A_{M_2}$, by condition (2) in Definition 2.2.1 we know that any $(P, g) \in \mathcal{P}(M_{12})$ must satisfy $g \in \operatorname{Nor}_{M_2}(M_{12,l})(\mathbb{Q})$. Conversely, for any $g \in \operatorname{Nor}_{M_2}(M_{12,l})(\mathbb{Q})$, we have $(P_2, g), (P_{12}, g) \in \mathcal{P}(M_{12})$. The statement about $\mathcal{P}(M_{12}, P_2)$ immediately follows.

To show the last statement about $\mathcal{P}(M_{12}, P_{12})$, we know from the above discussion that we have a surjection

$$\begin{aligned}\operatorname{Nor}_{M_2}(M_{12,l})(\mathbb{Q}) &\longrightarrow \mathcal{P}(M_{12}, P_{12})/\sim\\ g &\longmapsto (P_{12}, g).\end{aligned}$$

This surjection restricts to a surjection

$$\operatorname{Nor}_{M_{2,l}}(M_{12,l})(\mathbb{Q}) \longrightarrow \mathcal{P}(M_{12}, P_{12})/\sim,$$

which induces a bijection (see Definition 1.1.1 for the notation)

$$\mathcal{W}_{M_{12,l}}^{M_{2,l}} \xrightarrow{\sim} \mathcal{P}(M_{12}, P_{12})/\sim.$$

Now note that $\mathrm{GL}(V_2/V_1) \cong \mathbb{G}_m$ is the split component of $M_{12,l}$. As in Remark 1.1.2, we have an injective homomorphism

$$\begin{aligned} \mathcal{W}_{M_{12,l}}^{M_{2,l}} &\longrightarrow \mathrm{Aut}(\mathrm{GL}(V_2/V_1)) \cong \mathbb{Z}/2\mathbb{Z} \\ g &\longmapsto \mathrm{Int}(g)|_{\mathrm{GL}(V_2/V_1)}. \end{aligned}$$

The desired statement follows from the fact that for all $g_0 \in M_{2,l}(\mathbb{Q})^\sharp$, we have $g_0 \in \mathrm{Nor}_{M_{2,l}}(M_{12,l})(\mathbb{Q})$, and $\mathrm{Int}(g_0)|_{\mathrm{GL}(V_2/V_1)}$ is non-trivial. □

2.3. Definitions related to Kottwitz's fixed point formula

2.3.1. — Let M_h be the reductive group GL_i over $\mathbb{Q}$, where $i = 1$ or 2. We equip M_h with the Siegel Shimura datum $\mathcal{H}_{2(2-i)}$ (see [**Pin90**, 2.7, 2.8]). We define some group-theoretic terms that appear in Kottwitz's fixed point formula for the Shimura varieties associated to $(M_h, \mathcal{H}_{2(2-i)})$. The main reference is [**Kot90**, Part I]; see also [**Mor10b**, §1.6]. We fix a prime p, and an integer $a \geq 1$.

Define a cocharacter μ of M_h as follows. When $M_h = \mathbb{G}_m$, let μ be the identity cocharacter. When $M_h = \mathrm{GL}_2$, let μ be $z \mapsto \mathrm{diag}(z, 1)$. Thus μ is a Hodge cocharacter for the Shimura datum $(M_h, \mathcal{H}_{2(2-i)})$.

The following definition is equivalent to the standard definition as in [**Kot92b**, §19] or [**Mor10b**, §1.6]; it appears simpler since in the group M_h stable conjugacy is the same as conjugacy.

Definition 2.3.2. — A *Kottwitz triple* in M_h (of level p^a, for the Shimura datum $(M_h, \mathcal{H}_{2(2-i)})$) is a triple $(\gamma_0, \gamma, \delta)$, with $\gamma_0 \in M_h(\mathbb{Q})$, $\gamma \in M_h(\mathbb{A}_f^p)$, $\delta \in M_h(\mathbb{Q}_{p^a})$, satisfying the following conditions:

(1) The element γ_0 is semi-simple and $\mathbb{R}$-elliptic (see Definition 1.1.8).

(2) The element γ is conjugate to γ_0 in $M_h(\mathbb{A}_f^p)$.

(3) The element $\mathrm{N}\,\delta := \delta\sigma(\delta)\cdots\sigma^{a-1}(\delta) \in M_h(\mathbb{Q}_{p^a})$ is conjugate to γ_0 in $M_h(\mathbb{Q}_{p^a})$.

(4) If $M_h = \mathbb{G}_m$, then the p-adic valuation of $\delta \in \mathbb{Q}_{p^a}^\times$ is -1. If $M_h = \mathrm{GL}_2$, then the p-adic valuation of the determinant of $\delta \in \mathrm{GL}_2(\mathbb{Q}_{p^a})$ is -1.

Two Kottwitz triples $(\gamma_0, \gamma, \delta)$ and $(\gamma_0', \gamma', \delta')$ are said to be *equivalent*, if γ_0 is conjugate to γ_0' in $M_h(\mathbb{Q})$, and δ is σ-conjugate to δ' inside $M_h(\mathbb{Q}_{p^a})$. In the sequel, it is understood that whenever Kottwitz triples appear in a summation, they are taken up to equivalence.

Remark 2.3.3. — Abstractly, condition (4) in Definition 2.3.2 says that the image of δ in $\pi_1(M_h)_{\Gamma_p}$ under the Kottwitz map is equal to that of $-\mu$.

2.3.4. — Let $(\gamma_0, \gamma, \delta)$ be a Kottwitz triple. Let $I_0 = M_{h,\gamma_0}$ be the centralizer (which is connected) of γ_0 in M_h. Define $\mathfrak{K}(I_0/\mathbb{Q})$ to be the finite abelian group consisting of those elements of $\pi_0([Z(\widehat{I_0})/Z(\widehat{M_h})]^{\Gamma_\mathbb{Q}})$ whose images in $\mathbf{H}^1(\Gamma_\mathbb{Q}, Z(\widehat{M_h}))$ are locally trivial; see [**Kot86**, §4.6]. In [**Kot90**, §2] Kottwitz defines an invariant

$$\alpha(\gamma_0, \gamma, \delta) \in \mathfrak{K}(I_0/\mathbb{Q})^D$$

of the triple $(\gamma_0, \gamma, \delta)$. Here $\mathfrak{K}(I_0/\mathbb{Q})^D$ is the Pontryagin dual of $\mathfrak{K}(I_0/\mathbb{Q})$.

Lemma 2.3.5. — *For $M_h = \mathrm{GL}_1$ or GL_2, we always have $\mathfrak{K}(I_0/\mathbb{Q}) = 0$.*

Proof. — If $I_0 = M_h$ then obviously $\mathfrak{K}(I_0/\mathbb{Q}) = 0$. Thus we may assume that $M_h = \mathrm{GL}_2$ and that γ_0 in non-central. Then I_0 is a maximal torus T in GL_2 defined over $\mathbb{Q}$. In this case $Z(\widehat{I_0}) = \widehat{I_0} = \widehat{T}$. Since the Galois action on $Z(\widehat{\mathrm{GL}_2})$ is trivial, by Chebotarev's density theorem the only locally trivial element of $\mathbf{H}^1(\Gamma_\mathbb{Q}, Z(\widehat{\mathrm{GL}_2}))$ is the trivial element. In view of the exact sequence

$$\pi_0(Z(\widehat{\mathrm{GL}_2})^{\Gamma_\mathbb{Q}}) \to \pi_0(\widehat{T}^{\Gamma_\mathbb{Q}}) \to \pi_0([\widehat{T}/Z(\widehat{\mathrm{GL}_2})]^{\Gamma_\mathbb{Q}}) \to \mathbf{H}^1(\Gamma_\mathbb{Q}, Z(\widehat{\mathrm{GL}_2})),$$

it suffices to show that

$$\widehat{T}^{\Gamma_\mathbb{Q}} \subset Z(\widehat{\mathrm{GL}_2}).$$

Since γ_0 is $\mathbb{R}$-elliptic, $T_\mathbb{R}$ is an elliptic maximal torus in $\mathrm{GL}_{2,\mathbb{R}}$. Hence there exists an identification $\widehat{T} \cong \mathbb{C}^\times \times \mathbb{C}^\times$ such that the non-trivial element of Γ_∞ acts on $\widehat{T}$ by switching the two coordinates. It follows that $\widehat{T}^{\Gamma_\infty} \subset Z(\widehat{\mathrm{GL}_2})$, and *a fortiori* $\widehat{T}^{\Gamma_\mathbb{Q}} \subset Z(\widehat{\mathrm{GL}_2})$. □

2.3.6. — Let $(\gamma_0, \gamma, \delta)$ be a Kottwitz triple. By Lemma 2.3.5, the invariant $\alpha(\gamma_0, \gamma, \delta)$ automatically vanishes. Hence as in [**Kot90**, §3], there is an inner form I of I_0 over $\mathbb{Q}$ satisfying the following conditions.

– The group $I_\mathbb{R}$ is anisotropic modulo center.
– For any finite place v of $\mathbb{Q}$ not equal to p, $I_{\mathbb{Q}_v}$ is the trivial inner form of $I_{0,\mathbb{Q}_v}$.
– The inner form $I_{\mathbb{Q}_p}$ of $I_{0,\mathbb{Q}_p}$ is isomorphic (as an inner from) to the σ-centralizer $(M_h)_{\delta\sigma}$ of δ in M_h (which is denoted by $I(p)$ in *loc. cit.*).

We refer the reader to *loc. cit.* for more details.

Fix Haar measures on $I(\mathbb{Q}_p)$, $I(\mathbb{A}_f^p)$, and $I(\mathbb{R})$ such that the product Haar measure on $I(\mathbb{A})$ is the Tamagawa measure. Fix a Haar measure on $M_h(\mathbb{Q}_{p^a})$ such that $M_h(\mathbb{Z}_{p^a})$ has volume 1. Fix Haar measures on $M_h(\mathbb{R})$ and $M_h(\mathbb{A}_f^p)$ arbitrarily.

Definition 2.3.7. — In the setting of §2.3.6, we define

$$c(\gamma_0, \gamma, \delta) := c_1(\gamma_0, \gamma, \delta) c_2(\gamma_0, \gamma, \delta),$$

where

$$c_1(\gamma_0, \gamma, \delta) = \mathrm{vol}(I(\mathbb{Q}) \backslash I(\mathbb{A}_f)) = \tau(I)\, \mathrm{vol}(A_{M_h}(\mathbb{R})^0 \backslash I(\mathbb{R}))^{-1},$$
$$c_2(\gamma_0, \gamma, \delta) = \left| \ker(\ker^1(\mathbb{Q}, I_0) \to \ker^1(\mathbb{Q}, M_h)) \right|.$$

Here $\tau(I)$ is the Tamagawa number of I.

Definition 2.3.8. — In the setting of §2.3.6, we define the *orbital integral along* γ to be the functional

$$O_\gamma : C_c^\infty(M_h(\mathbb{A}_f^p)) \longrightarrow \mathbb{C}$$
$$f \longmapsto O_\gamma(f) = \int_{M_{h,\gamma}(\mathbb{A}_f^p)\backslash M_h(\mathbb{A}_f^p)} f(g^{-1}\gamma g),$$

with respect to the fixed Haar measure on $M_h(\mathbb{A}_f^p)$ and the Haar measure on $M_{h,\gamma}(\mathbb{A}_f^p)$ transferred from $I(\mathbb{A}_f^p)$. We define the *twisted orbital integral along* δ to be the functional

$$TO_\delta : C_c^\infty(M_h(\mathbb{Q}_{p^a})) \longrightarrow \mathbb{C}$$
$$f \longmapsto TO_\delta(f) = \int_{(M_h)_{\delta\sigma}(\mathbb{Q}_p)\backslash M_h(\mathbb{Q}_{p^a})} f(g^{-1}\delta\sigma(g)),$$

with respect to the fixed Haar measure on $M_h(\mathbb{Q}_{p^a})$ and the Haar measure on $(M_h)_{\delta\sigma}(\mathbb{Q}_p)$ transferred from $I(\mathbb{Q}_p)$. For more details see [**Kot90**, §3].

Definition 2.3.9. — Let $\phi_a^{M_h} : M_h(\mathbb{Q}_{p^a}) \to \mathbb{Q}$ be the characteristic function of $M_h(\mathbb{Z}_{p^a})\mu(p)^{-1}M_h(\mathbb{Z}_{p^a})$.

2.4. Definition of Tr_M

In this section, let P be a standard parabolic subgroup of G, and let $M = M_P$ be the standard Levi component of P. We define the term $\mathrm{Tr}_M(f^{p,\infty}dg^{p,\infty}, K, a)$ in (1.8.4.1).

Definition 2.4.1. — For $\gamma_0 \in M_h(\mathbb{R})$ and $\gamma_L \in M_l(\mathbb{R})$, we write $\gamma_0 \sim_\mathbb{R} \gamma_L$, if one of the following conditions holds.

(1) We have $M_h \cong \mathrm{GL}_2$.
(2) We have $M_h \cong \mathbb{G}_m$, $\gamma_0 \in M_h(\mathbb{R})^0$, and $\gamma_L \in M_l(\mathbb{R})^0$.
(3) We have $M_h \cong \mathbb{G}_m$, $\gamma_0 \notin M_h(\mathbb{R})^0$, and $\gamma_L \notin M_l(\mathbb{R})^0$.

Remark 2.4.2. — When $M = M_1$, we have $M_h = \mathrm{GL}_2$, and so the condition $\gamma_0 \sim_\mathbb{R} \gamma_L$ is by definition automatic. When $M = M_{12}$ or M_2, we have $\pi_0(M_h(\mathbb{R})) \cong \pi_0(M_l(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$. Thus the condition $\gamma_0 \sim_\mathbb{R} \gamma_L$ depends only on the $M_h(\mathbb{R})$-conjugacy class of γ_0 and the $M_l(\mathbb{R})$-conjugacy class of γ_L.

Definition 2.4.3. — Let K be a compact open subgroup of $G(\mathbb{A}_f)$. Let p be a hyperspecial prime for K, and let K_p, K^p be as in Definition 1.8.2. Let $f^{p,\infty}dg^{p,\infty} \in$

$\mathcal{H}(G(\mathbb{A}_f^p) /\!\!/ K^p)_{\mathbb{Q}}$, and let $a \in \mathbb{Z}_{\geq 1}$. We define the complex number

$$\mathrm{Tr}_M(f^{p,\infty}dg^{p,\infty}, K, a) := \sum_{\gamma_L} \iota^{M_l}(\gamma_L)^{-1}\chi((M_{l,\gamma_L})^0) \sum_{(\gamma_0,\gamma,\delta)} c(\gamma_0,\gamma,\delta) \cdot \delta_{P(\mathbb{Q}_p)}(\gamma_0)^{1/2} O_{\gamma_L\gamma}(f_M^{p,\infty}) O_{\gamma_L}(1_{M_l(\mathbb{Z}_p)}) TO_\delta(\phi_a^{M_h}) L_M(\gamma_L\gamma_0), \tag{2.4.3.1}$$

where γ_L runs through the semi-simple conjugacy classes in $M_l(\mathbb{Q})$ that are $\mathbb{R}$-elliptic (see Definition 1.1.8; if no such γ_L exists, then the sum is empty), and $(\gamma_0, \gamma, \delta)$ runs through the equivalence classes of Kottwitz triples in M_h of level p^a (see Definition 2.3.2) such that $\gamma_0 \sim_{\mathbb{R}} \gamma_L$ (see Definition 2.4.1 and Remark 2.4.2). The other terms are defined as follows:

(1) We write $\iota^{M_l}(\gamma_L)$ for $\left|M_{l,\gamma_L}(\mathbb{Q})/(M_{l,\gamma_L})^0(\mathbb{Q})\right|$.

(2) We write $\chi((M_{l,\gamma_L})^0)$ for the *Euler characteristic* of the reductive group $(M_{l,\gamma_L})^0$ over $\mathbb{Q}$, as defined in [**GKM97**, §7.10].

(3) The term $c(\gamma_0, \gamma, \delta)$ is as in Definition 2.3.7.

(4) We let $f_M^{p,\infty} \in C_c^\infty(M(\mathbb{A}_f^p))$ be the *constant term* of $f^{p,\infty}$ as defined in [**GKM97**, §7.13]. This function depends on auxiliary choices, but its orbital integrals are well defined once all the relevant Haar measures are fixed.

(5) We have a canonical identification

$$C_c^\infty(M(\mathbb{A}_f^p)) \cong C_c^\infty(M_h(\mathbb{A}_f^p)) \otimes_{\mathbb{C}} C_c^\infty(M_l(\mathbb{A}_f^p)).$$

In view of this, we define the functional $O_{\gamma_L\gamma} : C_c^\infty(M(\mathbb{A}_f^p)) \to \mathbb{C}$ to be the tensor product of the functional $O_\gamma : C_c^\infty(M_h(\mathbb{A}_f^p)) \to \mathbb{C}$ in Definition 2.3.8 and the functional

$$\begin{aligned} O_{\gamma_L} : C_c^\infty(M_l(\mathbb{A}_f^p)) &\longrightarrow \mathbb{C} \\ f &\longmapsto O_{\gamma_L}(f) = \int_{M_{l,\gamma_L}(\mathbb{A}_f^p)\backslash M_l(\mathbb{A}_f^p)} f(g^{-1}\gamma_L g)dg, \end{aligned} \tag{2.4.3.2}$$

where the relevant Haar measures are to be specified in Remark 2.4.4 below.

(6) We let $M_l(\mathbb{Z}_p)$ be the hyperspecial subgroup of $M_l(\mathbb{Q}_p)$ given by

$$M_l(\mathbb{Z}_p) := [K_p \cap (M_l(\mathbb{Q}_p)N_P(\mathbb{Q}_p))]/(K_p \cap N_P(\mathbb{Q}_p)). \tag{2.4.3.3}$$

See Remark 2.4.5 below for more explanations.

(7) We define

$$O_{\gamma_L}(1_{M_l(\mathbb{Z}_p)}) := \int_{M_{l,\gamma_L}(\mathbb{Q}_p)\backslash M_l(\mathbb{Q}_p)} 1_{M_l(\mathbb{Z}_p)}(g^{-1}\gamma_L g)dg, \tag{2.4.3.4}$$

where the relevant Haar measures are to be specified in Remark 2.4.4 below.

(8) The term $TO_\delta(\phi_a^{M_h})$ is as in Definitions 2.3.8 and 2.3.9.

(9) The term $L_M(\cdot)$ is as in Definition 2.2.3.

Remark 2.4.4. — We make precise the choices of various Haar measures in Definition 2.4.3. We choose an arbitrary Haar measure on $M_l(\mathbb{A}_f^p)$, and choose arbitrary

Haar measures on $M_{l,\gamma_L}(\mathbb{A}_f^p)$ and $M_{l,\gamma_L}(\mathbb{Q}_p)$ for each γ_L. We then define the Haar measure on $M(\mathbb{A}_f^p) = M_h(\mathbb{A}_f^p) \times M_l(\mathbb{A}_f^p)$ to be the product of the Haar measure on $M_l(\mathbb{A}_f^p)$ chosen above and the Haar measure on $M_h(\mathbb{A}_f^p)$ chosen in §2.3.6. We then specify various normalizations:

(1) Use the Haar measure on $M(\mathbb{A}_f^p)$ as above and the Haar measure $dg^{p,\infty}$ on $G(\mathbb{A}_f^p)$ to define the constant term $f_M^{p,\infty}$.

(2) Use the Haar measures on $M_l(\mathbb{A}_f^p)$ and $M_{l,\gamma_L}(\mathbb{A}_f^p)$ chosen above to define (2.4.3.2).

(3) Use the Haar measure on $M_l(\mathbb{Q}_p)$ giving volume 1 to $M_l(\mathbb{Z}_p)$, and the Haar measure on $M_{l,\gamma_L}(\mathbb{Q}_p)$ chosen above, to define (2.4.3.4).

(4) Use the Haar measures on $M_{l,\gamma_L}(\mathbb{A}_f^p)$ and $M_{l,\gamma_L}(\mathbb{Q}_p)$ chosen above to define the product measure on $(M_{l,\gamma_L})^0(\mathbb{A}_f)$, and use the latter to define $\chi((M_{l,\gamma_L})^0)$ as in [**GKM97**, §7.10].

Remark 2.4.5. — We explain why $M_l(\mathbb{Z}_p)$ defined by (2.4.3.3) is a hyperspecial subgroup of $M_l(\mathbb{Q}_p)$ by collecting standard facts about reductive group schemes from [**SGA70**, XXVI]. Since K_p is a hyperspecial subgroup of $G(\mathbb{Q}_p)$, there is a reductive group scheme $\mathcal{G}$ over $\mathbb{Z}_p$ with generic fiber $G_{\mathbb{Q}_p}$ such that $K_p = \mathcal{G}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$. By [**SGA70**, XXVI, Cor. 3.5], the parabolic subgroup $P_{\mathbb{Q}_p}$ of $G_{\mathbb{Q}_p}$ extends to a unique parabolic subgroup $\mathcal{P}$ of $\mathcal{G}$. Since parabolic subgroups are closed (see [**SGA70**, XXVI, Prop. 1.2]), we have $\mathcal{P}(\mathbb{Z}_p) = P(\mathbb{Q}_p) \cap K_p$. Now the reductive quotient $\mathcal{M}$ of $\mathcal{P}$ (see [**SGA70**, XXVI, Cor. 1.5, Prop. 1.6]) is a reductive group scheme over $\mathbb{Z}_p$ whose generic fiber is M. Since $\operatorname{Spec}\mathbb{Z}_p$ is affine, by [**SGA70**, XXVI, Cor. 2.3] we know that $\mathcal{P}$ admits a Levi component. It follows that the natural map $\mathcal{P}(\mathbb{Z}_p) \to \mathcal{M}(\mathbb{Z}_p)$ is surjective. Therefore, the subgroup $\mathcal{M}(\mathbb{Z}_p)$ of $M(\mathbb{Q}_p)$ is equal to the image of $P(\mathbb{Q}_p) \cap K_p$ under $P(\mathbb{Q}_p) \to M(\mathbb{Q}_p)$. Now since $M = M_h \times M_l$, any hyperspecial subgroup of $M(\mathbb{Q}_p)$ (such as $\mathcal{M}(\mathbb{Z}_p)$) must be the direct product of a hyperspecial subgroup of $M_h(\mathbb{Q}_p)$ and a hyperspecial subgroup of $M_l(\mathbb{Q}_p)$. Hence the kernel of $\mathcal{M}(\mathbb{Z}_p) \hookrightarrow M(\mathbb{Q}_p) \to M_h(\mathbb{Q}_p)$, which is $M_l(\mathbb{Z}_p)$, must be a hyperspecial subgroup of $M_l(\mathbb{Q}_p)$.

Remark 2.4.6. — When $M = M_1$ or M_{12}, every element of $M_l(\mathbb{Q})$ is semi-simple $\mathbb{R}$-elliptic, because $M_{l,\mathbb{R}}$ is isomorphic to either $\mathrm{SO}(n-2,0)$ or $\mathbb{G}_{m,\mathbb{R}} \times \mathrm{SO}(n-2,0)$, and $\mathrm{SO}(n-2,0)(\mathbb{R})$ is compact. When d is even and $M = M_2$, we know that $M_{l,\mathbb{R}} \cong \mathrm{SO}(n-1,1)$ does not have elliptic maximal tori (as n is even and at least 4), so there are no $\mathbb{R}$-elliptic elements of $M_l(\mathbb{Q})$ in the sense of Definition 1.1.8. In this case it is understood that $\mathrm{Tr}_M(f^{p,\infty}dg^{p,\infty}, K, a) = 0$.

2.5. An equivalent form of Morel's formula

At this point we have defined the terms in (1.8.4.1). In this section we give an equivalent form of (1.8.4.1). It is this equivalent form that we shall prove in §3. In the following, we fix K, p, $f^{p,\infty}dg^{p,\infty}$, and a as in Definition 2.4.3, and we shall omit them from the notation when convenient. For instance, we shall write Tr_M for $\mathrm{Tr}_M(f^{p,\infty}dg^{p,\infty}, K, a)$.

2.5.1. — Let $M \in \{M_1, M_2, M_{12}\}$. We set

$$\mathrm{StdLev}(M_l) := \begin{cases} \{M_{1,l}\}, & M = M_1, \\ \{M_{2,l}, M_{12,l}\}, & M = M_2, \\ \{M_{12,l}\}, & M = M_{12}. \end{cases}$$

Thus in each case $\mathrm{StdLev}(M_l)$ is a set of representatives of the $M_l(\mathbb{Q})$-conjugacy classes of Levi subgroups of M_l.

Definition 2.5.2. — Let $M \in \{M_1, M_2, M_{12}\}$ and let $(Q, g) \in \mathcal{P}(M)$ (see Definition 2.2.1). We define $\mathrm{Tr}_{M,Q,g}$ by the same formula (2.4.3.1) used to define Tr_M, but with $L_M(\cdot)$ replaced by $L_{M,Q,g}(\cdot)$ (see Definition 2.2.2). Thus

$$\begin{aligned}(2.5.2.1)\quad \mathrm{Tr}_{M,Q,g} := \sum_{\gamma_L} \iota^{M_l}(\gamma_L)^{-1}\chi((M_{l,\gamma_L})^0) \sum_{(\gamma_0,\gamma,\delta)} c(\gamma_0,\gamma,\delta) \\ \cdot \delta_{P(\mathbb{Q}_p)}(\gamma_0)^{1/2} O_{\gamma_L\gamma}(f_M^{p,\infty}) O_{\gamma_L}(1_{M_l(\mathbb{Z}_p)}) TO_\delta(\phi_a^{M_h}) L_{M,Q,g}(\gamma_L\gamma_0),\end{aligned}$$

Definition 2.5.3. — For $Q \in \{P_1, P_2, P_{12}\}$, we define

$$\mathrm{T}'_Q := \sum_M \mathrm{Tr}_{M,Q,1},$$

where the sum is over $M \in \{M_1, M_2, M_{12}\}$ such that $M_l \in \mathrm{StdLev}(M_{Q,l})$. Indeed, for each such M, we have $(Q, 1) \in \mathcal{P}(M)$, and so $\mathrm{Tr}_{M,Q,1}$ is defined as in Definition 2.5.2.

Lemma 2.5.4. — *We have*

$$\mathrm{Tr}_{M_1} + \mathrm{Tr}_{M_2} + \mathrm{Tr}_{M_{12}} = \mathrm{T}'_{P_1} + \mathrm{T}'_{P_2} + \mathrm{T}'_{P_{12}}.$$

Proof. — By (2.2.3.1), for each $M \in \{M_1, M_2, M_{12}\}$ we have

$$\mathrm{Tr}_M = \sum_{(Q,g)\in\mathcal{P}(M)/\sim} |\mathcal{P}(M,Q)/\sim|^{-1} \mathrm{Tr}_{M,Q,g}\,.$$

By Propositions 2.2.4 and 2.2.8, if $(Q, g) \in \mathcal{P}(M)$, then $(Q, 1) \in \mathcal{P}(M)$. We claim that in this case $\mathrm{Tr}_{M,Q,g} = \mathrm{Tr}_{M,Q,1}$. Indeed, by definition $L_{M,Q,g}(\gamma_L\gamma_0) = L_{M,Q,1}(g\gamma_L g^{-1}\gamma_0)$, so it suffices to show that the expression

$$\iota^{M_l}(\gamma_L)^{-1}\chi((M_{l,\gamma_L})^0) O_{\gamma_L\gamma}(f_M^{p,\infty}) O_{\gamma_L}(1_{M_l(\mathbb{Z}_p)})$$

on the RHS of (2.5.2.1) is invariant under the replacement $\gamma_L \mapsto g\gamma_L g^{-1}$. The invariance of $\iota^{M_l}(\gamma_L)$ and $\chi((M_{l,\gamma_L})^0)$ follows from the fact that g normalizes M_l. To show the invariance of $O_{\gamma_L\gamma}(f_M^{p,\infty})$, it suffices to show that $f_M^{p,\infty}$ and its composition with the automorphism $\mathrm{Int}(g)$ of $M(\mathbb{A}_f^p)$ have equal orbital integrals at all elements. By

Kazhdan's density result [**Kaz86**] it suffices to check this only at regular semi-simple elements. Since orbital integrals are locally constant on the regular semi-simple locus, we further reduce to (G, M)-regular semi-simple elements. That is, we only need to show the invariance of $O_{\gamma_L\gamma}(f_M^{p,\infty})$ under $\gamma_L \mapsto g\gamma_L g^{-1}$ under the assumption that $\gamma_L\gamma$ is (G, M)-regular. This follows from the descent formula (see [**ST16**, Lem. 6.1] or [**vD72**])

$$O_{\gamma_L\gamma}(f_M^{p,\infty}) = \left|D_M^G(\gamma_L\gamma)\right|_{\mathbb{A}_f^p}^{1/2} O_{\gamma_L\gamma}(f^{p,\infty})$$

and the fact that g normalizes M.(1) Finally, to show the invariance of $O_{\gamma_L}(1_{M_l(\mathbb{Z}_p)})$, it suffices to show that $gM_l(\mathbb{Z}_p)g^{-1}$ is conjugate to $M_l(\mathbb{Z}_p)$ in $M_l(\mathbb{Q}_p)$. By the descriptions in Propositions 2.2.4 and 2.2.8, we are left to show that for any hyperspecial subgroup $U \subset \mathrm{SO}(W_2)(\mathbb{Q}_p)$ and any $x \in \mathrm{O}(W_2)(\mathbb{Q}_p) - \mathrm{SO}(W_2)(\mathbb{Q}_p)$, we have xUx^{-1} is conjugate to U in $\mathrm{SO}(W_2)(\mathbb{Q}_p)$. For this it suffices to exhibit one element of $\mathrm{O}(W_2)(\mathbb{Q}_p) - \mathrm{SO}(W_2)(\mathbb{Q}_p)$ normalizing U. But U is the stabilizer of a $\mathbb{Z}_p$-lattice Λ in W_2 (cf. [**LS20**, §2]). Since $p > 2$, if we take $v \in \Lambda$ such that $v_p(\langle v, v\rangle)$ is minimal, then the projection $w \mapsto w - \frac{\langle v,w\rangle}{\langle v,v\rangle} v$ preserves Λ, and hence Λ is the orthogonal direct sum of $\mathbb{Z}_p v$ and its orthogonal complement in Λ. We can therefore take the desired element of $\mathrm{O}(W_2)(\mathbb{Q}_p) - \mathrm{SO}(W_2)(\mathbb{Q}_p)$ to be the reflection along v, which stabilizes Λ. This finishes the proof of the claim.

By the claim we have

$$\mathrm{Tr}_M = \sum_Q \mathrm{Tr}_{M,Q,1},$$

where the sum is over $Q \in \{P_1, P_2, P_{12}\}$ such that $\mathcal{P}(M, Q) \neq \emptyset$. We finish the proof by noting that for $M \in \{M_1, M_2, M_{12}\}$ and $Q \in \{P_1, P_2, P_{12}\}$, we have $\mathcal{P}(M, Q) \neq \emptyset$ if and only if $M_l \in \mathrm{StdLev}(M_{Q,l})$. □

Definition 2.5.5. — For $Q \in \{P_1, P_2, P_{12}\}$, we define

$$\begin{aligned}(2.5.5.1)\quad \mathrm{T}_Q = \mathtt{m}_{M_Q} & \sum_{\mathbf{L}\in \mathrm{StdLev}(M_{Q,l})} (-1)^{\dim A_{\mathbf{L}}/A_{M_{Q,l}}} (n_{\mathbf{L}}^{M_{Q,l}})^{-1} \\ & \cdot \sum_{\gamma_L} \iota^{\mathbf{L}}(\gamma_L)^{-1} \chi(\mathbf{L}^0_{\gamma_L}) \left|D_{\mathbf{L}}^{M_{Q,l}}(\gamma_L)\right|_{\mathbb{R}}^{1/2} \\ & \cdot \sum_{(\gamma_0,\gamma,\delta)} c(\gamma_0,\gamma,\delta)\delta_{Q(\mathbb{Q}_p)}(\gamma_0)^{1/2} O_{\gamma_L\gamma}(f^{p,\infty}_{M_{\mathbf{L}}}) O_{\gamma_L}(1_{\mathbf{L}(\mathbb{Z}_p)}) TO_\delta(\phi_a^{M_{Q,h}}) \\ & \cdot \delta_{Q(\mathbb{R})}(\gamma_L\gamma_0)^{1/2}\, \mathrm{Tr}(\gamma_L\gamma_0 \mid R\Gamma(\mathrm{Lie}\, N_Q, \mathbb{V})_{>t_Q}).\end{aligned}$$

Here, for each $\mathbf{L} \in \mathrm{StdLev}(M_{Q,l})$, we let $M_{\mathbf{L}}$ be the unique element of $\{M_1, M_2, M_{12}\}$ such that $M_{\mathbf{L},l} = \mathbf{L}$. In other words, $M_{\mathbf{L}} = M_{Q,h} \times \mathbf{L}$. The second sum is over all

(1) The above argument of reducing to the (G, M)-regular case and then applying the descent formula is quite standard. In fact, one uses a similar argument to show, in the first place, that the choices made in the definition of the constant term do not affect its orbital integrals; cf. [**ST16**, §6.1].

semi-simple conjugacy classes γ_L in $\mathbf{L}(\mathbb{Q})$ which are $\mathbb{R}$-elliptic in the sense of Definition 1.1.8. (If no such element exists, then the summand labeled by $\mathbf{L}$ is zero.) The third sum is over equivalence classes of Kottwitz triples $(\gamma_0, \gamma, \delta)$ in $M_{Q,h}$ with $\gamma_0 \sim_{\mathbb{R}} \gamma_L$. The definition of $\mathbf{L}(\mathbb{Z}_p)$ is given by (2.4.3.3) applied to $M := M_{\mathbf{L}}$. All the other terms are defined in the same way as in Definition 2.4.3.

Lemma 2.5.6. — *For $Q \in \{P_1, P_2, P_{12}\}$, we have $\mathrm{T}'_Q = \mathrm{T}_Q$.*

Proof. — For each $\mathbf{L} \in \mathrm{StdLev}(M_{Q,l})$, let $P_{\mathbf{L}}$ be the unique element of $\{P_1, P_2, P_{12}\}$ such that $M_{P_{\mathbf{L}}} = M_{\mathbf{L}}$. Combining Definitions 2.2.2, 2.5.2, 2.5.3, and using the fact that for $\mathbf{L} \in \mathrm{StdLev}(M_{Q,l})$ we have $M_{\mathbf{L},h} = M_{Q,h}$, we obtain

$$\begin{aligned}\mathrm{T}'_Q = \mathtt{m}_{M_Q} \sum_{\mathbf{L} \in \mathrm{StdLev}(M_{Q,l})} & (-1)^{\dim A_{M_{\mathbf{L}}}/A_{M_Q}} (n_{M_{\mathbf{L}}}^{M_Q})^{-1} \\ & \cdot \sum_{\gamma_L} \iota^{\mathbf{L}}(\gamma_L)^{-1} \chi(\mathbf{L}^0_{\gamma_L}) \left| D_{M_{\mathbf{L}}}^{M_Q}(\gamma_L \gamma_0) \right|_{\mathbb{R}}^{1/2} \\ \cdot \sum_{(\gamma_0, \gamma, \delta)} & c(\gamma_0, \gamma, \delta) \delta_{P_{\mathbf{L}}(\mathbb{Q}_p)}(\gamma_0)^{1/2} O_{\gamma_L \gamma}(f_{M_{\mathbf{L}}}^{p,\infty}) O_{\gamma_L}(1_{\mathbf{L}(\mathbb{Z}_p)}) TO_\delta(\phi_a^{M_{Q,h}}) \\ & \cdot \delta_{Q(\mathbb{R})}(\gamma_L \gamma_0)^{1/2} \operatorname{Tr}(\gamma_L \gamma_0 \mid R\Gamma(\operatorname{Lie} N_Q, \mathbb{V})_{>t_Q}).\end{aligned}$$

Here the three summations are the same as on the RHS of (2.5.5.1). To finish the proof, we only need to check the following four identities for each $\mathbf{L} \in \mathrm{StdLev}(M_{Q,l})$:

(1) $\dim A_{M_{\mathbf{L}}}/A_{M_Q} = \dim A_{\mathbf{L}}/A_{M_{Q,l}}$.
(2) $n_{M_{\mathbf{L}}}^{M_Q} = n_{\mathbf{L}}^{M_{Q,l}}$.
(3) $D_{M_{\mathbf{L}}}^{M_Q}(\cdot) = D_{\mathbf{L}}^{M_{Q,l}}(\cdot)$.
(4) $\delta_{P_{\mathbf{L}}(\mathbb{Q}_p)}(\gamma_0) = \delta_{Q(\mathbb{Q}_p)}(\gamma_0)$.

The first three identities follow from the fact that $M_{\mathbf{L}} = M_{Q,h} \times \mathbf{L}$ and $M_Q = M_{Q,h} \times M_{Q,l}$. For the fourth identity, we have $P_{\mathbf{L}} \subset Q$, and the subgroup $N_{P_{\mathbf{L}}}/N_Q$ of $Q/N_Q = M_Q$ is contained inside $M_{Q,l} \subset M_Q$. Hence $\gamma_0 \in M_{Q,h}(\mathbb{Q})$ acts trivially on $\operatorname{Lie} N_{P_{\mathbf{L}}} / \operatorname{Lie} N_Q$, and the desired identity follows. □

Proposition 2.5.7. — *The formula (1.8.4.1) in Theorem 1.8.4 is equivalent to the following formula.*

(2.5.7.1)
$$\begin{aligned}\operatorname{Tr}(\mathrm{Frob}_p^a \times f^\infty dg^\infty \mid \mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})) - \operatorname{Tr}(\mathrm{Frob}_p^a \times f^\infty dg^\infty \mid \mathbf{H}_c^*(\mathrm{Sh}_K, \mathbb{V})) \\ = \mathrm{T}_{P_1} + \mathrm{T}_{P_2} + \mathrm{T}_{P_{12}}.\end{aligned}$$

Proof. — This follows from Lemmas 2.5.4 and 2.5.6. □

CHAPTER 3

PROOF OF MOREL'S FORMULA

In this chapter we prove Theorem 1.8.4.

3.1. Introduction to the proof

3.1.1. — Our goal is to prove the formula (1.8.4.1). In Proposition 2.5.7, we have shown that (1.8.4.1) is equivalent to (2.5.7.1). This last formula is a variant of [**Mor10b**, Thm. 1.7.1], and our proof will be a modification of the proof in *loc. cit.*.

First we review some key ingredients in [**Mor10b**, Thm. 1.7.1]. The proof is axiomatic in nature, building on the earlier work of Morel [**Mor06, Mor08**], and the work of Pink [**Pin92a**]. Other ingredients needed in this axiomatic approach include:

(1) Deligne's conjecture on local terms in the Grothendieck–Lefschetz–Verdier trace formula, which was proved in special cases that are already enough for Shimura varieties by Pink [**Pin92b**], and in general by Fujiwara [**Fuj97**] and Varshavsky [**Var05**].

(2) The fixed point formula of Goresky–Kottwitz–MacPherson [**GKM97**].

(3) The fixed point formula of Kottwitz [**Kot92b**].

The ingredient (1) is of course still valid in our case. As regards (2), we will need the original formula as well as a variant of it (see Proposition 3.2.3 below). As regards (3), we will apply this formula to the boundary pure Shimura data $(\mathbb{G}_m, \mathcal{H}_0)$ and $(\mathrm{GL}_2, \mathcal{H}_2)$. The Shimura datum $(\mathrm{GL}_2, \mathcal{H}_2)$ gives rise to the usual modular curves, and Kottwitz's formula is valid. For $(\mathbb{G}_m, \mathcal{H}_0)$, we need a version of Kottwitz's fixed point formula for certain variants of the usual zero-dimensional Shimura varieties associated to the datum (see Proposition 3.3.14 below). Finally, note that in Theorem 1.8.4 we have not provided a formula for the term $\mathrm{Tr}(\cdot \mid \mathbf{H}_c^*(\mathrm{Sh}_K, \mathbb{V}))$. Such a formula is eventually needed in order to fully understand the LHS of (1.8.4.1). This ingredient is provided in [**KSZ**] (for all Shimura varieties of abelian type), and is treated as a black box in the present paper when we prove Corollary 8.17.5 below.

3.1.2. — Let P be a standard proper parabolic subgroup of G. There are the following differences between our T_P in Definition 2.5.5 and Morel's definition [**Mor10b**, p. 23]. We do not explicitly assume that the Kottwitz triples should have trivial Kottwitz invariant, but this is automatic by Lemma 2.3.5. Also, in the first summation in (2.5.5.1) we do not explicitly assume that $\mathbf{L}$ is cuspidal (see Definition 1.1.6), but in our case if $\mathbf{L}$ is non-cuspidal then the sum over γ_L is empty. (Indeed, the possible choices of $\mathbf{L}$ are $M_{1,l}, M_{2,l}, M_{12,l}$. In the odd case all of them are cuspidal. In the even case, $M_{1,l}$ and $M_{12,l}$ are cuspidal, whereas $(M_{2,l})_{\mathbb{R}}$ does not contain elliptic maximal tori, as noted in Remark 2.4.6.) The sole essential difference is that we impose the condition $\gamma_0 \sim_{\mathbb{R}} \gamma_L$, which is not imposed by Morel, and this is due to the fact that our orthogonal Shimura datum $\mathbf{O}(V)$ does not satisfy the axioms in [**Mor10b**, §1.1].

Recall that Morel's axioms require that for each $P \in \mathrm{AdmPar}(G)$, the Levi quotient M_P of P should admit a decomposition $M_P = G_P \times L_P$ such that $G_P(\mathbb{R})$ acts transitively on $\mathcal{X}_P$ and $L_P(\mathbb{R})$ acts trivially on $\mathcal{X}_P$, among other things. In our case, by Proposition 1.5.2 (5), such a decomposition is clearly impossible for $P = P_2$. This is in fact related to the following geometric phenomenon. In general, each boundary stratum of the Baily–Borel compactification of a Shimura variety can be identified with the quotient of a smaller Shimura variety by the action of a finite group. If Morel's axioms are satisfied, then this finite quotient can be "absorbed" by a change of level. By contrast, in our case, the zero-dimensional boundary strata corresponding to P_2 cannot be identified as Shimura varieties without taking quotients.

To resolve this problem, we need to systematically modify the arguments in [**Mor10b**, Chap. 1] whenever they concern zero-dimensional boundary strata. Roughly speaking, Morel's formula for T_P is a mixture of two formulas: the fixed point formula of Kottwitz for a Shimura variety associated to G_P, and the fixed point formula of Goresky–Kottwitz–MacPherson for a locally symmetric space associated to L_P. In our case, we need to replace the "Shimura variety associated to G_P" by a finite quotient of it, and meanwhile replace the "locally symmetric space associated to L_P" by a finite covering of it. Fortunately, we only need these generalizations in very simple situations, and the extra complication is mainly of a combinatorial nature.

3.1.3. — We now discuss another ingredient in Morel's proof of [**Mor10b**, Thm. 1.7.1], namely the construction of suitable integral models. In [**Mor10b**, §1.3] Morel provides two approaches to the construction of the integral model of the Baily–Borel compactification, for which Pink's formula (see [**Mor10b**, Thm. 1.2.3] and [**Mor10b**, p.8 item (6)]) holds, among other things. The first approach, [**Mor10b**, Prop. 1.3.1], applies Lan's work [**Lan13**] to construct the integral model away from a controlled finite set of bad primes. This approach is valid in the PEL-type case. The second approach, [**Mor10b**, Prop. 1.3.4], is applicable in much more general situations, but it only constructs the integral model away from an *uncontrolled* finite set of primes. Although Lan's work has been generalized by Madapusi Pera [**MP19**] to the case of Hodge type, our Shimura datum $\mathbf{O}(V)$ of abelian type is still beyond the applicability.[(1)] Hence we have to follow Morel's second approach, losing control of the set of bad primes. This explains why in Theorem 1.8.4 the set Σ is not made specific and may also depend on λ and f^∞.

Nevertheless, we shall show (see Lemma 3.5.7 below) that the localizations to almost all primes of the abstract integral models constructed by Morel's second approach can be compared with other known integral models of Shimura varieties in the expected way. In particular, for sufficiently large primes we are in a position to apply the result of Lan–Stroh [**LS18**, Thm. 4.19], which relates the intersection cohomology and compact support cohomology of the special fiber of the integral model to those of the generic fiber respectively.

Outline of the proof. — In §3.2, we prove an analogue of the fixed point formula of Goresky–Kottwitz–MacPherson for certain double coverings of locally symmetric spaces. The main result is Proposition 3.2.3. In §3.3, we study certain finite quotients of zero-dimensional Shimura varieties that will appear on the boundary of $\overline{\mathrm{Sh}_K}$. We develop the analogues of various constructions in [**Mor10b**, Chap. 1] for these quotients. The main results are Propositions 3.3.14 and 3.3.16. In §3.4, we explain how Morel's axioms in [**Mor10b**, §1.1] should be modified to suit our situation. In §3.5, we construct the integral models away from an uncontrolled set of bad primes, and compare the localizations of these models at almost all primes with other known integral models. In §3.6, we assemble all the ingredients and explain how to modify the proof of [**Mor10b**, Thm. 1.7.1] to prove our Theorem 1.8.4.

3.2. A fixed point formula for some double coverings of locally symmetric spaces

3.2.1. — Let L be a reductive group over $\mathbb{Q}$. We assume that $\pi_0(L(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$. By the real approximation theorem, $L(\mathbb{Q})^+ := L(\mathbb{Q}) \cap L(\mathbb{R})^0$ is of index 2 in $L(\mathbb{Q})$. We also assume that a minimal Levi subgroup L_0 of $L_\mathbb{R}$ satisfies $\pi_0(L_0(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$. Then by Matsumoto's theorem (see [**BT65**, 14.4]), for any Levi subgroup L' of $L_\mathbb{R}$, the inclusion $L'(\mathbb{R}) \hookrightarrow L(\mathbb{R})$ induces an isomorphism $\pi_0(L'(\mathbb{R})) \xrightarrow{\sim} \pi_0(L(\mathbb{R}))$. Now for each Levi subgroup L' of L defined over $\mathbb{Q}$, we set

$$L'(\mathbb{Q})^+ := L'(\mathbb{Q}) \cap L'(\mathbb{R})^+,$$

which is of index 2 in $L'(\mathbb{Q})$.

(1) In [**LS18**], Lan–Stroh have given a "crude" construction of the integral models of the Baily–Borel compactifications in the case of abelian type. However, since good integral models of *toroidal compactifications* are also implicitly needed in order to verify Pink's formula, their construction does not seem sufficient for our purpose.

3.2.2. — Let L be as in §3.2.1. Let K be a neat compact open subgroup of $L(\mathbb{A}_f)$. Let X_L be the symmetric space associated to $L_{\mathbb{R}}$ as in Definition 1.1.4. We have the usual *locally symmetric space*

$$\mathrm{M}^K := L(\mathbb{Q})\backslash X_L \times L(\mathbb{A}_f)/K,$$

as considered in [**GKM97**, §7] and [**Mor10b**, Chap. 1]. We shall consider the following variant of M^K:

$$\mathrm{M}^K_{\mathrm{sh}} := L(\mathbb{Q})^+\backslash X_L \times L(\mathbb{A}_f)/K.$$

We call $\mathrm{M}^K_{\mathrm{sh}}$ a *shallower locally symmetric space*. Both M^K and $\mathrm{M}^K_{\mathrm{sh}}$ are smooth real manifolds, and the natural map $\mathrm{M}^K_{\mathrm{sh}} \to \mathrm{M}^K$ is easily seen to be a double covering.

Let $\mathbb{W}$ be an algebraic representation of $L_{\mathbb{C}}$. Denote by $\mathcal{F}^K\mathbb{W}$ the sheaf on M^K of local sections of the map

$$L(\mathbb{Q})\backslash(\mathbb{W} \times X_L \times L(\mathbb{A}_f)/K) \longrightarrow \mathrm{M}^K.$$

Denote by $R\Gamma_c(K, \mathbb{W})$ the virtual alternating sum of the compact support cohomology $\mathbf{H}^*_c(\mathrm{M}^K, \mathcal{F}^K\mathbb{W})$. Similarly, we let $\mathcal{F}^K_{\mathrm{sh}}\mathbb{W}$ be the sheaf on $\mathrm{M}^K_{\mathrm{sh}}$ of local sections of the map

$$L^+(\mathbb{Q})\backslash(\mathbb{W} \times X_L \times L(\mathbb{A}_f)/K) \longrightarrow \mathrm{M}^K_{\mathrm{sh}},$$

and denote by $R\Gamma_{c,\mathrm{sh}}(K, \mathbb{W})$ the virtual alternating sum of the compact support cohomology $\mathbf{H}^*_c(\mathrm{M}^K_{\mathrm{sh}}, \mathcal{F}^K_{\mathrm{sh}}\mathbb{W})$, cf. [**Mor10b**, §1.2].

Fix $g \in L(\mathbb{A}_f)$, and let $K' \subset L(\mathbb{A}_f)$ be another compact open subgroup such that $K' \subset K \cap gKg^{-1}$. Analogous to [**Mor10b**, p. 22], we have finite étale Hecke operators

$$T_1, T_g : \mathrm{M}^{K'}_{\mathrm{sh}} \longrightarrow \mathrm{M}^K_{\mathrm{sh}}.$$

As in [**Mor10b**, Thm. 1.6.6], the natural cohomological correspondence

$$T_g^*\mathcal{F}^K_{\mathrm{sh}}\mathbb{W} \longrightarrow T_1^!\mathcal{F}^K_{\mathrm{sh}}\mathbb{W}$$

gives rise to an endomorphism u_g of $R\Gamma_{c,\mathrm{sh}}(K, \mathbb{W})$. [(2)]

Let l_0 denote the non-trivial element of $L(\mathbb{Q})/L(\mathbb{Q})^+$. We have a natural action of $L(\mathbb{Q})/L(\mathbb{Q})^+$ on $\mathrm{M}^K_{\mathrm{sh}}$, induced by the diagonal left action of $L(\mathbb{Q})$ on $X_L \times L(\mathbb{A}_f)$. Under this action the covering $\mathrm{M}^K_{\mathrm{sh}} \to \mathrm{M}^K$ is a $L(\mathbb{Q})/L(\mathbb{Q})^+$-torsor. The sheaf $\mathcal{F}^K_{\mathrm{sh}}\mathbb{W}$ has a natural $L(\mathbb{Q})/L(\mathbb{Q})^+$-equivariant structure, and so l_0 induces an endomorphism, still denoted by l_0, of $R\Gamma_{c,\mathrm{sh}}(K, \mathbb{W})$. This endomorphism commutes with u_g. The following result is a variant of [**Mor10b**, Thm. 1.6.6], the latter being a special case of [**GKM97**, Thm. 7.14 B].

(2) Note that Morel [**Mor10b**, Thm. 1.6.6] and Goresky–Kottwitz–MacPherson [**GKM97**] follow different conventions concerning the definition of u_g; see [**Mor10b**, Rmk. 1.6.7]. We follow Morel's convention here.

Proposition 3.2.3. — *In the setting of §3.2.2, we have*

(3.2.3.1)

$$\mathrm{Tr}(u_g \mid R\Gamma_{c,\mathrm{sh}}(K,\mathbb{W})) = 2\sum_{L'}(-1)^{\dim(A_{L'}/A_L)}(n_{L'}^L)^{-1}\sum_{\gamma}\iota^{L'}(\gamma)^{-1}\chi((L'_\gamma)^0) \cdot O_\gamma(f_{L'}^\infty)\left|D_{L'}^L(\gamma)\right|^{1/2}\mathrm{Tr}(\gamma \mid \mathbb{W}),$$

and

(3.2.3.2)

$$\mathrm{Tr}(u_g l_0 \mid R\Gamma_{c,\mathrm{sh}}(K,\mathbb{W})) = 2\sum_{L'}(-1)^{\dim(A_{L'}/A_L)}(n_{L'}^L)^{-1}\sum_{\gamma}\iota^{L'}(\gamma)^{-1}\chi((L'_\gamma)^0) \cdot O_\gamma(f_{L'}^\infty)\left|D_{L'}^L(\gamma)\right|^{1/2}\mathrm{Tr}(\gamma \mid \mathbb{W}).$$

Here:

– *In both (3.2.3.1) and (3.2.3.2), the first sum is over $L(\mathbb{Q})$-conjugacy classes of Levi subgroups L' of L.*

– *In (3.2.3.1) (resp. (3.2.3.2)), the second sum is over $L'(\mathbb{Q})$-conjugacy classes γ in $L'(\mathbb{Q})^+$ (resp. $L'(\mathbb{Q}) - L'(\mathbb{Q})^+$) that are $\mathbb{R}$-elliptic in $L'(\mathbb{R})$ in the sense of Definition 1.1.8.*

– *We denote by f^∞ the function $1_{gK}/\operatorname{vol}(K') \in C_c^\infty(L(\mathbb{A}_f))$, and let $f_{L'}^\infty$ be the constant term of f^∞ along L', cf. Definition 2.4.3.*

– *All the other terms on the right hand sides of (3.2.3.1) and (3.2.3.2) are defined in the same way as in* [**Mor10b**, Thm. 1.6.6]*, cf. Definition 2.4.3.*

Proof. — The formula (3.2.3.1) follows from similar arguments as in [**GKM97**, §7]. The key point is that the main tools used in *loc. cit.*, namely the reductive Borel–Serre compactification and the weighted complexes on it, are still available in the current setting. In fact, these objects were studied in [**GHM94**] in the non-adelic setting, where one is allowed to replace any given arithmetic subgroup by an arbitrary finite-index subgroup. Hence by the standard translation between the adelic and the non-adelic languages, we can consider the reductive Borel–Serre compactification of $\mathrm{M}_{\mathrm{sh}}^K$, as well as weighted complexes on it. The arguments in [**GKM97**, §7] can be easily transported to this new setting.

We explain some more details. Fix a minimal parabolic subgroup P_0 of L, and fix a Levi component L_0 of P_0. For any standard parabolic subgroup P of L (i.e. one that contains P_0), we denote by L_P the Levi component of P containing L_0, and denote by N_P the unipotent radical of P. As in [**GKM97**, §7], the reductive Borel–Serre compactification of the usual locally symmetric space M^K has a stratification indexed by the standard parabolic subgroups P of L. The stratum indexed by P is of the form

(3.2.3.3)
$$L_P(\mathbb{Q})\backslash[(N_P(\mathbb{A}_f)\backslash L(\mathbb{A}_f)/K) \times X_{L_P}].$$

In [**GKM97**, §7], one considers the spaces $\mathrm{Fix}(P, x_0, \gamma)$, where P runs through the standard parabolic subgroups of L, x_0 runs through representatives of the double cosets in $P(\mathbb{A}_f)\backslash L(\mathbb{A}_f)/K'$, and γ runs through conjugacy classes in $L_P(\mathbb{Q})$. Each space $\mathrm{Fix}(P, x_0, \gamma)$ is of the form

$$\mathrm{Fix}(P, x_0, \gamma) = L_{P,\gamma}(\mathbb{Q})\backslash(Y^\infty \times Y_\infty).$$

We refer the reader to [**GKM97**, p. 523] for the definition of Y^∞ and Y_∞.

For us, the reductive Borel–Serre compactification of $\mathrm{M}^K_{\mathrm{sh}}$ still has a stratification indexed by the standard parabolic subgroups P of L, and the stratum indexed by P is of the form

$$L_P(\mathbb{Q})^+\backslash[(N_P(\mathbb{A}_f)\backslash L(\mathbb{A}_f)/K) \times X_{L_P}]. \tag{3.2.3.4}$$

Comparing (3.2.3.3) and (3.2.3.4), it is clear that if one is to count the fixed points of the cohomological correspondence in the same way as in [**GKM97**, §7], one should consider

$$\coprod_{P,x_0,\gamma} \mathrm{Fix}'(P, x_0, \gamma), \tag{3.2.3.5}$$

where P runs through the standard parabolic subgroups of L, x_0 runs through representatives of the double cosets in $P(\mathbb{A}_f)\backslash L(\mathbb{A}_f)/K'$, γ runs through conjugacy classes in $L_P(\mathbb{Q})^+$, and

$$\mathrm{Fix}'(P, x_0, \gamma) := L_P(\mathbb{Q})^+_\gamma\backslash(Y^\infty \times Y_\infty).$$

Here $L_P(\mathbb{Q})^+_\gamma$ denotes the centralizer of γ in $L_P(\mathbb{Q})^+$

Let P be a standard parabolic subgroup of L. For $\gamma \in L_P(\mathbb{Q})^+$, we say that γ is *of first kind* if $L_{P,\gamma}(\mathbb{Q}) \subset L_P(\mathbb{Q})^+$, and *of second kind* if otherwise. When γ is of first kind, the $L_P(\mathbb{Q})$-conjugacy class of γ is the disjoint union of two $L_P(\mathbb{Q})^+$-conjugacy classes, and we have $\mathrm{Fix}'(P, x_0, \gamma) = \mathrm{Fix}(P, x_0, \gamma)$. When γ is of second kind, the $L_P(\mathbb{Q})$-conjugacy class of γ is the same as the $L_P(\mathbb{Q})^+$-conjugacy class of γ, and $\mathrm{Fix}'(P, x_0, \gamma)$ is a double covering of $\mathrm{Fix}(P, x_0, \gamma)$. From this discussion, we see that the space (3.2.3.5) is the same as

$$\coprod_{P,x_0,\gamma} \mathrm{Fix}''(P, x_0, \gamma), \tag{3.2.3.6}$$

where P and x_0 run through the same indexing sets as before, γ runs through $L_P(\mathbb{Q})$-conjugacy classes in $L_P(\mathbb{Q})^+$, and $\mathrm{Fix}''(P, x_0, \gamma)$ is the disjoint union of two copies of $\mathrm{Fix}(P, x_0, \gamma)$ if γ is of first kind, and is equal to $\mathrm{Fix}'(P, x_0, \gamma)$ if γ is of second kind.

From the above discussion, the compact support Euler characteristic (see [**GKM97**, §7.10, §7.11]) of $\mathrm{Fix}''(P, x_0, \gamma)$ is equal to twice that of $\mathrm{Fix}(P, x_0, \gamma)$.

In the qualitative discussion in [**GKM97**, §7.12], the contribution from $\mathrm{Fix}(P, x_0, \gamma)$ to the Lefschetz formula is a product of three factors (1), (2), and (3), where factor (1) is the compact support Euler characteristic of $\mathrm{Fix}(P, x_0, \gamma)$. For us the contribution from $\mathrm{Fix}''(P, x_0, \gamma)$ is also a product of three analogous factors, where our factors (2) and (3) are identical to those in *loc. cit.*, and our factor (1) is two times the factor (1) in *loc. cit.* as we have already seen. Therefore, analogous to [**GKM97**, (7.12.1)], we have the following expression for the Lefschetz formula:

$$\sum_P \sum_\gamma 2(-1)^{\dim A_I/\dim A_{L_P}} \left|L_{P,\gamma}(\mathbb{Q})/L_\gamma^0(\mathbb{Q})\right|^{-1} \chi(L_\gamma^0)\mathrm{L}_P^{\mathrm{GKM}}(\gamma)O_\gamma(f_P), \tag{3.2.3.7}$$

where γ runs through the $L_P(\mathbb{Q})$-conjugacy classes in $L_P(\mathbb{Q})^+$ (instead of $L_P(\mathbb{Q})$), and the other notations are the same as in *loc. cit.* except that we write $\mathrm{L}_P^{\mathrm{GKM}}(\cdot)$ for the function denoted by $L_P(\cdot)$ in *loc. cit.*.

Now the rest of the arguments in [**GKM97**, §7] that deduce [**GKM97**, Thm. 7.14 B] from [**GKM97**, (7.12.1)] can be applied to (3.2.3.7). Also the elementary translation from [**GKM97**, Thm. 7.14 B, §7.17] to the formula of [**Mor10b**, Thm. 1.6.6] carry over to imply (3.2.3.1).

We have proved (3.2.3.1). We now prove (3.2.3.2). We claim that

$$\mathrm{Tr}(u_g \mid R\Gamma_{c,\mathrm{sh}}(K,\mathbb{W})) + \mathrm{Tr}(u_g l_0 \mid R\Gamma_{c,\mathrm{sh}}(K,\mathbb{W})) = 2\,\mathrm{Tr}(u_g \mid R\Gamma_c(K,\mathbb{W})). \tag{3.2.3.8}$$

Here we abuse notation and write u_g also for the endomorphism of $R\Gamma_c(K,\mathbb{W})$ induced by g. Once (3.2.3.8) is proved, the desired identity (3.2.3.2) follows from (3.2.3.1), (3.2.3.8), and the formula for $\mathrm{Tr}(u_g \mid R\Gamma_c(K,\mathbb{W}))$ given in [**Mor10b**, Thm. 1.6.6].

We now prove (3.2.3.8). Let π denote the double covering map $\mathrm{M}_{\mathrm{sh}}^K \to \mathrm{M}^K$. We write $\mathscr{F}_{\mathrm{sh}}$ (resp. $\mathscr{F}$) for the sheaf $\mathcal{F}_{\mathrm{sh}}^K\mathbb{W}$ (resp. $\mathcal{F}^K\mathbb{W}$) on $\mathrm{M}_{\mathrm{sh}}^K$ (resp. M^K). Since $\mathscr{F}_{\mathrm{sh}} = \pi^*\mathscr{F}$, and since π is a finite covering, we have

$$\mathbf{H}_c^*(\mathrm{M}_{\mathrm{sh}}^K, \mathscr{F}_{\mathrm{sh}}) = \mathbf{H}_c^*(\mathrm{M}_{\mathrm{sh}}^K, \pi^*\mathscr{F}) = \mathbf{H}_c^*(\mathrm{M}^K, \pi_*\pi^*\mathscr{F}). \tag{3.2.3.9}$$

For each character χ of the deck group $\Delta = \mathbb{Z}/2\mathbb{Z}$ of π, we let $\mathscr{G}_\chi$ be the local system on M^K given by the covering π and the character χ. Combining (3.2.3.9) and the projection formula

$$\pi_*\pi^*\mathscr{F} \cong \mathscr{F} \otimes \bigoplus_{\chi:\Delta\to\mathbb{C}^\times} \mathscr{G}_\chi,$$

we obtain a decomposition

$$\mathbf{H}_c^*(\mathrm{M}_{\mathrm{sh}}^K, \mathscr{F}_{\mathrm{sh}}) \cong \bigoplus_{\chi:\Delta\to\mathbb{C}^\times} \mathbf{H}_c^*(\mathrm{M}^K, \mathscr{F}\otimes\mathscr{G}_\chi).$$

This decomposition is equivariant with respect to u_g, and the direct summand $\mathbf{H}_c^*(\mathrm{M}^K, \mathscr{F}\otimes\mathscr{G}_\chi)$ corresponds to the χ-eigenspace for the Δ-action on the left hand side. The desired (3.2.3.8) follows. □

3.3. Cohomological correspondences on some zero-dimensional Shimura varieties

3.3.1. — Let $(\mathbb{G}_m, \mathcal{H}_0)$ be the zero-dimensional Siegel Shimura datum as in [**Pin90**, 2.8]. Recall that $\mathcal{H}_0$ consists of two elements, and $\mathbb{G}_m(\mathbb{R})$ acts on $\mathcal{H}_0$ via the unique non-trivial action of $\pi_0(\mathbb{G}_m(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$. We now recall the construction of the associated zero-dimensional Shimura varieties, following [**Pin90**, 11.3, 11.4] and [**Pin92a**, §5.5].

As usual, we fix a neat compact open subgroup K of $\mathbb{G}_m(\mathbb{A}_f)$, and define the set of $\mathbb{C}$-points of the Shimura variety as

$$\mathrm{Sh}_K(\mathbb{C}) = \mathrm{Sh}_K(\mathbb{G}_m, \mathcal{H}_0)(\mathbb{C}) := \mathbb{G}_m(\mathbb{Q})\backslash\mathcal{H}_0 \times \mathbb{G}_m(\mathbb{A}_f)/K.$$

There is a natural action of $\pi_0(\mathbb{G}_m(\mathbb{A})/\mathbb{G}_m(\mathbb{Q}))$ on the finite set $\mathrm{Sh}_K(\mathbb{C})$, from which we obtain an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathrm{Sh}_K(\mathbb{C})$ via the isomorphism

$$\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \xrightarrow{\sim} \pi_0(\mathbb{G}_m(\mathbb{A})/\mathbb{G}_m(\mathbb{Q})) \tag{3.3.1.1}$$

from class field theory (normalized such that geometric Frobenius elements correspond to uniformizers). The *canonical model*

$$\mathrm{Sh}_K = \mathrm{Sh}_K(\mathbb{G}_m, \mathcal{H}_0)$$

is by definition the finite étale $\mathbb{Q}$-scheme corresponding to the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-set $\mathrm{Sh}_K(\mathbb{C})$.

In fact, using the transitivity of the $\pi_0(\mathbb{G}_m(\mathbb{A})/\mathbb{G}_m(\mathbb{Q}))$-action on $\mathrm{Sh}_K(\mathbb{C})$, we can describe Sh_K more explicitly as follows. The inclusion $\hat{\mathbb{Z}}^\times \subset \mathbb{G}_m(\mathbb{A}_f)$ induces an isomorphism $\hat{\mathbb{Z}}^\times \xrightarrow{\sim} \pi_0(\mathbb{G}_m(\mathbb{A})/\mathbb{G}_m(\mathbb{Q}))$. We thus identify $\hat{\mathbb{Z}}^\times$ with $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ via (3.3.1.1). (According to our normalization, this identification is induced by the Gauss isomorphisms $(\mathbb{Z}/m\mathbb{Z})^\times \xrightarrow{\sim} \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}), k+m\mathbb{Z} \mapsto (\zeta_m \mapsto \zeta_m^k)$.) Let $F_K/\mathbb{Q}$ be the finite abelian extension corresponding to the open subgroup $K \subset \hat{\mathbb{Z}}^\times \cong \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$. Then we have a canonical identification

$$\mathrm{Sh}_K \cong \mathrm{Spec}\, F_K.$$

From this description, it is clear that $\varprojlim_K \mathrm{Sh}_K \cong \mathrm{Spec}\, \mathbb{Q}^{\mathrm{ab}}$.

Observe that the non-identity bijection $\mathcal{H}_0 \to \mathcal{H}_0$ induces a bijection $\mathrm{Sh}_K(\mathbb{C}) \to \mathrm{Sh}_K(\mathbb{C})$ which is $\pi_0(\mathbb{G}_m(\mathbb{A})/\mathbb{G}_m(\mathbb{Q}))$-equivariant. From this we obtain an automorphism of the $\mathbb{Q}$-scheme Sh_K, denoted by σ_∞. If we identify Sh_K with $\mathrm{Spec}\, F_K$ as above, then σ_∞ is given by the complex conjugation acting on F_K. Moreover, since K is neat, we have $\mathbb{Q}^\times \cap K = \{1\}$, and it follows that σ_∞ is always a non-trivial automorphism of Sh_K (or equivalently, F_K is always totally complex).

We denote by $\mathrm{Sh}_K^\flat$ the quotient of Sh_K by σ_∞. Thus $\mathrm{Sh}_K^\flat \cong \mathrm{Spec}\, F_K^\flat$, where $F_K^\flat$ is the maximal totally real subfield of F_K. Alternatively, $\mathrm{Sh}_K^\flat$ is the Shimura variety at level K associated to the Shimura datum $(\mathbb{G}_m, \{\mathrm{N}_{\mathbb{C}/\mathbb{R}} : \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}\})$.

We shall need a common generalization of the $\mathbb{Q}$-schemes Sh_K and $\mathrm{Sh}_K^\flat$. First we define the generalization of a level subgroup.

Definition 3.3.2. — We say that a subgroup U of $\mathbb{G}_m(\mathbb{A}_f) \times \mathbb{Z}/2\mathbb{Z}$ is an *admissible level*, if there are neat compact open subgroups K_1 and K_2 of $\mathbb{G}_m(\mathbb{A}_f)$ such that

$$K_1 \times \{0\} \subset U \subset K_2 \times \mathbb{Z}/2\mathbb{Z}.$$

3.3.3. — Note that for any neat compact open subgroup $K \subset \mathbb{G}_m(\mathbb{A}_f)$, we have $K \subset \hat{\mathbb{Z}}^\times$, and the element $-1 \in \hat{\mathbb{Z}}^\times$ is not in K. Thus $K \times \mathbb{Z}/2\mathbb{Z}$ can be identified with a subgroup of $\hat{\mathbb{Z}}^\times$, where the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ corresponds to $-1 \in \hat{\mathbb{Z}}^\times$. It follows that every admissible level U as in Definition 3.3.2 can be canonically identified with an open subgroup of $\hat{\mathbb{Z}}^\times \cong \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$, and thus determines a finite abelian extension $F_U/\mathbb{Q}$. We define

$$\mathrm{Sh}_U = \mathrm{Sh}_U(\mathbb{G}_m, \mathcal{H}_0) := \mathrm{Spec}\, F_U.$$

When $U \subset \mathbb{G}_m(\mathbb{A}_f)$, the current definition of Sh_U agrees with the one in §3.3.1. Also, if K is a neat compact open subgroup of $\mathbb{G}_m(\mathbb{A}_f)$, then $K \times \mathbb{Z}/2\mathbb{Z}$ is an admissible level and we have $\mathrm{Sh}_{K\times\mathbb{Z}/2\mathbb{Z}} = \mathrm{Sh}^\flat_K$.

The usual Hecke operators can be generalized to this new setting as follows. Let U be an admissible level, and let $g \in \mathbb{G}_m(\mathbb{A}_f) \times \mathbb{Z}/2\mathbb{Z}$. We shall define an automorphism

$$[\cdot g]_U : \mathrm{Sh}_U \longrightarrow \mathrm{Sh}_U\, .$$

For this we identify g with an element of $\mathbb{G}_m(\mathbb{A})$ by identifying $\mathbb{Z}/2\mathbb{Z}$ with $\{\pm 1\} \subset \mathbb{R}^\times$. Then g determines an element $\rho(g) \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ via the inverse of (3.3.1.1). We define $[\cdot g]_U$ to be the automorphism of $\mathrm{Sh}_U = \mathrm{Spec}\, F_U$ corresponding to the restriction of $\rho(g)$ to F_U.

If U' is another admissible level contained in U, then we have a natural map $\mathrm{Sh}_{U'} \to \mathrm{Sh}_U$, and the two compositions

$$\mathrm{Sh}_{U'} \longrightarrow \mathrm{Sh}_U \xrightarrow{[\cdot g]_U} \mathrm{Sh}_U,$$

$$\mathrm{Sh}_{U'} \xrightarrow{[\cdot g]_{U'}} \mathrm{Sh}_{U'} \longrightarrow \mathrm{Sh}_U$$

are equal. We denote them by $[\cdot g]_{U',U}$.

If K is a neat compact open subgroup of $\mathbb{G}_m(\mathbb{A}_f)$ and $g \in \mathbb{G}_m(\mathbb{A}_f)$, then the above definition of $[\cdot g]_K$ recovers the usual Hecke operator on Sh_K. If ϵ denotes the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$, then $[\cdot \epsilon]_K$ is the automorphism σ_∞ of Sh_K as in §3.3.1.

For an admissible level U, we define

$$\mathscr{S}_U = \mathscr{S}_U(\mathbb{G}_m, \mathcal{H}_0) := \mathrm{Spec}\, \mathcal{O}_{F_U},$$

and call it the *canonical integral model* of Sh_U. The Hecke operators $[\cdot g]_U$ and $[\cdot g]_{U',U}$ as above uniquely extend to the canonical integral models.

Lemma 3.3.4. — *Let U_1 and U_2 be two admissible levels with $U_1 \subset U_2$. Then the following statements hold.*

(1) *The natural map $\mathrm{Sh}_{U_1} \to \mathrm{Sh}_{U_2}$ is a Galois covering and a U_2/U_1-torsor.*

(2) *Let p be a prime number such that $\mathbb{Z}_p^\times \subset U_1$. (Here $\mathbb{Z}_p^\times$ is viewed as a subgroup of $\mathbb{G}_m(\mathbb{A}_f) \subset \mathbb{G}_m(\mathbb{A}_f) \times \mathbb{Z}/2\mathbb{Z}$.) Then $\mathscr{S}_{U_i} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ are finite étale over $\mathbb{Z}_{(p)}$ for*

$i = 1, 2$. *Moreover, the natural map* $\mathscr{S}_{U_1} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \to \mathscr{S}_{U_2} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ *is a Galois covering and a* U_2/U_1*-torsor.*

Proof. — Statement (1) is just Galois theory. To show (2), we observe that p is unramified in F_{U_1} and F_{U_2} by class field theory. □

3.3.5. — Let L be a reductive group over $\mathbb{Q}$, and fix a continuous action of $L(\mathbb{R})$ on the set $\mathcal{H}_0$. We write $L(\mathbb{Q})^\natural$ for $\mathrm{Cent}_{L(\mathbb{Q})}\mathcal{H}_0$. Thus $L(\mathbb{Q})^\natural$ is a normal subgroup of $L(\mathbb{Q})$ of index at most 2. We have a canonical injection

$$L(\mathbb{Q})/L(\mathbb{Q})^\natural \hookrightarrow \mathrm{Aut}(\mathcal{H}_0) = \mathbb{Z}/2\mathbb{Z}. \tag{3.3.5.1}$$

Let $M = \mathbb{G}_m \times L$. Thus the group $M(\mathbb{R})$ acts on $\mathcal{H}_0$, where we let $\mathbb{G}_m(\mathbb{R})$ act as in §3.3.1. Let K_M be a neat compact open subgroup of $M(\mathbb{A}_f)$. Define

$$K_{M,\diamond} := K_M/(K_M \cap L(\mathbb{A}_f)).$$

We identify $K_{M,\diamond}$ with the image of K_M under the projection $M(\mathbb{A}_f) \to \mathbb{G}_m(\mathbb{A}_f)$. Since K_M is a neat compact open subgroup of $M(\mathbb{A}_f)$, we know that $K_{M,\diamond}$ is a neat compact open subgroup of $\mathbb{G}_m(\mathbb{A}_f)$. Define the following subgroups[(3)] of $M(\mathbb{A}_f)$:

$$\bar{H} := K_M \cap (\mathbb{G}_m(\mathbb{A}_f)L(\mathbb{Q})), \tag{3.3.5.2}$$

$$\bar{H}_L^\natural := K_M \cap L(\mathbb{Q})^\natural. \tag{3.3.5.3}$$

Note that $\bar{H}_L^\natural$ is a normal subgroup of $\bar{H}$. We define

$$\check{H} := \bar{H}/\bar{H}_L^\natural.$$

We have a natural homomorphism $\check{H} \to \mathbb{G}_m(\mathbb{A}_f)$ induced by the projection map $K_M \to \mathbb{G}_m(\mathbb{A}_f)$, and a natural homomorphism $\check{H} \to \mathbb{Z}/2\mathbb{Z}$ induced by the composition

$$\mathbb{G}_m(\mathbb{A}_f)L(\mathbb{Q}) \to L(\mathbb{Q}) \to L(\mathbb{Q})/L(\mathbb{Q})^\natural \xrightarrow{(3.3.5.1)} \mathbb{Z}/2\mathbb{Z},$$

where the first map is the projection to the second factor. Taking the product, we obtain a homomorphism $\check{H} \to \mathbb{G}_m(\mathbb{A}_f) \times \mathbb{Z}/2\mathbb{Z}$ which is injective. We use it to view $\check{H}$ as a subgroup of $\mathbb{G}_m(\mathbb{A}_f) \times \mathbb{Z}/2\mathbb{Z}$.

Lemma 3.3.6. — *In the setting of §3.3.5, the following statements hold.*

(1) *We have* $\bar{H}_L^\natural = K_M \cap (\mathrm{Cent}_{M(\mathbb{Q})}\mathcal{H}_0)$.

(2) *The subgroup* $\check{H}$ *of* $\mathbb{G}_m(\mathbb{A}_f) \times \mathbb{Z}/2\mathbb{Z}$ *is an admissible level.*

Proof. — For (1), the containment $\bar{H}_L^\natural \subset K_M \cap (\mathrm{Cent}_{M(\mathbb{Q})}\mathcal{H}_0)$ is clear. For the reverse containment, let $g \in \mathbb{G}_m(\mathbb{Q})$ and $l \in L(\mathbb{Q})$ be such that $gl \in K_M \cap (\mathrm{Cent}_{M(\mathbb{Q})}\mathcal{H}_0)$. Then $g \in K_{M,\diamond} \cap \mathbb{G}_m(\mathbb{Q})$, which is the trivial group by the neatness of $K_{M,\diamond}$. Hence

(3) In the application, typically M will be the Levi quotient of a parabolic subgroup P of a reductive group G, and we reserve the notations $H, H_L^\natural$ for certain subgroups of $P(\mathbb{A}_f)$ whose images in $M(\mathbb{A}_f)$ are the subgroups $\bar{H}, \bar{H}_L^\natural$ defined here, cf. §3.4.5.

$gl = l$, and $l \in L(\mathbb{Q})^\natural$. This shows (1). For (2), we let $K_1 = K_M \cap \mathbb{G}_m(\mathbb{A}_f)$ and $K_2 = K_{M,\diamond}$. Then K_1 and K_2 are neat compact open subgroups of $\mathbb{G}_m(\mathbb{A}_f)$, and we have $K_1 \times \{0\} \subset \check{H} \subset K_2 \times \mathbb{Z}/2\mathbb{Z}$. □

3.3.7. — We keep the setting of §3.3.5. By Lemma 3.3.6 (2), $\check{H}$ is an admissible level. Applying the construction in §3.3.3, we obtain a $\mathbb{Q}$-scheme $\mathrm{Sh}_{\check{H}}$ and a $\mathbb{Z}$-scheme $\mathscr{S}_{\check{H}}$.

By definition, the profinite Galois covering $\operatorname{Spec}\mathbb{Q}^{\mathrm{ab}} \to \mathrm{Sh}_{\check{H}}$ is a $\check{H}$-torsor. We may thus construct étale sheaves on $\mathrm{Sh}_{\check{H}}$ associated to suitable $\check{H}$-modules. More precisely, let Rep_M be the category of finite-dimensional algebraic representations of M on $\mathbb{E}_\lambda$-vector spaces (where $\mathbb{E}$ and λ are as in §1.7.1). Let $D^b(\mathrm{Rep}_M)$ be the bounded derived category of Rep_M (i.e., the category of graded objects of Rep_M of finite length, as Rep_M is semi-simple). As explained in [**Mor10b**, §1.2] and [**Mor06**, §2.1.4], we have an additive triangulated functor

$$\mathcal{F}^{\check{H}} R\Gamma(\bar{H}_L^\natural, -) : D^b(\mathrm{Rep}_M) \longrightarrow D^b_c(\mathrm{Sh}_{\check{H}}, \mathbb{E}_\lambda). \tag{3.3.7.1}$$

Roughly speaking, to compute this functor at $\mathbb{W} \in D^b(\mathrm{Rep_M})$, one first applies the right derived functor of $\mathbf{H}^0(\bar{H}_L^\natural, -)$ to $\mathbb{W}$ to get a complex of $\check{H}$-modules, and then uses this complex and the $\check{H}$-tower $\operatorname{Spec}\mathbb{Q}^{\mathrm{ab}} \to \mathrm{Sh}_{\check{H}}$ to construct a complex of $\mathbb{E}_\lambda$-sheaves on $\mathrm{Sh}_{\check{H}}$. We refer the reader to [**Mor06**, §2.1.4, Généralisation] for the precise construction.[4]

Using (3.3.7.1), we define the following functor, which can be viewed as a compact support analogue:

$$\begin{aligned} \mathcal{F}^{\check{H}} R\Gamma_c(\bar{H}_L^\natural, -) : D^b(\mathrm{Rep}_M) &\longrightarrow D^b_c(\mathrm{Sh}_{\check{H}'}, \mathbb{E}_\lambda), \\ \mathbb{W} &\longmapsto D\Big(\mathcal{F}^{\check{H}} R\Gamma(\bar{H}_L^\natural, \mathbb{W}^*)[2q(L_\mathbb{R})]\Big), \end{aligned} \tag{3.3.7.2}$$

where $D(\cdot)$ denotes the Verdier dual, $\mathbb{W}^*$ denotes the contragredient of $\mathbb{W}$, and $q(L_\mathbb{R})$ is as in Definition 1.1.4.

Similarly, let $\mathrm{Rep}_{\mathbb{G}_m \times \mathbb{Z}/2\mathbb{Z}}$ be the category of finite-dimensional algebraic representations of $\mathbb{G}_m \times \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{E}_\lambda$-vector spaces, and let $D^b(\mathrm{Rep}_{\mathbb{G}_m \times \mathbb{Z}/2\mathbb{Z}})$ be the bounded derived category. (Here we view $\mathbb{Z}/2\mathbb{Z}$ as a constant group scheme.) We have an additive triangulated functor

$$\mathcal{F}^{K_{M,\diamond} \times \mathbb{Z}/2\mathbb{Z}}(-) : D^b(\mathrm{Rep}_{\mathbb{G}_m \times \mathbb{Z}/2\mathbb{Z}}) \longrightarrow D^b_c(\mathrm{Sh}_{K_{M,\diamond} \times \mathbb{Z}/2\mathbb{Z}}, \mathbb{E}_\lambda) = D^b_c(\mathrm{Sh}^\flat_{K_{M,\diamond}}, \mathbb{E}_\lambda) \tag{3.3.7.3}$$

given as follows. Let $\mathbb{W} \in \mathrm{Rep}_{\mathbb{G}_m \times \mathbb{Z}/2\mathbb{Z}}$. First viewing $\mathbb{W}$ as an algebraic representation of $\mathbb{G}_m$, we obtain the associated automorphic $\mathbb{E}_\lambda$-sheaf on $\mathrm{Sh}_{K_{M,\diamond}}$ as usual (see §1.7.1). We then use the $\mathbb{Z}/2\mathbb{Z}$-action on $\mathbb{W}$ to define the descent datum

[4] It is assumed in *loc. cit.* that $L(\mathbb{Q}) = L(\mathbb{Q})^\natural$, but this assumption can be removed without affecting any of the arguments.

with respect to the double covering $\mathrm{Sh}_{K_{M,\diamond}} \to \mathrm{Sh}^{\flat}_{K_{M,\diamond}}$, and obtain an $\mathbb{E}_\lambda$-sheaf on $\mathrm{Sh}^{\flat}_{K_{M,\diamond}}$. Equivalently, we let the Galois group Γ of $\operatorname{Spec}\mathbb{Q}^{\mathrm{ab}} \to \mathrm{Sh}^{\flat}_{K_{M,\diamond}}$, namely $\Gamma = K_{M,\diamond} \times \mathbb{Z}/2\mathbb{Z} \subset \hat{\mathbb{Z}}^\times$, act on $\mathbb{W}$ via the projection $\Gamma \to \mathbb{G}_m(\mathbb{Q}_\ell) \times \mathbb{Z}/2\mathbb{Z}$ followed by the canonical action of $\mathbb{G}_m(\mathbb{Q}_\ell) \times \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{W}$. We then obtain an $\mathbb{E}_\lambda$-sheaf on $\mathrm{Sh}^{\flat}_{K_{M,\diamond}}$ via the Γ-torsor $\operatorname{Spec}\mathbb{Q}^{\mathrm{ab}} \to \mathrm{Sh}^{\flat}_{K_{M,\diamond}}$ and the Γ-representation $\mathbb{W}$.

3.3.8. — We keep the setting of §3.3.5. Let $D^b(\mathrm{Rep}_M)$ and $D^b(\mathrm{Rep}_{\mathbb{G}_m\times\mathbb{Z}/2\mathbb{Z}})$ be as in §3.3.7. For any neat compact open subgroup $U \subset L(\mathbb{A}_f)$, we shall construct a functor

$$R\Gamma_\natural(U,-) : D^b(\mathrm{Rep}_M) \longrightarrow D^b(\mathrm{Rep}_{\mathbb{G}_m\times\mathbb{Z}/2\mathbb{Z}}). \tag{3.3.8.1}$$

The construction is similar to the one described in [**Mor10b**, Rmk. 1.5.2 (1)]. Consider the space

$$\mathrm{M}^U_\natural := L(\mathbb{Q})^\natural \backslash X_L \times L(\mathbb{A}_f)/U,$$

where X_L is as in Definition 1.1.4. Thus $\mathrm{M}^U_\natural$ is a variant of the usual locally symmetric space M^U, cf. §3.2.2. We know that $\mathrm{M}^U_\natural$ is a smooth manifold, and the natural map $\mathrm{M}^U_\natural \to \mathrm{M}^U$ is a covering map of degree $[L(\mathbb{Q}) : L(\mathbb{Q})^\natural]$. (In our later application, L will satisfy the assumptions in §3.2.1 and we will have $L(\mathbb{Q})^\natural = L(\mathbb{Q})^+$, so $\mathrm{M}^U_\natural$ is the same as $\mathrm{M}^U_{\mathrm{sh}}$ discussed in §3.2.2.)

Fix a system of representatives $(l_i)_{i\in I}$ of the double cosets in $L(\mathbb{Q})^\natural\backslash L(\mathbb{A}_f)/U$. Here the indexing set I is finite, since the set $L(\mathbb{Q})\backslash L(\mathbb{A}_f)/U$ is finite and $[L(\mathbb{Q}) : L(\mathbb{Q})^\natural] \leq 2$. Then we have

$$\mathrm{M}^U_\natural \cong \coprod_{i\in I} \Gamma_i\backslash X_L,$$

where each $\Gamma_i := l_iUl_i^{-1} \cap L(\mathbb{Q})^\natural$ is a neat arithmetic subgroup of $L(\mathbb{Q})$. For $\mathbb{W} \in D^b(\mathrm{Rep}_M)$, we define

$$R\Gamma_\natural(U,\mathbb{W}) := \bigoplus_{i\in I} R\Gamma(\Gamma_i,\mathbb{W}), \tag{3.3.8.2}$$

where each $R\Gamma(\Gamma_i,-)$ is the functor $D^b(\mathrm{Rep}_M) \to D^b(\mathrm{Rep}_{\mathbb{G}_m})$ as in [**Mor10b**, Rmk. 1.2.2] such that the cohomology of $R\Gamma(\Gamma_i,\mathbb{W})$ computes the group cohomology $\mathbf{H}^*(\Gamma_i,\mathbb{W})$.

We further equip $R\Gamma_\natural(U,\mathbb{W})$ with a $\mathbb{Z}/2\mathbb{Z}$-action as follows: If $L(\mathbb{Q})/L(\mathbb{Q})^\natural$ is trivial, we define this action to be trivial. Assume that $L(\mathbb{Q})/L(\mathbb{Q})^\natural \cong \mathbb{Z}/2\mathbb{Z}$. Then left-multiplication by the non-trivial element of $L(\mathbb{Q})/L(\mathbb{Q})^\natural$ induces an involution on the set $L(\mathbb{Q})^\natural\backslash L(\mathbb{A}_f)/U$, and hence an involution on I. If $\{i,j\}$ is a size-two orbit in I under the involution, then there is a canonical coset in $\Gamma_j\backslash L(\mathbb{Q})$ consisting of $l \in L(\mathbb{Q})$ satisfying $ll_i \in l_jU$. For any such l, the isomorphism $\mathbb{W} \to \mathbb{W}$ given by the action of l intertwines with the isomorphism $\Gamma_i \xrightarrow{\sim} \Gamma_j$ given by $\mathrm{Int}(l)$, and we obtain an isomorphism $\tau_{i,j} : R\Gamma(\Gamma_i,\mathbb{W}) \xrightarrow{\sim} R\Gamma(\Gamma_j,\mathbb{W})$, which is independent of the choice of l. Moreover, the isomorphism $\tau_{j,i} : R\Gamma(\Gamma_j,\mathbb{W}) \xrightarrow{\sim} R\Gamma(\Gamma_i,\mathbb{W})$ obtained in the similar way is inverse to $\tau_{i,j}$. Now consider a size-one orbit $\{i\}$ in I under the involution. Then Γ_i is a subgroup of $l_i U l_i^{-1} \cap L(\mathbb{Q})$ of index 2. For any $l \in (l_i U l_i^{-1} \cap L(\mathbb{Q})) - \Gamma_i$, the isomorphism $\mathbb{W} \to \mathbb{W}$ given by the action of l intertwines with the isomorphism $\Gamma_i \xrightarrow{\sim} \Gamma_i$ given by $\mathrm{Int}(l)$, and we obtain an automorphism τ_i of $R\Gamma(\Gamma_i, \mathbb{W})$, which is independent of the choice of l and has order at most 2. The collection of $\tau_{i,j}$ and τ_i as above thus gives a canonical $\mathbb{Z}/2\mathbb{Z}$-action on $R\Gamma_\natural(U, \mathbb{W})$, and we thereby view $R\Gamma_\natural(U, \mathbb{W})$ as an object in $D^b(\mathrm{Rep}_{\mathbb{G}_m \times \mathbb{Z}/2\mathbb{Z}})$.

At this point, we have constructed the desired functor (3.3.8.1), after fixing the choice of a system of representatives $(l_i)_{i\in I}$. It can be checked that changing the system of representatives does not change the functor up to natural isomorphism.

Using (3.3.8.1), we define the following functor as a compact support analogue:

$$\begin{aligned} R\Gamma_{c,\natural}(U, -) : D^b(\mathrm{Rep}_M) &\longrightarrow D^b(\mathrm{Rep}_{\mathbb{G}_m \times \mathbb{Z}/2\mathbb{Z}}) \\ \mathbb{W} &\longmapsto \Big(R\Gamma_\natural(U, \mathbb{W}^*)[2q(L_\mathbb{R})] \Big)^*, \end{aligned} \tag{3.3.8.3}$$

where $*$ denotes taking contragredient, and $q(L_\mathbb{R})$ is as in Definition 1.1.4.

Remark 3.3.9. — For $\mathbb{W} \in \mathrm{Rep}_M$, the object $R\Gamma_\natural(U, \mathbb{W})$ (resp. $R\Gamma_{c,\natural}(U, \mathbb{W})$) is an incarnation of the cohomology (resp. cohomology with compact support) of the space $\mathrm{M}_\natural^U$ "with coefficients in $\mathbb{W}$". To explain this, fix a field embedding $\mathbb{E}_\lambda \hookrightarrow \mathbb{C}$. Then $\mathbb{W}$ determines an algebraic representation $\mathbb{W}_\mathbb{C}$ of $L_\mathbb{C}$ over $\mathbb{C}$. Consider the sheaf $\mathcal{F}_\natural^U(\mathbb{W}_\mathbb{C})$ of local sections of

$$L(\mathbb{Q})^\natural \backslash \mathbb{W}_\mathbb{C} \times X_L \times L(\mathbb{A}_f)/U \longrightarrow \mathrm{M}_\natural^U,$$

cf. [**Mor10b**, §1.2] and §3.2.2. Then for each $k \in \mathbb{Z}$, the base change to $\mathbb{C}$ of the k-th cohomology of $R\Gamma_\natural(U, \mathbb{W})$ (resp. $R\Gamma_{c,\natural}(U, \mathbb{W})$) is isomorphic to $\mathbf{H}^k(\mathrm{M}_\natural^U, \mathcal{F}_\natural^U(\mathbb{W}_\mathbb{C}))$ (resp. $\mathbf{H}_c^k(\mathrm{M}_\natural^U, \mathcal{F}_\natural^U(\mathbb{W}_\mathbb{C}))$).

3.3.10. — We keep the setting of §3.3.5. Consider the following situation, which is a special case of the situation described below [**Mor10b**, Notation 1.5.1]. Fix $m \in \mathbb{G}_m(\mathbb{A}_f)L(\mathbb{Q}) \subset M(\mathbb{A}_f)$. Let K'_M be a compact open subgroup of $M(\mathbb{A}_f)$ such that

$$K'_M \subset K_M \cap m K_M m^{-1}.$$

Let $\bar{H}'$ and $(\bar{H}_L^\natural)'$ be defined by the formulas (3.3.5.2) and (3.3.5.3, but with K_M replaced by K'_M. Note that we have $\bar{H}' \subset \bar{H} \cap m\bar{H}m^{-1}$.

Let $\check{H}' := \bar{H}'/(\bar{H}_L^\natural)'$. Let $\theta_1 : \check{H}' \to \check{H}$ be the homomorphism induced by $\mathrm{Int}(m^{-1}) : \bar{H}' \to \bar{H}$, and let $\theta_2 : \check{H}' \to \check{H}$ be the homomorphism induced by the inclusion $\bar{H}' \subset \bar{H}$. As a generalization of the functor (3.3.7.1), for $i \in \{1, 2\}$ we have a functor

$$\mathcal{F}^{\check{H}'} \theta_i^* R\Gamma(\bar{H}_L^\natural, -) : D^b(\mathrm{Rep}_M) \longrightarrow D_c^b(\mathrm{Sh}_{\check{H}'}, \mathbb{E}_\lambda). \tag{3.3.10.1}$$

To compute this functor at $\mathbb{W} \in D^b(\mathrm{Rep}_M)$, roughly speaking one first applies the right derived functor of $\mathbf{H}^0(\bar{H}_L^\natural, -)$ to $\mathbb{W}$ to get a complex of $\check{H}$-modules, then pulls this complex back via θ_i^* to obtain a complex of $\check{H}'$-modules, and then uses the last complex and the $\check{H}'$-tower $\operatorname{Spec}\mathbb{Q}^{\mathrm{ab}} \to \mathrm{Sh}_{\check{H}'}$ to construct a complex of $\mathbb{E}_\lambda$-sheaves on $\mathrm{Sh}_{\check{H}'}$. The precise construction of (3.3.10.1) is along similar lines as the construction of (3.3.7.1), for which we refer to [**Mor10b**, §1.5].

Let $\bar{m}$ be the image of m in $\mathbb{G}_m(\mathbb{A}_f) \times (L(\mathbb{Q})/L(\mathbb{Q})^\natural) \subset \mathbb{G}_m(\mathbb{A}_f) \times \mathbb{Z}/2\mathbb{Z}$. As in §3.3.3, we have Hecke operators

$$[\cdot\bar{m}]_{\check{H}',\check{H}} : \mathrm{Sh}_{\check{H}'} \longrightarrow \mathrm{Sh}_{\check{H}},$$
$$[\cdot 1]_{\check{H}',\check{H}} : \mathrm{Sh}_{\check{H}'} \longrightarrow \mathrm{Sh}_{\check{H}}.$$

In the sequel we denote them simply by $[\cdot m]$ and $[\cdot 1]$.

Let $\mathbb{W} \in D^b(\mathrm{Rep}_M)$. Applying the functor (3.3.7.1) to W, we obtain

$$\mathscr{L} := \mathcal{F}^{\check{H}} R\Gamma(\bar{H}_L^\natural, \mathbb{W}) \in D_c^b(\mathrm{Sh}_{\check{H}}, \mathbb{E}_\lambda).$$

As explained in [**Mor10b**, §1.5], it follows from [**Pin92a**, Prop. 1.11.5] that there are canonical isomorphisms

$$\mathcal{F}^{\check{H}'} \theta_1^* R\Gamma(\bar{H}_L^\natural, \mathbb{W}) \cong [\cdot m]^* \mathscr{L},$$
$$\mathcal{F}^{\check{H}'} \theta_2^* R\Gamma(\bar{H}_L^\natural, \mathbb{W}) \cong [\cdot 1]^* \mathscr{L}.$$

Using these isomorphisms, as in [**Mor10b**, §1.5] one constructs a canonical cohomological correspondence

$$c_{m,1} : [\cdot m]^* \mathscr{L} \longrightarrow [\cdot 1]^! \mathscr{L} = [\cdot 1]^* \mathscr{L}. \tag{3.3.10.2}$$

(Both sides are complexes of sheaves on $\mathrm{Sh}_{\check{H}'}$.) Similarly, applying the functor (3.3.7.2) we obtain

$$\mathscr{L}_c := \mathcal{F}^{\check{H}} R\Gamma_c(\bar{H}_L^\natural, \mathbb{W}) \in D_c^b(\mathrm{Sh}_{\check{H}}, \mathbb{E}_\lambda),$$

and there is a canonical cohomological correspondence

$$c_{m,1} : [\cdot m]^* \mathscr{L}_c \longrightarrow [\cdot 1]^! \mathscr{L}_c = [\cdot 1]^* \mathscr{L}_c. \tag{3.3.10.3}$$

Now let p be a prime number which is coprime to λ and hyperspecial for K_M (see Definition 1.8.2). Assume in addition that $m \in \mathbb{G}_m(\mathbb{A}_f^p)L(\mathbb{Q})$. Then there exists K_M' as in the above discussion such that p is also hyperspecial for K_M'. For such K_M', it is clear from Lemma 3.3.4 (2) that the Hecke operators $[\cdot m] : \mathrm{Sh}_{\check{H}'} \to \mathrm{Sh}_{\check{H}}$ and $[\cdot 1] : \mathrm{Sh}_{\check{H}'} \to \mathrm{Sh}_{\check{H}}$ extend to finite étale morphisms $\mathscr{S}_{\check{H}'} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \to \mathscr{S}_{\check{H}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ (still denoted by $[\cdot m]$ and $[\cdot 1]$), that $\mathscr{L}$ and $\mathscr{L}_c$ extend to complexes of lisse $\mathbb{E}_\lambda$-sheaves on $\mathscr{S}_{\check{H}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, and that the cohomological correspondences (3.3.10.2) and (3.3.10.3) also extend. We denote by $\overline{\mathscr{L}}$ (resp. $\overline{\mathscr{L}_c}$) the pull-back to $\mathscr{S}_{\check{H}} \otimes_{\mathbb{Z}} \mathbb{F}_p$ of the extension of $\mathscr{L}$ (resp. $\mathscr{L}_c$) over $\mathscr{S}_{\check{H}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. As in [**Mor10b**, Notation 1.5.1], for any $a \in \mathbb{Z}_{\geq 1}$ we can twist the reductions of (3.3.10.2) and (3.3.10.3) over $\mathbb{F}_p$ by the a-th power of the absolute Frobenius, and obtain cohomological correspondences

$$\Phi^a c_{m,1} : [\cdot m]^* \overline{\mathscr{L}} \longrightarrow [\cdot 1]^! \overline{\mathscr{L}},$$
$$\Phi^a c_{m,1} : [\cdot m]^* \overline{\mathscr{L}_c} \longrightarrow [\cdot 1]^! \overline{\mathscr{L}_c}.$$

Definition 3.3.11. — In the setting of §3.3.10, we define

$$\mathrm{Tr}_{\mathcal{H}}(a, m, K_M, K'_M, \mathbb{W}) := \sum_k (-1)^k \, \mathrm{Tr}(\Phi^a c_{m,1} \mid \mathbf{H}^k(\mathscr{S}_{\check{H}} \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p, \overline{\mathscr{L}})),$$
$$\mathrm{Tr}_{\mathcal{H},c}(a, m, K_M, K'_M, \mathbb{W}) := \sum_k (-1)^k \, \mathrm{Tr}(\Phi^a c_{m,1} \mid \mathbf{H}^k(\mathscr{S}_{\check{H}} \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p, \overline{\mathscr{L}_c})).$$

3.3.12. — We keep the setting of §3.3.5. Let $\mathbb{U} \in D^b(\mathrm{Rep}_{\mathbb{G}_m \times \mathbb{Z}/2\mathbb{Z}})$. Applying the functor (3.3.7.3) to $\mathbb{U}$ we obtain

$$\mathscr{M} := \mathcal{F}^{K_{M,\diamond} \times \mathbb{Z}/2\mathbb{Z}}(\mathbb{U}) \in D^b_c(\mathrm{Sh}_{K_{M,\diamond} \times \mathbb{Z}/2\mathbb{Z}}, \mathbb{E}_\lambda).$$

Let p be a prime number coprime to λ such that $\mathbb{Z}_p^\times \subset K_{M,\diamond}$. (For instance, if p is hyperspecial for K_M, then $\mathbb{Z}_p^\times \subset K_{M,\diamond}$.) Let $g \in \mathbb{G}_m(\mathbb{A}_f^p)$. As in §3.3.3, we have the Hecke operator

$$[\cdot g]_{K_{M,\diamond} \times \mathbb{Z}/2\mathbb{Z}} : \mathrm{Sh}_{K_{M,\diamond} \times \mathbb{Z}/2\mathbb{Z}} \longrightarrow \mathrm{Sh}_{K_{M,\diamond} \times \mathbb{Z}/2\mathbb{Z}},$$

which we denote simply by $[\cdot g]$. Similarly as in §3.3.10, we have a canonical cohomological correspondence

$$u(0, g) : [\cdot g]^* \mathscr{M} \longrightarrow \mathscr{M},$$

and we can pass to the special fiber of the canonical integral model mod p, twist by the a-th power of the absolute Frobenius (where $a \in \mathbb{Z}_{\geq 1}$), and obtain a cohomological correspondence

$$u(a, g) = \Phi^a u(0, g) : [\cdot g]^* \overline{\mathscr{M}} \longrightarrow \overline{\mathscr{M}}. \tag{3.3.12.1}$$

Definition 3.3.13. — In the setting of §3.3.12, we define

$$\mathrm{Tr}(a, g, K_{M,\diamond} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{U}) := \sum_k (-1)^k \, \mathrm{Tr}(u(a, g) \mid \mathbf{H}^k(\mathscr{S}_{K_{M,\diamond} \times \mathbb{Z}/2\mathbb{Z}} \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p, \overline{\mathscr{M}})).$$

The following result is a variant of [**Mor10b**, Rmk. 1.6.5].

Proposition 3.3.14. — *Keep the setting of §3.3.12. We have*

$$\mathrm{Tr}(a, g, K_{M,\diamond} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{U}) = \sum_{(\gamma_0, \gamma, \delta)} c(\gamma_0, \gamma, \delta) O_\gamma(f^p) TO_\delta(\phi_a^{\mathbb{G}_m}) \widetilde{\mathrm{Tr}}(\gamma_0 \mid \mathbb{U}). \tag{3.3.14.1}$$

Here the terms on the right are as follows.

(1) *The summation is over Kottwitz triples $(\gamma_0, \gamma, \delta)$ in $\mathbb{G}_m$ of level p^a, as in §2.3.*

(2) *The terms $c(\gamma_0, \gamma, \delta)$, $O_\gamma(\cdot)$, and $TO_\delta(\cdot)$ are defined as in §2.3.*

(3) *We define $f^p := 1_{gK^p_{M,\diamond}} / \mathrm{vol}(K^p_{M,\diamond}) \in C_c^\infty(\mathbb{G}_m(\mathbb{A}_f^p))$, where $K^p_{M,\diamond}$ is the subgroup of $\mathbb{G}_m(\mathbb{A}_f^p)$ such that $K_{M,\diamond} = \mathbb{Z}_p^\times K^p_{M,\diamond}$. The function $\phi_a^{\mathbb{G}_m}$ is as in Definition 2.3.9.*

(4) *For any* $\gamma_0 \in \mathbb{G}_m(\mathbb{Q}) = \mathbb{Q}^\times$, *we set*

$$\widetilde{\mathrm{Tr}}(\gamma_0 \mid \mathbb{U}) := \begin{cases} \mathrm{Tr}(\gamma_0 \mid \mathbb{U}), & \text{if } \gamma_0 > 0, \\ \mathrm{Tr}(\gamma_0 \times \epsilon \mid \mathbb{U}), & \text{if } \gamma_0 < 0, \end{cases}$$

where ϵ *denotes the non-trivial element of* $\mathbb{Z}/2\mathbb{Z}$.

Proof. — We write K for $K_{M,\diamond}$, and write S for the set $\mathscr{S}_{K_{M,\diamond}\times\mathbb{Z}/2\mathbb{Z}}(\overline{\mathbb{F}}_p)$. We identify the three sets S, $\mathrm{Sh}^\flat_K(\mathbb{C})$, and $\mathbb{G}_m(\mathbb{Q})\backslash\mathbb{G}_m(\mathbb{A}_f)/K = \mathbb{Q}^\times\backslash\mathbb{A}_f^\times/K$. Let Φ be the endomorphism of S induced by the absolute Frobenius on the $\mathbb{F}_p$-scheme $\mathscr{S}_{K_{M,\diamond}\times\mathbb{Z}/2\mathbb{Z}}$. We denote by p_p the image of p under the embedding $\mathbb{Q}_p^\times \hookrightarrow \mathbb{A}_f^\times$. Then the endomorphism $\Phi^a \circ [\cdot g]$ of S is given by the multiplication by $p_p^a g$ on $\mathbb{Q}^\times\backslash\mathbb{A}_f^\times/K$. Similarly, we write $\tilde{S}$ for $\mathscr{S}_K(\overline{\mathbb{F}}_p)$, and identify it with $\mathrm{Sh}_K(\mathbb{C}) = \mathbb{Q}^\times\backslash\mathcal{H}_0 \times \mathbb{A}_f^\times/K \cong \mathbb{Q}_{>0}\backslash\mathbb{A}_f^\times/K$.

Since we are in the zero-dimensional case, we can compute $\mathrm{Tr}(a, g, K_{M,\diamond}\times\mathbb{Z}/2\mathbb{Z}, \mathbb{U})$ by summing the naive local terms over the fixed points of S under $\Phi^a \circ [\cdot g]$.

Let $x \in S$ be a fixed point under $\Phi^a \circ [\cdot g]$. Then x has a representative $\tilde{x} \in \mathbb{A}_f^\times$ for which there exists $f_0 \in \mathbb{Q}^\times$ satisfying $f_0\tilde{x} \in p_p^a g\tilde{x}K$, or equivalently $f_0 \in p_p^a gK$. Hence the set of fixed points is non-empty if and only if $\mathbb{Q}^\times \cap p_p^a gK \neq \emptyset$, and when it is non-empty it is equal to S.

If $\mathbb{Q}^\times \cap p_p^a gK = \emptyset$, then $\mathrm{Tr}(a, g, K_{M,\diamond} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{U}) = 0$ since there are no fixed points. In this case it is straightforward to check that the RHS of (3.3.14.1) is also zero.

Assume that $\mathbb{Q}^\times \cap p_p^a gK \neq \emptyset$. In this case, this set has a unique element f_0, since we have $\mathbb{Q}^\times \cap K = \{1\}$ by the neatness of K. We have seen that in this case every point in S is a fixed point. There are two cases to consider.

First suppose that $f_0 > 0$. Then every point in $\tilde{S}$ is fixed by $\Phi^a \circ [\cdot g]$. Write $g_\ell \in \mathbb{G}_m(\mathbb{Q}_\ell)$ for the ℓ-adic component of g. In this case, the naive local term at each point in S is equal to the naive local term at any one of the two lifts of that point in $\tilde{S}$, and the latter is equal to the trace on the algebraic $\mathbb{G}_m(\mathbb{Q}_\ell)$-representation $\mathbb{U}$ of the product of g_ℓ^{-1} and the ℓ-adic component of $f_0^{-1}p_p^a g \in K$ (cf. the argument on [**Kot92b**, p. 433]). Hence the native local term is equal to $\mathrm{Tr}(f_0^{-1} \mid \mathbb{U})$, which is equal to $\widetilde{\mathrm{Tr}}(f_0^{-1} \mid \mathbb{U})$ since $f_0 > 0$.

Now suppose that $f_0 < 0$. Then for every point in S, the two lifts of it in $\tilde{S}$ are permuted non-trivially by $\Phi^a \circ [\cdot g]$. In this case, the naive local term at each point in S is equal to the trace on the algebraic $\mathbb{G}_m(\mathbb{Q}_\ell) \times \mathbb{Z}/2\mathbb{Z}$-representation $\mathbb{U}$ of the product of g_ℓ^{-1} and the projection to $\mathbb{Q}_\ell^\times \times \mathbb{Z}/2\mathbb{Z}$ of $f_0^{-1}p_p^a g \times \epsilon \in K \times \mathbb{Z}/2\mathbb{Z}$, which is $\mathrm{Tr}(f_0^{-1} \times \epsilon \mid \mathbb{U}) = \widetilde{\mathrm{Tr}}(f_0^{-1} \mid \mathbb{U})$.

We conclude that in both cases the naive local term at each point in S is equal to $\widetilde{\mathrm{Tr}}(f_0^{-1} \mid \mathbb{U})$. Hence

$$\mathrm{Tr}(a, g, K_{M,\diamond} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{U}) = \widetilde{\mathrm{Tr}}(f_0^{-1} \mid \mathbb{U})\,|S|\,.$$

To compute the RHS of (3.3.14.1), we note that every Kottwitz triple $(\gamma_0, \gamma, \delta)$ that makes a non-zero contribution must satisfy $\gamma_0 = f_0^{-1}$. (In fact all Kottwitz triples satisfying this condition are in one equivalence class.) On the other hand, by [**Mor10b**, Rmk. 1.6.5] we know that

$$\sum_{(\gamma_0,\gamma,\delta)} c(\gamma_0, \gamma, \delta) O_\gamma(f^p) TO_\delta(\phi_a^{\mathbb{G}_m}) = \frac{1}{2} \left|\tilde{S}\right|,$$

which is nothing but $|S|$. Hence the RHS of (3.3.14.1) is equal to $\widetilde{\mathrm{Tr}}(f_0^{-1} \mid \mathbb{U}) \, |S|$ as well. The proof is complete. □

3.3.15. — We now state a variant of [**Mor10b**, Prop. 1.7.2]. We keep the setting of §3.3.5. Let p be a prime number which is coprime to λ and hyperspecial for K_M. Fix $m \in M(\mathbb{A}_f^p)$ (not necessarily in $\mathbb{G}_m(\mathbb{A}_f^p)L(\mathbb{Q})$). Let K'_M be a compact open subgroup of $M(\mathbb{A}_f)$ such that p is hyperspecial for K'_M and such that

$$K'_M \subset K_M \cap m K_M m^{-1}.$$

Fix a system of representatives $(m_i)_{i\in I}$ of those double cosets e in

$$\mathbb{G}_m(\mathbb{A}_f)L(\mathbb{Q})\backslash M(\mathbb{A}_f)/K'_M$$

satisfying

$$emK_M = eK_M.$$

For every $i \in I$, let $g_i \in \mathbb{G}_m(\mathbb{A}_f)$ and $l_i \in L(\mathbb{Q})$ be such that

$$g_i l_i m_i \in m_i m K_M.$$

We may and shall assume that $m_i \in M(\mathbb{A}_f^p)$ and $g_i \in \mathbb{G}_m(\mathbb{A}_f^p)$ for each $i \in I$.

Let $\mathbb{W} \in D^b(\mathrm{Rep}_M)$, and let

$$\mathbb{U} := R\Gamma_\natural(K_M \cap L(\mathbb{A}_f), \mathbb{W}) \in D^b(\mathrm{Rep}_{\mathbb{G}_m\times\mathbb{Z}/2\mathbb{Z}}),$$
$$\mathbb{U}_c := R\Gamma_{c,\natural}(K_M \cap L(\mathbb{A}_f), \mathbb{W}) \in D^b(\mathrm{Rep}_{\mathbb{G}_m\times\mathbb{Z}/2\mathbb{Z}}),$$

where the notations are as in (3.3.8.1) and (3.3.8.3). The following result is a variant of [**Mor10b**, Prop. 1.7.2].

Proposition 3.3.16. — *Keep the setting and notation of §3.3.15. Write g for the projection of m in $\mathbb{G}_m(\mathbb{A}_f^p)$, and write $K'_{M,\diamond}$ for $K'_M/(K'_M \cap L(\mathbb{A}_f))$. Assume that $[L(\mathbb{Q}) : L(\mathbb{Q})^\natural] = 2$. Then for each $a \in \mathbb{Z}_{\geq 1}$ we have*

$$(3.3.16.1) \quad \sum_{i\in I} \mathrm{Tr}_{\mathcal{H}}(a, g_i l_i, m_i K_M m_i^{-1}, m_i K'_M m_i^{-1}, \mathbb{W}) = \mathrm{Tr}(a, g, K_{M,\diamond} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{U}) \cdot [K_{M,\diamond} : K'_{M,\diamond}],$$

and

$$(3.3.16.2)\qquad \sum_{i\in I}\mathrm{Tr}_{\mathcal{H},c}(a,g_il_i,m_iK_Mm_i^{-1},m_iK'_Mm_i^{-1},\mathbb{W}) = \mathrm{Tr}(a,g,K_{M,\diamond}\times\mathbb{Z}/2\mathbb{Z},\mathbb{U}_c)\cdot[K_{M,\diamond}:K'_{M,\diamond}].$$

Here the terms $\mathrm{Tr}_{\mathcal{H}}(\cdots)$, $\mathrm{Tr}_{\mathcal{H},c}(\cdots)$, *and* $\mathrm{Tr}(\cdots)$ *are as in Definitions 3.3.11 and 3.3.13.*

Remark 3.3.17. — The RHS of (3.3.16.1) is indeed the analogue of the RHS of [**Mor10b**, Prop. 1.7.2 (1)]. We have the seemingly extra factor $[K_{M,\diamond}:K'_{M,\diamond}]$, but this is due to the fact that in our definition of the cohomological correspondence (3.3.12.1) we used the Hecke operator $[\cdot g]$ as an endomorphism of $\mathrm{Sh}_{K_{M,\diamond}\times\mathbb{Z}/2\mathbb{Z}}$, as opposed to using the correspondence $\mathrm{Sh}_{K_{M,\diamond}\times\mathbb{Z}/2\mathbb{Z}}\xleftarrow{[\cdot g]}\mathrm{Sh}_{K'_{M,\diamond}\times\mathbb{Z}/2\mathbb{Z}}\xrightarrow{[\cdot 1]}\mathrm{Sh}_{K_{M,\diamond}\times\mathbb{Z}/2\mathbb{Z}}$.

Similarly, the RHS of (3.3.16.2) is the analogue of the RHS of [**Mor10b**, Prop. 1.7.2 (2)].

Proof. — By duality, (3.3.16.1) implies (3.3.16.2). The proof of (3.3.16.1) is essentially the same as that of [**Mor10b**, Prop. 1.7.2(1)], the only difference being that we need to modify Morel's functor $R\Gamma(K_M,-)$ (and its analogue for K'_M). Below we explain this modification.

Consider the space

$$\mathrm{M}_\natural^{K_M} := M(\mathbb{Q})\backslash(\mathcal{H}_0\times X_L\times M(\mathbb{A}_f))/K_M \cong (\mathrm{Cent}_{M(\mathbb{Q})}\mathcal{H}_0)\backslash X_L\times M(\mathbb{A}_f)/K_M,$$

where $M(\mathbb{Q})$ acts on $\mathcal{H}_0\times X_L\times M(\mathbb{A}_f)$ diagonally, and for the action of $M(\mathbb{Q})$ on $\mathcal{H}_0$ both the factors $\mathbb{G}_m(\mathbb{Q})$ and $L(\mathbb{Q})$ act non-trivially. (The action of $L(\mathbb{Q})$ on $\mathcal{H}_0$ is via the unique non-trivial action of $L(\mathbb{Q})/L(\mathbb{Q})^\natural\cong\mathbb{Z}/2\mathbb{Z}$.) Thus $\mathrm{M}_\natural^{K_M}$ is a double covering of the usual locally symmetric space

$$\mathrm{M}^{K_M} = M(\mathbb{Q})\backslash X_M\times M(\mathbb{A}_f)/K_M,$$

where $X_M=X_L$ since $X_{\mathbb{G}_m}$ is a point. Let $R\Gamma_\natural(K_M,\mathbb{W})$ be the "cohomology of $\mathrm{M}_\natural^{K_M}$ with coefficients in $\mathbb{W}$" (cf. Remark 3.3.9). Namely, we write $\mathrm{M}_\natural^{K_M}$ as

$$\coprod_{j\in J}(n_jK_Mn_j^{-1}\cap\mathrm{Cent}_{M(\mathbb{Q})}\mathcal{H}_0)\backslash X_L,$$

where $(n_j)_{j\in J}$ is a system of representatives of the double cosets in

$$(\mathrm{Cent}_{M(\mathbb{Q})}\mathcal{H}_0)\backslash M(\mathbb{A}_f)/K_M,$$

and define

$$R\Gamma_\natural(K_M,\mathbb{W}) := \bigoplus_{j\in J}R\Gamma(n_jK_Mn_j^{-1}\cap\mathrm{Cent}_{M(\mathbb{Q})}\mathcal{H}_0,\mathbb{W})$$

inside the derived category of finite-dimensional $\mathbb{E}_\lambda$-vector spaces.

Observe that we have a fibration

$$M_{\natural}^{K_M} \longrightarrow \mathbb{G}_m(\mathbb{Q})\backslash\mathbb{G}_m(\mathbb{A}_f)/K_{M,\diamond} \tag{3.3.17.1}$$

induced by the projection $\mathcal{H}_0 \times X_L \times M(\mathbb{A}_f) \to \mathbb{G}_m(\mathbb{A}_f)$. The fibers of (3.3.17.1) are naturally identified with

$$L(\mathbb{Q})\backslash\mathcal{H}_0 \times X_L \times L(\mathbb{A}_f)/(K_M \cap L(\mathbb{A}_f)),$$

which we observe is the same as $\mathrm{M}_{\natural}^{K_M\cap L(\mathbb{A}_f)}$ defined in §3.3.8, since $[L(\mathbb{Q}) : L(\mathbb{Q})^{\natural}] = 2$. The base of the fibration (3.3.17.1) is identified with $\mathrm{Sh}_{K_{M,\diamond}\times\mathbb{Z}/2\mathbb{Z}}(\mathbb{C})$. Hence we have identifications

(3.3.17.2)

$$R\Gamma_{\natural}(K_M, \mathbb{W}) \cong R\Gamma(\mathrm{Sh}_{K_{M,\diamond}\times\mathbb{Z}/2\mathbb{Z}} \otimes_{\mathbb{Q}} \mathbb{C}, \mathscr{M}) \cong R\Gamma(\mathscr{S}_{K_{M,\diamond}\times\mathbb{Z}/2\mathbb{Z}} \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p, \overline{\mathscr{M}}),$$

where $\mathscr{M} = \mathcal{F}^{K_{M,\diamond}\times\mathbb{Z}/2\mathbb{Z}}(\mathbb{U})$ and $\overline{\mathscr{M}}$ is the reduction of $\mathscr{M}$ (cf. §3.3.12).

On the other hand, we have a fibration

$$M_{\natural}^{K_M} \longrightarrow \mathbb{G}_m(\mathbb{A}_f)L(\mathbb{Q})\backslash M(\mathbb{A}_f)/K_M \tag{3.3.17.3}$$

induced by the projection $\mathcal{H}_0 \times X_L \times M(\mathbb{A}_f) \to M(\mathbb{A}_f)$. For each $e \in M(\mathbb{A}_f)$, we denote by $\mathrm{M}_{\natural}^{K_M}(e)$ the fiber of (3.3.17.3) over the double coset of e. Then $\mathrm{M}_{\natural}^{K_M}(e)$ is identified with

$$\mathbb{G}_m(\mathbb{Q})\backslash\mathcal{H}_0 \times \mathbb{G}_m(\mathbb{A}_f) \times X_L/\bar{H}_e,$$

where $\bar{H}_e := eK_Me^{-1} \cap (\mathbb{G}_m(\mathbb{A}_f)L(\mathbb{Q}))$ is the analogue of $\bar{H}$ in §3.3.5 with eK_Me^{-1} replacing the role of K_M, and the right action of $\bar{H}_e$ on $\mathcal{H}_0 \times \mathbb{G}_m(\mathbb{A}_f) \times X_L$ is given as follows. The action of $\bar{H}_e$ on $\mathcal{H}_0 \times \mathbb{G}_m(\mathbb{A}_f)$ is the restriction of the $\mathbb{G}_m(\mathbb{A}_f)L(\mathbb{Q})$-action, where $\mathbb{G}_m(\mathbb{A}_f)$ acts on $\mathbb{G}_m(\mathbb{A}_f)$ by multiplication and $L(\mathbb{Q})$ acts on $\mathcal{H}_0$ via the non-trivial action of $L(\mathbb{Q})/L(\mathbb{Q})^{\natural}$. The action of $\bar{H}_e$ on X_L is given by the restriction of the projection map $\mathbb{G}_m(\mathbb{A}_f)L(\mathbb{Q}) \to L(\mathbb{Q})$ followed by the inversion on $L(\mathbb{Q})$ and followed by the natural left $L(\mathbb{Q})$-action on X_L.

Let $\bar{H}_{L,e}^{\natural} := eK_Me^{-1} \cap L(\mathbb{Q})^{\natural}$ and $\check{H}_e := \bar{H}_e/\bar{H}_{L,e}^{\natural}$, which are the analogues of $\bar{H}_L^{\natural}$ and $\check{H}$ in §3.3.5 with eK_Me^{-1} replacing the role of K_M. Then we have a fibration

$$\mathrm{M}_{\natural}^{K_M}(e) \longrightarrow \mathbb{G}_m(\mathbb{Q})\backslash\mathcal{H}_0 \times \mathbb{G}_m(\mathbb{A}_f)/\check{H}_e, \tag{3.3.17.4}$$

where the $\check{H}_e$-action on $\mathcal{H}_0\times\mathbb{G}_m(\mathbb{A}_f)$ is induced by the $\bar{H}_e$-action on $\mathcal{H}_0\times\mathbb{G}_m(\mathbb{A}_f)\times X_L$ in the above. The fibers of (3.3.17.4) are identified with $X_L/\bar{H}_{L,e}^{\natural}$, while the base is identified with $\mathrm{Sh}_{\check{H}_e}(\mathbb{C})$. Hence we have an identification

$$R\Gamma_{\natural}(K_M, \mathbb{W}) \cong \bigoplus_e R\Gamma(\mathrm{Sh}_{\check{H}_e} \otimes_{\mathbb{Q}} \mathbb{C}, \mathscr{L}(e)) \cong \bigoplus_e R\Gamma(\mathscr{S}_{\check{H}_e} \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p, \overline{\mathscr{L}(e)}), \tag{3.3.17.5}$$

where e runs through a system of representatives of the double cosets in

$$\mathbb{G}_m(\mathbb{A}_f)L(\mathbb{Q})\backslash M(\mathbb{A}_f)/K_M,$$

and for each e we define $\mathscr{L}(e) := \mathcal{F}^{\check{H}_e} R\Gamma(\bar{H}^{\natural}_{L,e}, \mathbb{W})$ and define $\overline{\mathscr{L}(e)}$ to be its reduction (cf. §3.3.10).

In view of (3.3.17.2) and (3.3.17.5), we can replace Morel's functor $R\Gamma(K_M, -)$ by $R\Gamma_{\natural}(K_M, -)$ (and also for K'_M) and proceed in exactly the same way as in [**Mor10b**, Prop. 1.7.2] to conclude the proof. □

3.4. Modifying Morel's axioms

3.4.1. — Let $(G, \mathcal{X})$ be a pure Shimura datum. We keep the notation in §1.3. We replace the axioms on p. 2 of [**Mor10b**] by the following axioms:

A0. — For each $P \in \mathrm{AdmPar}(G)$, the Levi quotient M_P of P admits a decomposition $M_P = G_P \times L_P$, where G_P is the image of $P^{\mathrm{Pink}} \subset P$ as in §1.3.

A1. — For each $P \in \mathrm{AdmPar}(G)$, the set $\mathcal{RBC}_P(G, \mathcal{X})$ is a singleton. In particular, $\mathcal{X}_P$ is equal to $\mathcal{X}_{\mathcal{Y}}$ for the unique element $(P, \mathcal{Y}) \in \mathcal{RBC}_P(G, \mathcal{X})$, and we have a Shimura datum $(G_P, \mathcal{X}_P)$; cf. §1.3.6.

A2. — For each $P \in \mathrm{AdmPar}(G)$, the action of $L_P(\mathbb{R})$ on $\mathcal{X}_P$ (see Proposition 1.3.7) is trivial unless $\mathcal{X}_P$ is zero-dimensional.

A3. — For each $P \in \mathrm{AdmPar}(G)$, let $L_P(\mathbb{Q})^{\natural} := \mathrm{Cent}_{L_P(\mathbb{Q})} \mathcal{X}_P$. For each neat compact open subgroup K_M of $M_P(\mathbb{A}_f)$, we have $K_M \cap \mathrm{Cent}_{M_P(\mathbb{Q})} \mathcal{X}_P = K_M \cap L_P(\mathbb{Q})^{\natural}$.

Remark 3.4.2. — Our axiom **A0** is slightly more restrictive than the first two conditions on p. 2 of [**Mor10b**], where G_P is allowed to be different from the image of P^{Pink}. Assuming **A0**, our axiom **A1** is equivalent to the first part of the fourth condition in *loc. cit.*, and our axiom **A2** is weaker than the second part of that condition. Our axiom **A3** is identical to the fifth condition in *loc. cit.*. We have deleted the third condition in *loc. cit.* from the axioms as a general correction. Indeed, this condition is neither used in [**Mor10b**] nor satisfied by any of the Shimura data considered in [**Mor10b**], [**Mor11**], or the present paper.

3.4.3. — From now on we assume the axioms in §3.4.1. Let $P \in \mathrm{AdmPar}(G)$ and $g \in G(\mathbb{A}_f)$. On p. 2 of [**Mor10b**], Morel defines the groups H_P, H_L, K_Q, K_N associated to the pair (P, g). We define:

$$\begin{aligned} H_P &:= gKg^{-1} \cap P(\mathbb{Q}) P^{\mathrm{Pink}}(\mathbb{A}_f), \\ H_L^{\natural} &:= gKg^{-1} \cap L_P(\mathbb{Q})^{\natural} N_P(\mathbb{A}_f), \\ K_Q &:= gKg^{-1} \cap P^{\mathrm{Pink}}(\mathbb{A}_f), \\ K_N &:= gKg^{-1} \cap N_P(\mathbb{A}_f). \end{aligned}$$

Our H_P, K_Q, K_N are the same as Morel's definitions, and our $H_L^{\natural}$ is equal to Morel's H_L (defined to be $gKg^{-1} \cap L_P(\mathbb{Q}) N_P(\mathbb{A}_f)$) when $L_P(\mathbb{Q}) = L_P(\mathbb{Q})^{\natural}$ (which is always true under Morel's axioms). In general, $H_L^\natural$ may be different from H_L, and $H_L^\natural$ is the correct replacement of H_L in the discussion on the structure of the boundary strata in [**Mor10b**, Chap. 1]. The point is that under the axioms in §3.4.1, the group $H_L^\natural$ is always equal to Pink's group H_C in [**Pin92a**, §3.7], which has a canonical definition. More precisely, as on p. 2 of [**Mor10b**], the boundary stratum in the Baily–Borel compactification corresponding to (P, g) is of the form[5]

(3.4.3.1) $$\mathrm{Sh}_{K_Q/K_N}(G_P, \mathcal{X}_P)/H_P,$$

and the action of H_P factors through the finite quotient group $H_P/H_L^\natural K_Q$ (instead of $H_P/H_L K_Q$).

In Table 1 below we compare Pink's notation in [**Pin92a**, §3.7], Morel's notation in [**Mor10b**, p. 2], and our notation. The symbols in the first column all have canonical definitions, independent of the axioms in [**Mor10b**] or §3.4.1. Under the axioms in §3.4.1, the three symbols in every row denote the same object, with the only exception that Morel's H_L is not equal to Pink's H_C in general.

TABLE 1. Comparison of notations

Pink's notation	Morel's notation	Our notation
Q	P	P
P_1	Q_P	P^{Pink}
W_1	N_P	N_P
G_1	G_P	G_P
$\mathcal{X}_1$	$\mathcal{X}_P$	$\mathcal{X}_P$
$\mathcal{X}_Q$	$\mathcal{X}_P$	$\mathcal{X}_P$
$\mathrm{Stab}_{Q(\mathbb{Q})}\mathcal{X}_1$	$P(\mathbb{Q})$	$P(\mathbb{Q})$
H_Q	H_P	H_P
H_C	H_L ☡	$H_L^\natural$
K_W	K_N	K_N
K_P	K_Q	K_Q
$\pi_1(K_P) = K_P/K_W$	K_Q/K_N	K_Q/K_N

☡ : $H_L \neq H_L^\natural$ unless $L_P(\mathbb{Q}) = L_P(\mathbb{Q})^\natural$.

3.4.4. — We make the following crucial assumption **CA**, in addition to the axioms in §3.4.1.

CA. — If $P \in \mathrm{AdmPar}(G)$ is such that $\mathcal{X}_P$ is zero-dimensional, then the Shimura datum $(G_P, \mathcal{X}_P)$ is the Siegel Shimura datum $(\mathbb{G}_m, \mathcal{H}_0)$. For such P we also assume that L_P satisfies the assumptions in §3.2.1. Namely, we assume that $\pi_0(L_P(\mathbb{R})) \cong$

[5] We systematically replace Morel's notation $M^{K_Q/K_N}(G_P, \mathcal{X}_P)$ for the Shimura variety by the notation $\mathrm{Sh}_{K_Q/K_N}(G_P, \mathcal{X}_P)$.

$\pi_0(L_0(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$, where L_0 is any minimal Levi subgroup of $L_{P,\mathbb{R}}$. Moreover, for such P we assume that $\pi_0(L_P(\mathbb{R}))$ acts non-trivially on $\mathcal{H}_0$. In particular, we have $L_P(\mathbb{Q})^\natural = L_P(\mathbb{Q})^+$.

3.4.5. — Under **CA**, we know that for any $P \in \mathrm{AdmPar}(G)$ such that $\mathcal{X}_P$ is zero-dimensional, and for any $g \in G(\mathbb{A}_f)$, the boundary stratum (3.4.3.1) corresponding to (P, g) is related to the generalized Shimura varieties in §3.3.3 in the following way. In §3.3.5, we identify $\mathbb{G}_m$ with G_P, and take $L = L_P$, $M = M_P$. Let K_M be the image of $gKg^{-1} \cap P(\mathbb{A}_f)$ under the projection $P(\mathbb{A}_f) \to M(\mathbb{A}_f)$, and define $\bar{H}, \bar{H}_L^\natural, \check{H}$ as in §3.3.5. Then $\bar{H}$ (resp. $\bar{H}_L^\natural$) is the image of H_P (resp. $H_L^\natural$) under $P(\mathbb{A}_f) \to M(\mathbb{A}_f)$, and (3.4.3.1) is the same as $\mathrm{Sh}_{\check{H}}$ defined in §3.3.7.

3.4.6. — Our orthogonal Shimura datum $\mathbf{O}(V)$ satisfies **A0**–**A3** in §3.4.1, and **CA** in §3.4.4. Indeed, it suffices to verify these conditions for the standard maximal proper parabolic subgroups $P_i,\ i = 1, 2$. We take L_{P_i} to be $M_{i,l}$. Then the desired conditions follow from Proposition 1.5.2 and Lemma 3.3.6 (1).

3.5. Integral models

3.5.1. — We now turn to construct the integral models of the Baily–Borel compactification $\overline{\mathrm{Sh}_K}$ and its strata. For this let us specialize to the orthogonal Shimura datum $(G, \mathcal{X}) = \mathbf{O}(V)$. Recall that the standard maximal proper parabolic subgroups of G are P_1 and P_2. We write $(G_i, \mathcal{X}_i)$ for the Shimura datum $(G_{P_i} = M_{i,h}, \mathcal{X}_{P_i})$ for $i \in \{1, 2\}$, and write $(G_0, \mathcal{X}_0)$ for $(G, \mathcal{X})$. (Our numbering of the P_i and G_i is the same as the abstract numbering in [**Mor10b**, §1.1].) For $i \in \{1, 2\}$, we set L_{P_i} to be $M_{i,l}$. In accordance with *loc. cit.*, we define $L_{P_{12}}$ to be $M_{12,l}$, so that M_{12} is the direct product of G_2 and $L_{P_{12}}$.

Without loss of generality, we assume that the function f^∞ in Theorem 1.8.4 is of the form $1_{KgK}/\operatorname{vol}(K \cap gKg^{-1})$ for some fixed $g \in G(\mathbb{A}_f)$. Since $\mathbf{O}(V)$ is of abelian type, we can apply [**Mor10b**, Prop. 1.3.4] to construct the following objects:

– a finite set Σ of prime numbers containing Σ_0 (where Σ_0 is as in §1.8.3).

– a set $\mathcal{K}_i$ of neat compact open subgroups of $G_i(\mathbb{A}_f)$ for $i \in \{0, 1\}$ such that K and $K \cap gKg^{-1}$ are elements of $\mathcal{K}_0$.

– a set $\mathcal{K}_2$ of admissible levels, in the sense of Definition 3.3.2.

– a subset A_i of $G_i(\mathbb{A}_f)$ for $i \in \{0, 1, 2\}$ such that 1 and g are elements of A_0.

– a smooth quasi-projective scheme $\mathscr{S}_U(G_i, \mathcal{X}_i)$ over $\mathbb{Z}[1/\Sigma]$ with generic fiber $\mathrm{Sh}_U(G_i, \mathcal{X}_i)$, for each $i \in \{0, 1, 2\}$ and each $U \in \mathcal{K}_i$. Here when $i = 2$, the Shimura variety $\mathrm{Sh}_U(G_2, \mathcal{X}_2)$ at the admissible level U is understood as in §3.3.3.

– a normal projective scheme $\overline{\mathscr{S}_U}(G_i, \mathcal{X}_i)$ over $\mathbb{Z}[1/\Sigma]$ containing $\mathscr{S}_U(G_i, \mathcal{X}_i)$ as a dense open subscheme, whose generic fiber is the Baily–Borel compactification $\overline{\mathrm{Sh}_U}(G_i, \mathcal{X}_i)$ of $\mathrm{Sh}_U(G_i, \mathcal{X}_i)$, for each $i \in \{0, 1\}$ and each $U \in \mathcal{K}_i$.

These objects should satisfy all the requirements in [**Mor10b**, Prop. 1.3.4] and the paragraph following it. To be more precise, the formulations of these requirements need to be suitably modified when they concern zero-dimensional boundary strata. In the above, we have already modified the formulation of [**Mor10b**, Prop. 1.3.4] when it concerns $\mathcal{K}_2$, i.e., our $\mathcal{K}_2$ is a set of admissible levels, which are more general than neat compact open subgroups of $G_2(\mathbb{A}_f) = \mathbb{G}_m(\mathbb{A}_f)$. The conditions (a), (b), and (1)–(7) in [**Mor10b**, §1.3] also need to be modified as follows.

– In condition (a), if $j = 2$, we need to replace $L_{P'}(\mathbb{Q})$ with $L_{P'}(\mathbb{Q})^+$. (Here P' is either P_2 or P_{12}, and $L_{P'}(\mathbb{Q})^+$ is the same as $L_{P'}(\mathbb{Q}) \cap L_{P_2}(\mathbb{Q})^\natural$.) After this replacement, the quotient group in question is naturally a subgroup of $G_2(\mathbb{A}_f) \times \mathbb{Z}/2\mathbb{Z} = \mathbb{G}_m(\mathbb{A}_f) \times \mathbb{Z}/2\mathbb{Z}$, and the requirement is that this subgroup should be a member of $\mathcal{K}_2$.

– As in the paragraph following [**Mor10b**, Prop. 1.3.4], we may and shall assume that the $\mathcal{K}_i$ are minimal in the following sense. We assume that $\mathcal{K}_0$ is the union of the $G(\mathbb{A}_f)$-conjugacy class of K and that of $K \cap gKg^{-1}$. Then we determine $\mathcal{K}_1$ as the minimal set that is stable under $G_1(\mathbb{A}_f)$-conjugacy and such that condition (a) is satisfied for $(i, j) = (0, 1)$. Having determined $\mathcal{K}_0$ and $\mathcal{K}_1$, we determine $\mathcal{K}_2$ as the minimal set such that the modified version of condition (a) as above is satisfied for $(i, j) \in \{(0, 2), (1, 2)\}$. In particular, $\mathcal{K}_1$ is finite modulo $G_1(\mathbb{A}_f)$-conjugacy, and $\mathcal{K}_2$ is finite.

– In condition (b), if $j = 2$, we still keep $L_P(\mathbb{Q})$, and do *not* replace it with $L_P(\mathbb{Q})^\natural$.

– In conditions (3) and (4), if $i < 2$, then the relevant requirements about zero-dimensional boundary strata should be reinterpreted in the obvious way, taking into account that in the generic fiber these strata are given by the generalized Shimura varieties $\mathrm{Sh}_U(\mathbb{G}_m, \mathcal{H}_0)$ at admissible levels U; cf. §3.4.5.

– In conditions (5)–(7), for $i = 2$ and $U \in \mathcal{K}_2$, the sheaves on the integral model $\mathscr{S}_U(G_2, \mathcal{X}_2)$ in question should be extensions of those sheaves on the generic fiber $\mathrm{Sh}_U(G_2, \mathcal{X}_2)$ that are constructed by the functors (3.3.7.1), (3.3.7.2), and (3.3.7.3). (Indeed, by the minimality of $\mathcal{K}_2$ assumed above, each $U \in \mathcal{K}_2$ is of the form either $\bar{H}/\bar{H}_L^\natural$ or $K_{M,\diamond}$, for a suitable choice of L and K_M as in §3.3.5; cf. §3.4.5.)

With the above modifications, the same proof of [**Mor10b**, Prop. 1.3.4] still goes through.

Remark 3.5.2. — The construction in §3.5.1 can be easily generalized to an arbitrary abelian-type Shimura datum satisfying **A0**–**A3** in §3.4.1 and **CA** in §3.4.4.

Next we would like to to compare the localizations of the integral models constructed in §3.5.1 with other known integral models, at least at almost all primes. We need some preparations.

Definition 3.5.3. — Let S be a scheme of finite type over $\mathbb{Q}$.

(1) By a *family of local integral models* of S, we mean the choice of an integral model $\mathcal{S}_p$ of S over $\mathbb{Z}_p$ (i.e. a $\mathbb{Z}_p$-scheme with generic fiber $S \otimes_{\mathbb{Q}} \mathbb{Q}_p$) for almost all primes p. Two such families $(\mathcal{S}_p)_{p\gg 0}$ and $(\mathcal{S}'_p)_{p\gg 0}$ are called *equivalent*, if for almost all p there exists a $\mathbb{Z}_p$-isomorphism $\mathcal{S}_p \xrightarrow{\sim} \mathcal{S}'_p$ extending the identity on the generic fiber.

(2) Given a finite-type $\mathbb{Z}$-scheme $\mathcal{S}$ with generic fiber S, we obtain a family of local integral models $(\mathcal{S} \otimes_{\mathbb{Z}} \mathbb{Z}_p)_{p\gg 0}$ of S. Any family of local integral models equivalent to such a family is called *eventually globalizable*.

Remark 3.5.4. — By the "spreading out" property of isomorphisms (see [**Gro66**, Thm. (8.10.5) (i)] or [**Poo17**, Thm. 3.2.1]), the eventually globalizable condition characterizes the family of local integral models up to equivalence.

Lemma 3.5.5. — *Let R be an integral domain, with fraction field F. Let $\mathcal{Y}$ be a scheme flat and locally of finite presentation over R. Let X be a scheme over F, and let $\pi : \mathcal{Y} \otimes_R F \to X$ be an F-morphism. Then there exists at most one separated R-scheme $\mathcal{X}$ with generic fiber X such that π extends to an fppf R-morphism $\pi_0 : \mathcal{Y} \to \mathcal{X}$.*

Proof. — Let $\mathcal{X}$ and $\mathcal{X}'$ be two separated R-schemes with generic fiber X, together with fppf R-morphisms $\pi_0 : \mathcal{Y} \to \mathcal{X}$ and $\pi'_0 : \mathcal{Y} \to \mathcal{X}'$ extending π. We claim that π'_0 factors uniquely through π_0. The lemma follows from the claim by symmetry.

To prove the claim, form the fiber product $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ with respect to $\pi_0 : \mathcal{Y} \to \mathcal{X}$. Since π_0 is an fpqc covering and therefore a universal effective epimorphism, it suffices to check the equality of the two morphisms

$$g_i : \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \xrightarrow{\mathrm{pr}_i} \mathcal{Y} \xrightarrow{\pi'_0} \mathcal{X}', \qquad i = 1, 2.$$

Since both π_0 and the structure morphism $\mathcal{Y} \to \operatorname{Spec} R$ are flat and locally of finite presentation, the same holds for the structure morphism $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \to \operatorname{Spec} R$, which implies that it is open. Hence the generic fiber of $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ is dense in $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$. Since the R-morphisms g_1 and g_2 agree on the dense generic fiber, and since the target $\mathcal{X}'$ is separated over R (which implies that the locus where $g_1 = g_2$ is closed), we conclude that $g_1 = g_2$ on a closed subscheme of $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ whose underlying topological space is that of $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$. In particular g_1 and g_2 induce the same map at the level of topolgical spaces. To finish the proof, we can reduce to the affine case, namely we can replace $\mathcal{X}'$ by an affine R-scheme $\operatorname{Spec} A$, and replace $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ by an affine R-scheme $\operatorname{Spec} B$ flat over R. We know that $g_1, g_2 : A \to B$ induce the same map $A \otimes_R F \to B \otimes_R F$. Hence we can conclude that $g_1 = g_2$ since $B \to B \otimes_R F$ is injective. □

3.5.6. — We keep the notation in §3.5.1. In the following, by "enlarging Σ" we always mean replacing Σ by a finite set of primes containing Σ. Also, when we write $p \notin \Sigma$ it is understood that p is a prime.

Let $(\mathrm{GSpin}(V), \mathcal{X}')$ be the GSpin Shimura datum associated to the quadratic space V, which is of Hodge type and has reflex field $\mathbb{Q}$. The natural homomorphism $\mathrm{GSpin}(V) \to G$ extends to a morphism $(\mathrm{GSpin}(V), \mathcal{X}') \to (G, \mathcal{X})$ of Shimura data, inducing an isomorphism between the adjoint Shimura data. For more details see [**MP16**, §3].

We fix a neat compact open subgroup $\tilde{K} \subset \mathrm{GSpin}(V)(\mathbb{A}_f)$ such that its image in $G(\mathbb{A}_f)$ is contained in K. We denote by $\mathrm{Sh}_{\tilde{K}}$ the canonical model over $\mathbb{Q}$ of the Shimura variety associated to $(\mathrm{GSpin}(V), \mathcal{X}')$ at level $\tilde{K}$, and denote by $\overline{\mathrm{Sh}_{\tilde{K}}}$ the Baily–Borel compactification over $\mathbb{Q}$. Thus $\mathrm{Sh}_{\tilde{K}}$ is smooth quasi-projective over $\mathbb{Q}$, and $\overline{\mathrm{Sh}_{\tilde{K}}}$ is normal projective over $\mathbb{Q}$. There are natural $\mathbb{Q}$-morphisms $\pi : \mathrm{Sh}_{\tilde{K}} \to \mathrm{Sh}_K$ and $\bar{\pi} : \overline{\mathrm{Sh}_{\tilde{K}}} \to \overline{\mathrm{Sh}_K}$.

Note that π is finite étale surjective. Indeed, by fpqc descent, it suffices to check these properties for the base change of π to $\mathbb{C}$, which is clear from the adelic description of the Shimura varieties over $\mathbb{C}$ and Hilbert 90 applied to $\ker(\mathrm{GSpin}(V) \to G) = \mathbb{G}_m$; cf. [**MP16**, §3.2].

Recall that $K \in \mathcal{K}_0$. We let $\mathscr{S}_K = \mathscr{S}_K(G, \mathcal{X})$ be the smooth quasi-projective scheme over $\mathbb{Z}[1/\Sigma]$ with generic fiber Sh_K as given in §3.5.1. By standard "spreading out" (see [**Poo17**, Thm. 3.2.1]), we may and shall assume that the following objects exist after enlarging Σ:

– a smooth quasi-projective scheme $\mathscr{S}_{\tilde{K}}$ over $\mathbb{Z}[1/\Sigma]$ with generic fiber $\mathrm{Sh}_{\tilde{K}}$.
– a normal projective scheme $\overline{\mathscr{S}_{\tilde{K}}}$ over $\mathbb{Z}[1/\Sigma]$ with generic fiber $\overline{\mathrm{Sh}_{\tilde{K}}}$.
– a dense open embedding $\mathscr{S}_{\tilde{K}} \hookrightarrow \overline{\mathscr{S}_{\tilde{K}}}$ extending the embedding $\mathrm{Sh}_{\tilde{K}} \hookrightarrow \overline{\mathrm{Sh}_{\tilde{K}}}$.
– a finite étale surjective morphism $\pi_0 : \mathscr{S}_{\tilde{K}} \to \mathscr{S}_K$ extending π.

We also enlarge Σ so that the following condition holds:

– For each $p \notin \Sigma$, there are reductive group schemes $\tilde{\mathcal{G}}_p$ and $\mathcal{G}_p$ over $\mathbb{Z}_p$ with generic fibers $\mathrm{GSpin}(V)_{\mathbb{Q}_p}$ and $G_{\mathbb{Q}_p}$ respectively such that the homomorphism $\mathrm{GSpin}(V)_{\mathbb{Q}_p} \to G_{\mathbb{Q}_p}$ extends to a homomorphism $\tilde{\mathcal{G}}_p \to \mathcal{G}_p$. Moreover, we have $\tilde{K} = \tilde{\mathcal{G}}_p(\mathbb{Z}_p)\tilde{K}^p$ and $K = \mathcal{G}_p(\mathbb{Z}_p)K^p$ for some compact open subgroups $\tilde{K}^p \subset \mathrm{GSpin}(V)(\mathbb{A}_f^p)$ and $K^p \subset G(\mathbb{A}_f^p)$.

Lemma 3.5.7. — *In the setting of §3.5.6, it is possible to further enlarge Σ and find a number field F unramified outside Σ such that the following conditions hold for all $p \notin \Sigma$. Here all isomorphisms between integral models are required to extend the identity on the generic fiber.*

(1) *For each $U \in \mathcal{K}_2$, $\mathscr{S}_U(G_2, \mathcal{X}_2) \otimes \mathbb{Z}_p$ is isomorphic to the base change to $\mathbb{Z}_p$ of the canonical integral model of $\mathrm{Sh}_U(G_2, \mathcal{X}_2)$ in §3.3.7.*

(2) *For each $U \in \mathcal{K}_1$, $\mathscr{S}_U(G_1, \mathcal{X}_1) \otimes \mathbb{Z}_p$ is isomorphic to the canonical hyperspecial integral model over $\mathbb{Z}_p$ of the modular curve $\mathrm{Sh}_U(G_1, \mathcal{X}_1)$.*

(3) *The integral model* $\mathscr{S}_{\tilde{K}} \otimes \mathbb{Z}_p$ *(resp.* $\overline{\mathscr{S}_{\tilde{K}}} \otimes \mathbb{Z}_p$*) is isomorphic to the canonical hyperspecial integral model* $\mathscr{S}_{\tilde{K},p,\mathrm{can}}$ *(resp.* $\overline{\mathscr{S}_{\tilde{K},p,\mathrm{can}}}$*) over* $\mathbb{Z}_p$ *constructed in* [**Kis10**] *(resp. in* [**MP19**]*).*

(4) *The integral model* $\mathscr{S}_K \otimes \mathbb{Z}_p$ *is isomorphic to the canonical hyperspecial integral model* $\mathscr{S}_{K,p,\mathrm{can}}$ *over* $\mathbb{Z}_p$ *constructed in* [**Kis10**].

(5) *For each place* v *of* F *above* p, $\overline{\mathscr{S}_K} \otimes_{\mathbb{Z}} \mathcal{O}_{F,v}$ *is isomorphic to the base change to* $\mathcal{O}_{F,v}$ *of the integral model over* $\mathbb{Z}_p$ *of* $\overline{\mathrm{Sh}_K}$ *constructed in* [**LS18**, Prop. 2.4].

Proof. — First note that for (1) and (2) it suffices to show that we can enlarge Σ for each U separately, since $\mathcal{K}_1$ is finite modulo $G_1(\mathbb{A}_f)$-conjugacy and $\mathcal{K}_2$ is finite. (Indeed, as is implicit in the proof of [**Mor10b**, Prop. 1.3.4], the integral models at conjugate levels are by construction isomorphic to each other.)

For (1)–(3), we know that the canonical integral models in each case form an eventually globalizable family of local integral models (Definition 3.5.3) as p varies. We are done by Remark 3.5.4.

For (4), we would like to apply Lemma 3.5.5 to characterize $\mathscr{S}_{K,p,\mathrm{can}}$ in terms of $\mathscr{S}_{\tilde{K},p,\mathrm{can}}$. Let $p \notin \Sigma$. By the construction in [**Kis10**] (cf. [**LS18**, Prop. 2.4, Remark 2.6]) and by the surjectivity of π, the morphism $\pi_{\mathbb{Q}_p} : \mathrm{Sh}_{\tilde{K}} \otimes_{\mathbb{Q}} \mathbb{Q}_p \to \mathrm{Sh}_K \otimes_{\mathbb{Q}} \mathbb{Q}_p$ extends to a finite étale surjective (hence fppf) morphism $\mathscr{S}_{\tilde{K},p,\mathrm{can}} \to \mathscr{S}_{K,p,\mathrm{can}}$. We also know that $\mathscr{S}_{\tilde{K},p,\mathrm{can}}$ is flat of finite presentation over $\mathbb{Z}_p$. By [**LS18**, Prop. 2.4], $\mathscr{S}_{K,p,\mathrm{can}}$ is quasi-projective and hence separated over $\mathbb{Z}_p$. By part (3), we may assume that $\mathscr{S}_{\tilde{K},p,\mathrm{can}} = \mathscr{S}_{\tilde{K}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Since $\mathscr{S}_{\tilde{K}} \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathscr{S}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is also finite étale surjective and since $\mathscr{S}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is also separated over $\mathbb{Z}_p$ (as it is quasi-projective), we know from Lemma 3.5.5 that $\mathscr{S}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is isomorphic to $\mathscr{S}_{K,p,\mathrm{can}}$ as integral models of Sh_K.

For (5), we let $(C_{i,\mathrm{geom}})_{i\in I}$ be the connected components of $\overline{\mathrm{Sh}_{\tilde{K}}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$, and let $(D_{j,\mathrm{geom}})_{j\in J}$ be the connected components of $\overline{\mathrm{Sh}_K} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$. For each $i \in I$, let $C^0_{i,\mathrm{geom}}$ be the intersection of $C_{i,\mathrm{geom}}$ with $\mathrm{Sh}_{\tilde{K}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$, and similarly define $D^0_{j,\mathrm{geom}}$. The morphism $\bar{\pi} : \overline{\mathrm{Sh}_{\tilde{K}}} \to \overline{\mathrm{Sh}_K}$ induces a surjection $I \to J$, which we still denote by $\bar{\pi}$. As in the proof of [**LS18**, Prop. 2.4], we know that for each $i \in I$, the morphism $C_{i,\mathrm{geom}} \to D_{\bar{\pi}(i),\mathrm{geom}}$ induced by π is the quotient by a finite group Δ_i acting on $C_{i,\mathrm{geom}}$, in the sense of [**LS18**, Rmk. 2.6]. Moreover, Δ_i acts freely on $C^0_{i,\mathrm{geom}}$ and the Galois étale cover $C^0_{i,\mathrm{geom}} \to D^0_{\bar{\pi}(i),\mathrm{geom}}$ is a Δ_i-torsor. We pick a number field F such that each $C_{i,\mathrm{geom}}$ is the base change of a connected component C_i of $\overline{\mathrm{Sh}_{\tilde{K}}} \otimes_{\mathbb{Q}} F$, and such that the action of Δ_i on $C_{i,\mathrm{geom}}$ descends to C_i. For each $i \in I$, define D_i to be the quotient C_i/Δ_i, in the sense of [**LS18**, Rmk. 2.6]. We fix a section $\iota : J \to I$ of the surjection $I \to J$. Then it is clear that $\overline{\mathrm{Sh}_K} \otimes_{\mathbb{Q}} F$ can be identified with $\coprod_{j\in J} D_{\iota(j)}$.

Since our choice of F is independent of Σ, we can enlarge Σ such that F is unramified outside Σ. After further enlarging Σ, we may and shall assume that each C_i is contained in a unique connected component $\mathscr{C}_i$ of $\overline{\mathscr{S}_{\tilde{K}}} \otimes_{\mathbb{Z}} \mathcal{O}_F$, and that the action of Δ_i on C_i extends to $\mathscr{C}_i$. Since the formation of the quotient of a quasi-projective scheme by the action of a finite group commutes with flat base change, we know that the generic fiber of $\coprod_{j\in J}\mathscr{D}_{\iota(j)}$ is the same as $\coprod_{j\in J} D_{\iota(j)}$, which we have already identified with $\overline{\mathrm{Sh}_K}\otimes_{\mathbb{Q}} F$. Thus $\coprod_{j\in J}\mathscr{D}_{\iota(j)}$ and $\overline{\mathscr{S}_K}\otimes_{\mathbb{Z}}\mathcal{O}_F$ are two finite-type $\mathcal{O}_F\otimes_{\mathbb{Z}}\mathbb{Z}[1/\Sigma]$-schemes with the common generic fiber, and we can hence enlarge Σ to assume that they are $\mathcal{O}_F$-isomorphic. It is then clear from parts (3) and (4) above, and the construction in the proof of [**LS18**, Prop. 2.4], that the condition in (5) holds for all $p\notin\Sigma$ and all places v of F above p. □

3.6. Finish of the proof

Essentially all arguments in [**Mor10b**, Chap. 1] can be easily modified to suit our new axiomatic setting (i.e. **A0**–**A3** in §3.4.1 plus **CA** in §3.4.4). With each appearance of H_L replaced by $H_L^{\natural}$, the results of [**Mor10b**, §1.4, §1.5] all carry over. More precisely, in [**Mor10b**, Prop. 1.4.5], if the index n_r corresponds to zero-dimensional boundary data, then we replace the functor $\mathcal{F}^{H/H_L}R\Gamma(H_L/K_N,-)$ with the functor (3.3.7.1) (applied to $M=M_P$, $\bar{H}=$ the image of H under $P(\mathbb{A}_f)\to M_P(\mathbb{A}_f)$, and $\bar{H}_L^{\natural}=$ the intersection of $\bar{H}$ with $L_{n_r}(\mathbb{Q})^+$). We then modify [**Mor10b**, Cor. 1.4.6] correspondingly (by replacing the functor $\mathcal{F}^{H/H_L}R\Gamma_c(H_L/K_N,-)$ with the functor (3.3.7.2)), and modify the definitions of L_{C_1} and L_{C_2} on pp. 17–18 of [**Mor10b**, §1.5] correspondingly.

Let $\overline{\mathscr{S}_K}$ be the integral model constructed in §3.5.1. We now explain the modification of the proof of [**Mor10b**, Thm. 1.7.1], applied to the special fiber of $\overline{\mathscr{S}_K}$ modulo a prime $p\notin\Sigma$, where Σ is as in Lemma 3.5.7. We follow the notation in *loc. cit.*. Modifications are only needed when n_r corresponds to zero-dimensional boundary data. In this case, the definitions of v_h and $u_{h'}$ need to be modified in accordance with the modifications in [**Mor10b**, Cor. 1.4.6, §1.5] mentioned above. To get the relation between $\mathrm{Tr}(v_h)$ and $\mathrm{Tr}(u_{h'})$, we need to apply Proposition 3.3.16 in place of [**Mor10b**, Prop. 1.7.2]. Finally, in the calculation of $\mathrm{Tr}(v_h)$ on the bottom of [**Mor10b**, p. 25], we apply Proposition 3.3.14 in place of [**Mor10b**, Rmk. 1.6.4, Rmk. 1.6.5], and apply Proposition 3.2.3 in place of [**Mor10b**, Thm. 1.6.6]. Note that Proposition 3.3.14 is applicable thanks to condition (1) in Lemma 3.5.7. Also, the fixed point formula of Kottwitz for the one-dimensional boundary strata is applicable thanks to condition (2) in Lemma 3.5.7.

After calculating $\mathrm{Tr}(v_h)$, the same arguments as those on pp. 26–27 of [**Mor10b**] lead to a modified version of [**Mor10b**, Thm. 1.7.1], where the right hand side of the equality in that theorem is replaced by

$$\mathrm{Tr}(\mathrm{Frob}_p^a\times f^{\infty}dg^{\infty}\mid \mathbf{H}_c^*(\mathscr{S}_K\otimes_{\mathbb{Z}}\overline{\mathbb{F}}_p,\mathcal{F}^K\mathbb{V}))+\mathrm{T}_{P_1}+\mathrm{T}_{P_2}+\mathrm{T}_{P_{12}},$$

with the terms T_{P_1}, T_{P_2}, and $\mathrm{T}_{P_{12}}$ as in Definition 2.5.5.[(6)] From this, we deduce the analogue of the identity (2.5.7.1) for the special fiber. Namely we have proved (2.5.7.1) for a sufficiently large, but with $\overline{\mathrm{Sh}_K}$ and Sh_K replaced by the mod p reductions of the integral models.

To prove (2.5.7.1) itself, we apply [**LS18**, Thm. 4.19]. This result confirms Theorem 1.8.4 (1) and asserts that the terms $\mathrm{Tr}(\cdots \mid \mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V}))$ and $\mathrm{Tr}(\cdots \mid \mathbf{H}_c^*(\mathrm{Sh}_K, \mathbb{V}))$ in (2.5.7.1) are unchanged if we replace $\overline{\mathrm{Sh}_K}$ and Sh_K by the mod p reductions of the integral models.[(7)] Indeed, by [**LS18**, Thm. 4.19] and conditions (4), (5) in Lemma 3.5.7, we know that the compact support cohomology and the intersection cohomology (with coefficients in $\mathbb{V}$) of $\mathrm{Sh}_{K,\overline{\mathbb{Q}}_p}$ are respectively isomorphic to those of $\mathscr{S}_{K,\overline{\mathbb{F}}_p}$ under the canonical adjunction morphisms (which are Hecke-equivariant and $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-equivariant). Note that in Lemma 3.5.7 (5) we only compare the integral models over an extension of $\mathbb{Z}_p$, but this already suffices for the current purpose since whether the canonical adjunction morphisms are isomorphisms is insensitive to finite base change. This finishes the proof of Theorem 1.8.4 (1) and (2.5.7.1). In Proposition 2.5.7, we have already proved that (2.5.7.1) is equivalent to the identity (1.8.4.1) in Theorem 1.8.4 (2).

Finally, we explain why the two sides of (1.8.4.1) lie in $\mathbb{E}$ for all sufficiently large a. In the above proof of (2.5.7.1), it is already implicit that the LHS of (2.5.7.1) lies in the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ inside $\overline{\mathbb{Q}}_\ell$, and that the equality holds when we view the LHS as a number in $\mathbb{C}$ by choosing an arbitrary $\mathbb{E}$-algebra embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. (Remember that at the outset we fixed field embeddings $\mathbb{E}_\lambda \hookrightarrow \overline{\mathbb{Q}}_\ell$ and $\mathbb{E} \hookrightarrow \mathbb{C}$, and that the RHS of (2.5.7.1) is a number in $\mathbb{C}$.) Since the definition of the RHS of (2.5.7.1) depends only on the embedding $\mathbb{E} \hookrightarrow \mathbb{C}$ but not on the choice of $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, we see that both sides of (2.5.7.1) are in $\mathbb{E}$ since they must be fixed by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$. Thus it remains to check

$$\mathrm{Tr}(\mathrm{Frob}_p^a \times f^\infty dg^\infty \mid \mathbf{H}_c^*(\mathrm{Sh}_K, \mathbb{V})) \in \mathbb{E}.$$

But this follows from the point counting formula in [**KSZ**].

The proof of Theorem 1.8.4 is complete.

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(6) Note that the factor $\mathtt{m}_M$ in Definition 2.5.5 comes from the factor 2 in Proposition 3.2.3, which is an analogue of [**Mor10b**, Thm. 1.6.6]. By contrast, in Morel's case the extra factor 2 comes from [**Mor10b**, Rmk. 1.6.5].

(7) In [**LS18**, §3], an extra assumption is made on the relation between the level K and the prime ℓ. This assumption can be easily removed if we consider the system of levels in [**Pin92a**, §4.9] instead of the system $\mathcal{H}(\ell^r), r > 0$ in the notation of [**LS18**, §3].

CHAPTER 4

COMPARISON WITH DISCRETE SERIES CHARACTERS

4.1. Elliptic maximal tori in Levi subgroups

4.1.1. — We now pass to a local setting over $\mathbb{R}$. The symbols V, V_i, W_i, G, P_S, and M_S will now denote the base change to $\mathbb{R}$ of the corresponding objects in §1.4. We note that over $\mathbb{R}$, P_{12} is still a minimal parabolic subgroup of G, and P_{12}, P_1, P_2 are still the only proper parabolic subgroups of G containing P_{12}. Also note that the split component of M_S over $\mathbb{R}$ is just the base change to $\mathbb{R}$ of the split component over $\mathbb{Q}$. For this reason we still use the notation A_{M_S} for the split component over $\mathbb{R}$.

Note that W_2 and W_1 are quadratic spaces of signatures $(n-2, 0)$ and $(n-1, 1)$ respectively. We have

$$M_1 \cong \mathrm{GL}_2 \times \mathrm{SO}(W_2), \qquad M_2 \cong \mathrm{GL}_1 \times \mathrm{SO}(W_1), \qquad M_{12} \cong \mathbb{G}_m^2 \times \mathrm{SO}(W_2).$$

Hence M_1 and M_{12} always contain elliptic maximal tori (over $\mathbb{R}$), whereas M_2 contains elliptic maximal tori if and only if d is odd (recall that when d is even we assume that $n = d - 2 \geq 4$). We fix an elliptic maximal torus T_{W_2} in $\mathrm{SO}(W_2)$. We then obtain elliptic maximal tori:

$$T_1 := T_{\mathrm{GL}_2}^{\mathrm{std}} \times T_{W_2} \subset M_1 = \mathrm{GL}_2 \times \mathrm{SO}(W_2),$$
$$T_{12} := \mathbb{G}_m^2 \times T_{W_2} \subset M_{12} = \mathbb{G}_m^2 \times \mathrm{SO}(W_2),$$

where

$$T_{\mathrm{GL}_2}^{\mathrm{std}} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2 \mid a = d, b = -c \right\}.$$

When d is odd, we also fix an elliptic maximal torus T_{W_1} in $\mathrm{SO}(W_1)$, and obtain an elliptic maximal torus $T_2 = \mathbb{G}_m \times T_{W_1}$ in $M_2 = \mathbb{G}_m \times \mathrm{SO}(W_1)$.

4.1.2. — We define a maximal torus T' in $G_{\mathbb{C}}$ as follows. Remember that V is the orthogonal direct sum of span $\{e_1, e_1'\}$, span $\{e_2, e_2'\}$, and W_2. We choose a hyperbolic basis (see Definition 1.2.2) $\mathbb{B} = \{f_1, \cdots, f_d\}$ of the quadratic space $V_{\mathbb{C}}$ over $\mathbb{C}$ such that

$$f_1 = e_1, \qquad f_2 = e_2, \qquad f_d = e_1', \qquad f_{d-1} = e_2'.$$

As in §1.2.7, from $\mathbb{B}$ we obtain an embedding

$$\iota_{\mathbb{B}} : \mathbb{G}_{m,\mathbb{C}}^m \xrightarrow{\sim} T' \subset G_{\mathbb{C}},$$

and a Borel subgroup B of $G_{\mathbb{C}}$ containing T'. By construction, T' is contained in $M_{\mathbb{C}}$ for each $M \in \{M_1, M_2, M_{12}\}$, and B is contained in $P_{\mathbb{C}}$ for each $P \in \{P_1, P_2, P_{12}\}$. Moreover, $\iota_{\mathbb{B}}$ identifies the first two copies of $\mathbb{G}_m$ with the split component $\mathbb{G}_m^2 = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2/V_1)$ of M_{12}.

Let S be a non-empty subset of $\{1, 2\}$ and assume that $S \neq \{2\}$ if d is even. We fix an element $g_S \in M_S(\mathbb{C})$ such that $\mathrm{Int}(g_S)(T_{S,\mathbb{C}}) = T'$. Denote the standard characters of $\mathbb{G}_{m,\mathbb{C}}^m \cong T'$ by $\epsilon_1, \cdots, \epsilon_m$, and the standard cocharacters by $\epsilon_1^\vee, \cdots, \epsilon_m^\vee$. We transport them to $T_{S,\mathbb{C}}$ using $\mathrm{Int}(g_S)$, and retain the same notation.

For S as above, we let R_S be the subset of $\Phi(G_{\mathbb{C}}, T_{S,\mathbb{C}})$ consisting of real elements, and similarly we define $R_S^\vee \subset \Phi(G_{\mathbb{C}}, T_{S,\mathbb{C}})^\vee$. We view R_S and $R_S^\vee$ as subsets of $X^*(A_{M_S})$ and $X_*(A_{M_S})$ respectively. In Tables 2 and 3 below, we determine R_S and $R_S^\vee$ explicitly in the odd and even cases respectively. In the last rows of the two tables, we record the type of the root datum $(X^*(A_{M_S}), R_S, X_*(A_{M_S}), R_S^\vee)$.

Table 2. Real root systems in the odd case

S	$\{1\}$	$\{2\}$	$\{1,2\}$
R_S	$\{\pm(\epsilon_1+\epsilon_2)\}$	$\{\epsilon_1\}$	$\{\pm\epsilon_1, \pm\epsilon_2, \pm\epsilon_1 \pm \epsilon_2\}$
$R_S^\vee$	$\{\pm(\epsilon_1^\vee+\epsilon_2^\vee)\}$	$\{2\epsilon_1^\vee\}$	$\{\pm 2\epsilon_1^\vee, \pm 2\epsilon_2^\vee, \pm\epsilon_1^\vee \pm \epsilon_2^\vee\}$
$X^*(A_{M_S})$	$\frac{1}{2}\mathbb{Z}(\epsilon_1+\epsilon_2)$	$\mathbb{Z}\epsilon_1$	$\mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2$
$X_*(A_{M_S})$	$\mathbb{Z}(\epsilon_1^\vee+\epsilon_2^\vee)$	$\mathbb{Z}\epsilon_1^\vee$	$\mathbb{Z}\epsilon_1^\vee \oplus \mathbb{Z}\epsilon_2^\vee$
type	A_1	A_1	B_2

Table 3. Real root systems in the even case

S	$\{1\}$	$\{1,2\}$
R_S	$\{\pm(\epsilon_1+\epsilon_2)\}$	$\{\pm\epsilon_1 \pm \epsilon_2\}$
$R_S^\vee$	$\{\pm(\epsilon_1^\vee+\epsilon_2^\vee)\}$	$\{\pm\epsilon_1^\vee \pm \epsilon_2^\vee\}$
$X^*(A_{M_S})$	$\frac{1}{2}\mathbb{Z}(\epsilon_1+\epsilon_2)$	$\mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2$
$X_*(A_{M_S})$	$\mathbb{Z}(\epsilon_1^\vee+\epsilon_2^\vee)$	$\mathbb{Z}\epsilon_1^\vee \oplus \mathbb{Z}\epsilon_2^\vee$
type	A_1	$\mathsf{A}_1 \times \mathsf{A}_1$

4.2. Stable discrete series characters

4.2.1. — We keep the setting of §4.1. Fix an irreducible algebraic representation $\mathbb{V}$ of $G_{\mathbb{C}}$. This gives rise to an L-packet $\Pi(\mathbb{V})$ of discrete series representations of

$G(\mathbb{R})$. Let $\Theta = \Theta_{\mathbb{V}}$ be the stable character associated to $\Pi(\mathbb{V})$, i.e., the sum[(1)] of the characters of the members of $\Pi(\mathbb{V})$.

Let S be a non-empty subset of $\{1, 2\}$, and assume that $S \neq \{2\}$ if d is even. Let $M := M_S$. In §4.1 we fixed an elliptic maximal torus T_S in M. In the sequel, unless otherwise stated, we call an element $\gamma \in T_S(\mathbb{R})$ *regular* if it is regular in G, i.e., if $\alpha(\gamma) \neq 1$ for all $\alpha \in \Phi(G_{\mathbb{C}}, T_{S,\mathbb{C}})$.

The *normalized stable discrete series character* $\Phi_M^G(\cdot, \Theta)$ is defined and studied in [**Art89**] and [**GKM97**]; see also [**Mor10b**, §3.2]. It is a continuous function $T_S(\mathbb{R}) \to \mathbb{C}$ such that

$$\Phi_M^G(\gamma, \Theta) = \left|D_M^G(\gamma)\right|_{\mathbb{R}}^{1/2} \Theta(\gamma)$$

for all regular $\gamma \in T_S(\mathbb{R})$. In the following we recall a formula for $\Phi_M^G(\gamma, \Theta)$, for regular $\gamma \in T_S(\mathbb{R})$.

4.2.2. — In §4.1 we fixed a Borel pair (T', B') in $G_{\mathbb{C}}$, an elliptic maximal torus T_S in M, and an element $g_S \in M(\mathbb{C})$ such that $\mathrm{Int}(g_S)(T_{S,\mathbb{C}}) = T'$. We now denote by B the Borel subgroup $\mathrm{Int}(g_S)^{-1}(B')$ of $G_{\mathbb{C}}$ containing $T_{S,\mathbb{C}}$. Remember that $P_{S,\mathbb{C}} \supset B$. We let $B_M := M_{\mathbb{C}} \cap B$, which is a Borel subgroup of $M_{\mathbb{C}}$. We make the following definitions:

– Denote by Φ^+ the set of B-positive roots in $\Phi(G_{\mathbb{C}}, T_{S,\mathbb{C}})$.

– Denote by Φ_M^+ the set of B_M-positive roots in $\Phi(M_{\mathbb{C}}, T_{S,\mathbb{C}})$.

– Denote by $\rho \in X^*(T_S) \otimes \frac{1}{2}\mathbb{Z}$ the half sum of the elements of Φ^+.

– Denote by $\lambda \in X^*(T_S)$ the highest weight of the $G_{\mathbb{C}}$-representation $\mathbb{V}$ with respect to the Borel pair $(T_{S,\mathbb{C}}, B)$ in $G_{\mathbb{C}}$.

– Denote by Ω the complex Weyl group $\Omega_{\mathbb{C}}(G, T_S)$.

– For $\omega \in \Omega$, denote by ωB the Borel subgroup $\dot{\omega} B \dot{\omega}^{-1}$ of $G_{\mathbb{C}}$, where $\dot{\omega} \in \mathrm{Nor}_{G(\mathbb{C})}(T_S)$ is any representative of ω.

– Denote by Δ_M the Weyl denominator of $M_{\mathbb{C}}$ with respect to the Borel pair $(T_{S,\mathbb{C}}, B_M)$ in $M_{\mathbb{C}}$; see Definition 1.1.3. Thus $\Delta_M = \prod_{\alpha \in \Phi_M^+}(1 - \alpha^{-1})$.

– For $\omega \in \Omega$, define

$$\begin{aligned} \Phi(\omega) &:= \Phi^+ \cap (-\omega\Phi^+), \\ l(\omega) &:= |\Phi(\omega)|, \\ \epsilon(\omega) &:= (-1)^{l(\omega)}. \end{aligned}$$

Thus $l(\omega)$ and $\epsilon(\omega)$ are the length and sign of ω respectively.

(1) Our definition of the stable character is the same as [**Mor11**], whereas in [**GKM97**] a sign $(-1)^{q(G)}$ is included.

Recall that in §4.1 we explicitly identified the set R_S of real roots in $\Phi(G_{\mathbb{C}}, T_{S,\mathbb{C}})$. Since S is currently fixed, we simply write R for R_S. For $\gamma \in T_S(\mathbb{R})$, we define

$$\begin{aligned} R_\gamma &:= \{\alpha \in R \mid \alpha(\gamma) > 0\}, \\ R_\gamma^+ &:= \{\alpha \in R \mid \alpha(\gamma) > 1\}, \\ \epsilon_R(\gamma) &:= (-1)^{\left|\Phi^+ \cap (-R_\gamma^+)\right|} = (-1)^{\#\left\{\alpha \in \Phi^+ \cap R \mid 0 < \alpha(\gamma) < 1\right\}}. \end{aligned}$$

Then by the work of Harish-Chandra [**HC65**] and Herb [**Her79**], we have the following formula for $\Phi_M^G(\gamma, \Theta)$, for regular $\gamma \in T_S(\mathbb{R})$:

$$\begin{aligned} \Phi_M^G(\gamma, \Theta) = (-1)^{q(G)} \epsilon_R(\gamma) \delta_{P_S(\mathbb{R})}(\gamma)^{1/2} \Delta_M(\gamma)^{-1} \\ \cdot \sum_{\omega \in \Omega} \epsilon(\omega) n(\gamma, \omega B)(\omega\lambda)(\gamma) \prod_{\alpha \in \Phi(\omega)} \alpha^{-1}(\gamma). \end{aligned} \tag{4.2.2.1}$$

See also [**GKM97**, §4] and [**Mor11**, Fait 3.1.6]. Here $q(G)$ and $\delta_{P_S(\mathbb{R})}$ are defined in §1.1, and $n(\gamma, \omega B)$ are certain integers, whose definition we now explain following [**GKM97**, §4].

Let G^{SC} be the simply connected cover of G, and write $\mathrm{im}(G^{\mathrm{SC}}(\mathbb{R}))$ for the image of $G^{\mathrm{SC}}(\mathbb{R}) \to G(\mathbb{R})$. Firstly, if $\gamma \notin Z_G(\mathbb{R}) \,\mathrm{im}(G^{\mathrm{SC}}(\mathbb{R}))$, then $n(\gamma, \omega B) = 0$ for all $\omega \in \Omega$.

Remark 4.2.3. — In our case $Z_G(\mathbb{R}) \,\mathrm{im}(G^{\mathrm{SC}}(\mathbb{R})) = G(\mathbb{R})^0$. In fact, since G is semi-simple, we have $\mathrm{im}(G^{\mathrm{SC}}(\mathbb{R})) = G(\mathbb{R})^0$ by the connectedness of $G^{\mathrm{SC}}(\mathbb{R})$. Now in the odd case Z_G is trivial, and in the even case $Z_G(\mathbb{R}) = \{\pm \mathrm{id}_V\}$ is contained in $G(\mathbb{R})^0$ (see [**Kna02**, I.17]).

4.2.4. — We now assume that $\gamma \in T_S(\mathbb{R})$ is regular and lies in $Z_G(\mathbb{R}) \,\mathrm{im}(G^{\mathrm{SC}}(\mathbb{R}))$, and explain the definition of $n(\gamma, \omega B)$ in this case. First we need some preparations.

Let E^* be a finite-dimensional $\mathbb{R}$-vector space, and $U \subset E^*$ a root system. Let E_* denote the dual vector space of E^*, and let $U^\vee \subset E_*$ be the set of coroots. Assume that U spans E^*, and that the Weyl group of U contains $-1 \in \mathrm{GL}(E_*)$. Let $E_{*,\mathrm{reg}} \subset E_*$ and $E^*_{\mathrm{reg}} \subset E^*$ be the regular loci with respect to U and $U^\vee$ respectively. One associates to the datum (E^*, U) a function

$$\bar{c}_U : E_{*,\mathrm{reg}} \times E^*_{\mathrm{reg}} \longrightarrow \mathbb{Z}. \tag{4.2.4.1}$$

This function appeared in the work of Herb [**Her79**], and can be inductively characterized by the properties (1)–(5) listed in [**GKM97**, §3]. We will give explicit formulas for $\bar{c}_U$ in some special cases in Lemmas 4.2.8 and 4.2.10 below. Later in the paper (§§8.15 and 8.16), we will recall Herb's close formula for $\bar{c}_U$ in more complicated situations.

Now we write $X_*(A_M)_{\mathbb{R}}$ and $X^*(A_M)_{\mathbb{R}}$ for $X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}$ and $X^*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}$ respectively, and identify $X_*(A_M)_{\mathbb{R}}$ with $\mathrm{Lie}(A_M)$. We view the Weyl group of the root system R_γ as a subgroup of $\mathrm{GL}(X_*(A_M)_{\mathbb{R}})$. Let $X_*(A_M)_{\mathbb{R},\mathrm{reg}} \subset X_*(A_M)_{\mathbb{R}}$ and

$X^*(A_M)_{\mathbb{R},\mathrm{reg}} \subset X^*(A_M)_{\mathbb{R}}$ be the regular loci, with respect to the root systems R_γ and $R_\gamma^\vee$, respectively.

Lemma 4.2.5 **([GKM97, p. 499]).** — *For regular $\gamma \in T_S(\mathbb{R})$ which lies in $Z_G(\mathbb{R})\,\mathrm{im}(G^{\mathrm{SC}}(\mathbb{R}))$, the Weyl group of R_γ contains $-1 \in \mathrm{GL}(X_*(A_M)_{\mathbb{R}})$.* □

4.2.6. — Keep the setting of §4.2.4. In view of Lemma 4.2.5 and the general construction (4.2.4.1), we obtain a function

$$\bar{c}_{R_\gamma} : X_*(A_M)_{\mathbb{R},\mathrm{reg}} \times X^*(A_M)_{\mathbb{R},\mathrm{reg}} \longrightarrow \mathbb{Z}.$$

We can now define the integers $n(\gamma, \omega B)$ in terms of the function $\bar{c}_{R_\gamma}$. Let $T_S(\mathbb{R})_1$ be the maximal compact subgroup of $T_S(\mathbb{R})$. We have a canonical decomposition

$$T_S(\mathbb{R}) = A_M(\mathbb{R})^0 \times T_S(\mathbb{R})_1.$$

We write the projection of $\gamma \in T_S(\mathbb{R})$ in $A_M(\mathbb{R})^0$ as $\exp(x_\gamma)$, with $x_\gamma \in \mathrm{Lie}(A_M) = X_*(A_M)_{\mathbb{R}}$. Our assumption that γ is regular ensures that

$$x_\gamma \in X_*(A_M)_{\mathbb{R},\mathrm{reg}}.$$

Let $\wp : X^*(T_S)_{\mathbb{R}} \to X^*(A_M)_{\mathbb{R}}$ be the natural restriction map. Then for any $\omega \in \Omega$ we have

$$\wp(\omega\lambda + \omega\rho) \in X^*(A_M)_{\mathbb{R},\mathrm{reg}}.$$

Define

$$n(\gamma, \omega B) := \bar{c}_{R_\gamma}(x_\gamma, \wp(\omega\lambda + \omega\rho)). \tag{4.2.6.1}$$

This finishes our explanation of (4.2.2.1).

Corollary 4.2.7. — *Let $\gamma \in T_S(\mathbb{R})$ be a regular element such that the Weyl group of R_γ does not contain $-1 \in \mathrm{GL}(X_*(A_M)_{\mathbb{R}})$. Then $\Phi_M^G(\gamma, \Theta) = 0$.*

Proof. — By Lemma 4.2.5, we have $\gamma \notin Z_G(\mathbb{R})\,\mathrm{im}(G^{\mathrm{SC}})(\mathbb{R})$. Hence $n(\gamma, \omega B) = 0$ for all $\omega \in \Omega$, and we have $\Phi_M^G(\gamma, \Theta) = 0$ by (4.2.2.1). □

In the sequel we will need explicit descriptions of the function $\bar{c}_U$ for certain root systems U in $\mathbb{R}^1$ and $\mathbb{R}^2$. For $i \in \{1, 2\}$, we use the standard inner product on $\mathbb{R}^i$ to identify $\mathbb{R}^i$ with its own dual space.

Lemma 4.2.8. — *Let ϵ be the basis vector 1 of $\mathbb{R}^1$. The Weyl group of the root system $U = \{\pm\epsilon\}$ contains -1. The regular loci in $\mathbb{R}^1$ with respect to U and with respect to $U^\vee$ are both $\mathbb{R}^1 - \{0\}$. The function $\bar{c}_U : (\mathbb{R}^1 - \{0\}) \times (\mathbb{R}^1 - \{0\}) \to \mathbb{Z}$ is given by:*

$$\bar{c}_U(x\epsilon, y\epsilon) = \begin{cases} 2, & \text{if } xy < 0, \\ 0, & \text{if } xy > 0. \end{cases}$$

Proof. — This follows from a direct computation based on properties (1)–(5) listed in [**GKM97**, §3]. □

4.2.9. — We now consider certain root systems in $\mathbb{R}^2$. Let $\{\epsilon_1 = (1,0), \epsilon_2 = (0,1)\}$ be the natural basis of $\mathbb{R}^2$, and let x_1 and x_2 be the two coordinate functions on $\mathbb{R}^2$. Let[(2)]

$$\begin{aligned} U_{\mathrm{odd}} &:= \{\pm\epsilon_1, \pm\epsilon_2, \pm\epsilon_1 \pm \epsilon_2\}, \\ U_{\mathrm{eds}} &:= \{\pm\epsilon_1, \pm\epsilon_2\} \\ U_{\mathrm{even}} &:= \{\pm\epsilon_1 \pm \epsilon_2\}. \end{aligned}$$

For each subscript $? \in \{\mathrm{odd}, \mathrm{eds}, \mathrm{even}\}$, $U_?$ is a root system in $\mathbb{R}^2$. The regular locus in $\mathbb{R}^2$ with respect to $U_?$ is equal to the regular locus with respect to $U_?^\vee$. We denote this locus by $\mathbb{R}^2_?$.

Explicitly, $\mathbb{R}^2_{\mathrm{odd}}$ is the complement of the two coordinate axes and the two diagonal lines. Thus it is the disjoint union of eight open cones. We label the cone $\{(x_1, x_2) \mid 0 < x_2 < x_1\}$ by the symbol $(\mathcal{I})$, and label the other cones counterclockwise, by $(\mathcal{II})$, $(\mathcal{III})$, $\cdots$, $(\mathcal{VIII})$. See Figure 1.

Similarly, $\mathbb{R}^2_{\mathrm{eds}}$ is the complement of the two coordinate axes, and $\mathbb{R}^2_{\mathrm{even}}$ is the complement of the two diagonal lines $x_1 = \pm x_2$. We label the four open cones constituting $\mathbb{R}^2_{\mathrm{even}}$ counterclockwise, starting with the cone $\{(x_1, x_2) \mid x_1 > |x_2|\}$, by the symbols $(\mathscr{A}), (\mathscr{B}), (\mathscr{C}), (\mathscr{D})$. See Figure 2.

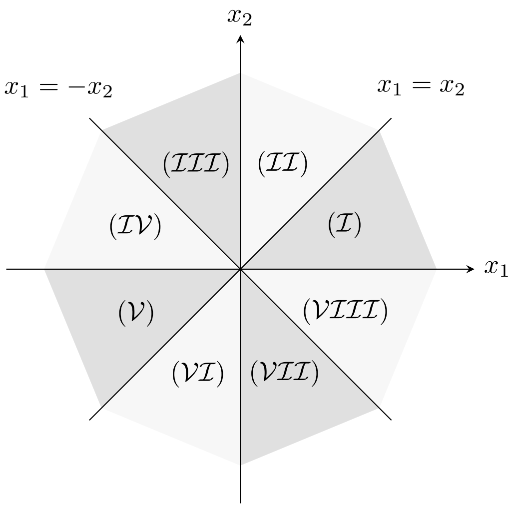

FIGURE 1. Labeling of the eight open cones complement to the two coordinate axes and the two diagonal lines in the x_1-x_2-plane. The union of the cones is denoted by $\mathbb{R}^2_{\mathrm{odd}}$.

(2) Here the subscript eds stands for "endoscopic".

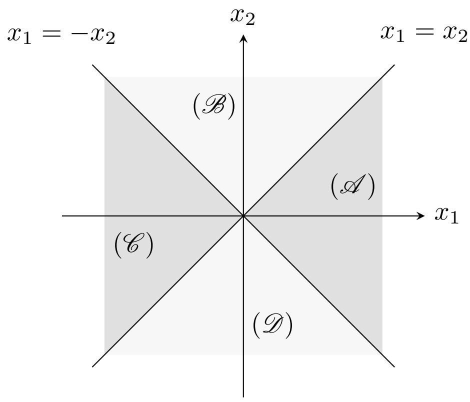

Figure 2. Labeling of the four open cones complement to the two diagonal lines in the x_1-x_2-plane. The union of the cones is denoted by $\mathbb{R}^2_{\mathrm{even}}$.

We shall use the same symbols $(\mathcal{I}), (\mathcal{II}), \cdots, (\mathscr{A}), (\mathscr{B}), \cdots,$ to denote the characteristic functions of the corresponding open cones. For each subscript $? \in \{\mathrm{odd}, \mathrm{eds}, \mathrm{even}\}$, the Weyl group of $U_?$ contains $-1 \in \mathrm{GL}(\mathbb{R}^2)$. Hence we have the associated function

$$\bar{c}_{U_?} : \mathbb{R}^2_? \times \mathbb{R}^2_? \longrightarrow \mathbb{Z}.$$

The following lemma describes this function. For each fixed $x \in \mathbb{R}^2_?$, we let $\mathbf{f}_{?,x} : \mathbb{R}^2_? \to \mathbb{Z}$ be the function that sends $x' \in \mathbb{R}^2_?$ to $\bar{c}_{U_?}(x, x')$.

Lemma 4.2.10. — *The following statements hold.*

(1) *If $x \in (\mathcal{V})$, then*

$$\frac{1}{4}\mathbf{f}_{\mathrm{odd},x} = (\mathcal{II}) + (\mathcal{VIII}). \tag{4.2.10.1}$$

If $x \in (\mathcal{IV})$, then

$$\frac{1}{4}\mathbf{f}_{\mathrm{odd},x} = (\mathcal{I}) + (\mathcal{VII}). \tag{4.2.10.2}$$

(2) *The function $\bar{c}_{U_{\mathrm{eds}}} : \mathbb{R}^2_{\mathrm{eds}} \times \mathbb{R}^2_{\mathrm{eds}} \to \mathbb{Z}$ is given by*

$$\bar{c}_{U_{\mathrm{eds}}}(x, x') = \begin{cases} 4, & \text{if } x \text{ and } x' \text{ lie in opposite quadrants,} \\ 0, & \text{otherwise.} \end{cases} \tag{4.2.10.3}$$

In particular, if $x \in (\mathcal{V})$, then

$$\frac{1}{4}\mathbf{f}_{\mathrm{eds},x}|_{\mathbb{R}^2_{\mathrm{odd}}} = (\mathcal{I}) + (\mathcal{II}). \tag{4.2.10.4}$$

If $x \in (\mathcal{IV})$, then

$$\frac{1}{4}\mathbf{f}_{\mathrm{eds},x}|_{\mathbb{R}^2_{\mathrm{odd}}} = (\mathcal{VII}) + (\mathcal{VIII}). \tag{4.2.10.5}$$

(3) *If* $x \in (\mathscr{C})$, *then*

$$\frac{1}{4}\mathbf{f}_{\mathrm{even},x} = (\mathscr{A}). \tag{4.2.10.6}$$

Proof. — This follows from a direct computation based on properties (1)–(5) listed in [**GKM97**, §3]. □

Remark 4.2.11. — The complete descriptions of $\bar{c}_{U_{\mathrm{odd}}}$ and $\bar{c}_{U_{\mathrm{even}}}$ follow immediately from Lemma 4.2.10 and the Weyl invariance of these functions (see property (5) in [**GKM97**, §3]).

4.3. Kostant's theorem

We apply Kostant's theorem [**Kos61**] to compute the character of the virtual representation in Definition 2.1.4.

Let S be a non-empty subset of $\{1,2\}$, and let $M := M_S$. Assume that $S \neq \{2\}$ in the even case. Let T_S be as in §4.1. We fix $\mathbb{V}$ as in §4.2.1, and continue to use the notations introduced in §4.2.2. Let $R\Gamma(\operatorname{Lie} N_S, \mathbb{V})_{>t_S}$ be as in Definition 2.1.4. Let ϖ_1 and ϖ_2 be as in Definition 2.1.1.

Lemma 4.3.1. — *For $\gamma \in T_S(\mathbb{C})$ regular in G (or more generally, regular in M), we have*

$$\operatorname{Tr}(\gamma \mid R\Gamma(\operatorname{Lie} N_S, \mathbb{V})_{>t_S}) = \Delta_M(\gamma)^{-1} \sum_{\substack{\omega \in \Omega \\ \langle \omega(\lambda+\rho), \varpi_i \rangle > 0, \ \forall i \in S}} \epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha \in \Phi(\omega)} \alpha^{-1}(\gamma).$$

Proof. — The proof is the same as a computation in the proof of [**Mor11**, Prop. 3.3.1]. Let $\Omega_S := \Omega_{\mathbb{C}}(M, T_S)$, which is naturally a subgroup of Ω. For $\omega_1 \in \Omega_S$ we define $l(\omega_1)$ and $\epsilon(\omega_1) = (-1)^{l(w)}$ by viewing ω_1 as in Ω; as a standard fact $l(w_1)$ is also the length of w_1 in Ω_S with respect to the simple roots in Φ_M^+. Consider

$$\begin{aligned}\Omega'_S &:= \left\{ \omega \in \Omega \mid \Phi(\omega) \subset \{\text{roots of } T_{S,\mathbb{C}} \text{ on } \operatorname{Lie}(N_S)_{\mathbb{C}}\} \right\} \\ &= \left\{ \omega \in \Omega \mid \Phi(\omega) \cap \Phi_M^+ = \emptyset \right\}.\end{aligned}$$

Then Ω'_S is the set of minimal length representatives of the cosets in $\Omega_S \backslash \Omega$; see [**Kos61**, p. 361] or [**GHM94**, p. 165]. In particular, multiplication induces a bijection

$$\Omega_S \times \Omega'_S \xrightarrow{\sim} \Omega. \tag{4.3.1.1}$$

We have fixed the positive system Φ_M^+ inside $\Phi(M_{\mathbb{C}}, T_{S,\mathbb{C}})$. As usual, we say that an element $\lambda' \in X^*(T_S)$ is *M-dominant*, if the pairing of λ' with any positive coroot in $\Phi(M_{\mathbb{C}}, T_{S,\mathbb{C}})^\vee$ is non-negative. For such λ', we let $V_{M,\lambda'}$ be the irreducible algebraic representation of $M(\mathbb{C})$ of highest weight λ'.

As recalled on p. 1700 of [**Mor11**], Kostant's theorem states that as an algebraic representation of $M(\mathbb{C})$, we have

$$\mathbf{H}^k(\mathrm{Lie}(N_S)_{\mathbb{C}}, \mathbb{V}) \cong \bigoplus_{\substack{\omega' \in \Omega'_S \\ l(\omega') = k}} V_{M, \omega'(\lambda+\rho)-\rho}.$$

Consequently,

$$\mathbf{H}^k(\mathrm{Lie}(N_S)_{\mathbb{C}}, \mathbb{V})_{>t_S} \cong \bigoplus_{\substack{\omega' \in \Omega'_S \\ l(\omega') = k \\ \langle \omega'(\lambda+\rho)-\rho, \varpi_i \rangle > t_i, \ \forall i \in S}} V_{M, \omega'(\lambda+\rho)-\rho}.$$

By a simple computation, we have $t_i = \langle -\rho, \varpi_i \rangle$ for $i = 1, 2$. Hence we have

$$\mathbf{H}^k(\mathrm{Lie}(N_S)_{\mathbb{C}}, \mathbb{V})_{>t_S} \cong \bigoplus_{\substack{\omega' \in \Omega'_S \\ l(\omega') = k \\ \langle \omega'(\lambda+\rho), \varpi_i \rangle > 0, \ \forall i \in S}} V_{M, \omega'(\lambda+\rho)-\rho}. \tag{4.3.1.2}$$

By the Weyl character formula (see for instance [**Mor11**, Fait 3.1.6]), for any M-dominant $\lambda' \in X^*(T_S)$ we have

$$\mathrm{Tr}(\gamma \mid V_{M,\lambda'}) = \Delta_M(\gamma)^{-1} \sum_{\omega_1 \in \Omega_S} \epsilon(\omega_1)(\omega_1 \lambda')(\gamma) \prod_{\alpha \in \Phi(\omega_1)} \alpha^{-1}(\gamma). \tag{4.3.1.3}$$

(Here we have used the fact that for each $\omega_1 \in \Omega_S$, the set $\Phi(\omega_1) = \Phi^+ \cap (-\omega_1 \Phi^+)$ is also equal to $\Phi^+_M \cap (-\omega_1 \Phi^+_M)$.)

Combining (4.3.1.2) and (4.3.1.3), we obtain

$$\begin{aligned}
\mathrm{Tr}(\gamma \mid R\Gamma(\mathrm{Lie}(N_S), \mathbb{V})_{>t_S}) &= \sum_{\substack{\omega' \in \Omega'_S \\ \langle \omega'(\lambda+\rho), \varpi_i \rangle > 0, \ \forall i \in S}} (-1)^{l(\omega')} \mathrm{Tr}(\gamma \mid V_{M, \omega'(\lambda+\rho)-\rho}) \\
&= \sum_{\substack{\omega' \in \Omega'_S \\ \langle \omega'(\lambda+\rho), \varpi_i \rangle > 0, \ \forall i \in S}} \epsilon(\omega') \Delta_M(\gamma)^{-1} \\
&\quad \cdot \sum_{\omega_1 \in \Omega_S} \epsilon(\omega_1) \Big(\omega_1(\omega'(\lambda+\rho)-\rho) \Big)(\gamma) \prod_{\alpha \in \Phi(\omega_1)} \alpha^{-1}(\gamma).
\end{aligned}$$

Since ϖ_i is invariant under Ω_S for every $i \in S$, and since we have the bijection (4.3.1.1), the above is equal to

$$\sum_{\substack{\omega \in \Omega \\ \langle \omega(\lambda+\rho), \varpi_i \rangle > 0, \ \forall i \in S}} \epsilon(\omega) \Delta_M(\gamma)^{-1} (\omega\lambda)(\gamma) \cdot \Big(\omega\rho - p_1(\omega)\rho \Big)(\gamma) \prod_{\alpha \in \Phi(p_1(\omega))} \alpha^{-1}(\gamma),$$

where for each $\omega \in \Omega$ we set $p_1(\omega)$ to be the unique element of Ω_S such that $\omega \in p_1(\omega)\Omega'_S$. To finish the proof, we just need to check that for all $\omega \in \Omega$, we have

$$\omega(\rho) - p_1(\omega)(\rho) - \sum_{\alpha\in\Phi(p_1(\omega_1))} \alpha = - \sum_{\alpha\in\Phi(\omega)} \alpha.$$

But this follows from the identity

$$\rho - \theta(\rho) = \sum_{\alpha\in\Phi(\theta)} \alpha$$

which holds for arbitrary $\theta \in \Omega$. □

4.4. Kostant–Weyl terms and discrete series characters, case M_1

4.4.1. — We keep the notations in §4.2 and §4.3. We take $S = \{1\}$ and $M = M_1$. Recall from §4.1 that we have fixed an elliptic maximal torus $T_1 = T^{\mathrm{std}}_{\mathrm{GL}_2} \times T_{W_2}$ in M. Consider a regular element $\gamma \in T_1(\mathbb{R})$. We write

$$\gamma = (\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \gamma_{W_2}) \in T^{\mathrm{std}}_{\mathrm{GL}_2}(\mathbb{R}) \times T_{W_2}(\mathbb{R}),$$

with $a, b \in \mathbb{R}$ and $a^2 + b^2 \neq 0$. Note that $(\epsilon_1 + \epsilon_2)(\gamma) = a^2 + b^2$. Hence we have

$$R_\gamma = \{\pm(\epsilon_1 + \epsilon_2)\}.$$

Let $L_M(\gamma)$ be as in Definition 2.2.3.

Proposition 4.4.2. — *Suppose $a^2 + b^2 < 1$. Then we have*

$$\Phi^G_M(\gamma, \Theta) = 2(-1)^{q(G)+1} L_M(\gamma).$$

Proof. — We first compute $\Phi^G_M(\gamma, \Theta)$ using (4.2.2.1). Clearly $T_1(\mathbb{R})$ is connected. Hence $\gamma \in G(\mathbb{R})^0$, and so the integers $n(\gamma, \omega B)$ in (4.2.2.1) are defined by (4.2.6.1).

The subgroup $A_M(\mathbb{R})^0 \subset T_1(\mathbb{R})$ consists of $\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \in T^{\mathrm{std}}_{\mathrm{GL}_2}(\mathbb{R}), z \in \mathbb{R}_{>0}$. The subgroup $T_1(\mathbb{R})_1 \subset T_1(\mathbb{R})$ is $\mathrm{U}(1)(\mathbb{R}) \times T_{W_2}(\mathbb{R})$, where $\mathrm{U}(1)(\mathbb{R})$ consists of

$$\begin{pmatrix} z_1 & z_2 \\ -z_2 & z_1 \end{pmatrix} \in T^{\mathrm{std}}_{\mathrm{GL}_2}(\mathbb{R}), \qquad z_1, z_2 \in \mathbb{R}, z_1^2 + z_2^2 = 1.$$

Hence the projection of γ in $A_M(\mathbb{R})^0 = \mathbb{R}_{>0}$ is $\sqrt{a^2 + b^2}$, and

$$x_\gamma = \log \sqrt{a^2 + b^2} \in \mathbb{R} \cong \mathrm{Lie}(A_M) = X_*(A_M)_\mathbb{R}.$$

Since $a^2 + b^2 < 1$, we have

$$x_\gamma \in \mathbb{R}_{<0} \cong \mathbb{R}_{>0}(-\epsilon_1^\vee - \epsilon_2^\vee).$$

Since $R_\gamma = \{\pm(\epsilon_1+\epsilon_2)\}$, by Lemma 4.2.8 we have

$$\bar{c}_{R_\gamma}(x_\gamma,\chi) = \begin{cases} 2, & \text{if } \chi \in \mathbb{R}_{>0}(\epsilon_1+\epsilon_2), \\ 0, & \text{if } \chi \in \mathbb{R}_{>0}(-\epsilon_1-\epsilon_2). \end{cases}$$

Hence by the definition (4.2.6.1), for $\omega \in \Omega$ we have

$$n(\gamma,\omega B) = \begin{cases} 2, & \text{if } \wp(\omega(\lambda+\rho)) \in \mathbb{R}_{>0}(\epsilon_1+\epsilon_2), \\ 0, & \text{if } \wp(\omega(\lambda+\rho)) \in \mathbb{R}_{>0}(-\epsilon_1-\epsilon_2). \end{cases}$$

Now the term $\epsilon_R(\gamma)$ in (4.2.2.1) is -1. By the above computation and by (4.2.2.1), we obtain

$$(4.4.2.1)\quad \Phi_M^G(\gamma,\Theta) = 2(-1)^{q(G)+1}\delta_{P_1(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1} \cdot \sum_{\substack{\omega\in\Omega \\ \wp(\omega(\lambda+\rho))\in\mathbb{R}_{>0}(\epsilon_1+\epsilon_2)}} \epsilon(\omega)(\omega\lambda)(\gamma)\prod_{\alpha\in\Phi(\omega)}\alpha^{-1}(\gamma).$$

Next we compute $2(-1)^{q(G)+1}L_M(\gamma)$. By Proposition 2.2.4 and Lemma 4.3.1, we have

$$(4.4.2.2)\quad 2(-1)^{q(G)+1}L_M(\gamma) = 2(-1)^{q(G)+1}\delta_{P_1(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1} \cdot \sum_{\substack{\omega\in\Omega \\ \langle\omega(\lambda+\rho),\varpi_1\rangle>0}} \epsilon(\omega)(\omega\lambda)(\gamma)\prod_{\alpha\in\Phi(\omega)}\alpha^{-1}(\gamma).$$

Comparing (4.4.2.1) and (4.4.2.2), we see that the proof reduces to checking that for all $\omega\in\Omega$, we have

$$\langle\omega(\lambda+\rho),\varpi_1\rangle > 0 \iff \wp(\omega(\lambda+\rho)) \in \mathbb{R}_{>0}(\epsilon_1+\epsilon_2).$$

This is obvious. □

4.5. Kostant–Weyl terms and discrete series characters, odd case M_2

4.5.1. — We keep the notations in §4.2 and §4.3. We take $S=\{2\}$ and $M=M_2$. Assume that d is odd. Recall from §4.1 that we have fixed an elliptic maximal torus $T_2 = \mathbb{G}_m \times T_{W_1}$ in M. Consider a regular element $\gamma\in T_2(\mathbb{R})$. We write

$$\gamma = (a,\gamma_{W_1}),$$

with $a\in\mathbb{R}^\times$. If $a>0$, then $R_\gamma=\{\pm\epsilon_1\}$. Otherwise $R_\gamma=\emptyset$. Let $L_M(\gamma)$ be as in Definition 2.2.3.

Proposition 4.5.2. — *When $a<0$, we have $\Phi_M^G(\gamma,\Theta)=0$. When $0<a<1$, we have*

$$\Phi_M^G(\gamma,\Theta) = (-1)^{q(G)+1}L_M(\gamma).$$

Proof. — When $a < 0$, we have $R_\gamma = \emptyset$. It follows from Corollary 4.2.7 that $\Phi^G_M(\gamma, \Theta) = 0$, as desired.

Now assume that $0 < a < 1$. We first compute $\Phi^G_M(\gamma, \Theta)$ using (4.2.2.1). We have $T_2 \cong \mathbb{G}_m \times \mathrm{U}(1)^{m-1}$, and $A_M \cong \mathbb{G}_m$, $T_2(\mathbb{R})_1 = \{\pm 1\} \times \mathrm{U}(1)(\mathbb{R})^{m-1}$. Hence the projection of γ in $A_M(\mathbb{R})^0 = \mathbb{R}_{>0}$ is a, and

$$x_\gamma = \log a \in \mathbb{R} \cong \mathrm{Lie}(A_M) = X_*(A_M)_{\mathbb{R}}.$$

Since $0 < a < 1$, we have

$$x_\gamma \in \mathbb{R}_{<0} \cong \mathbb{R}_{>0}(-\epsilon_1^\vee).$$

Since $R_\gamma = \{\pm\epsilon_1\}$, by Lemma 4.2.8 we have

$$\bar{c}_{R_\gamma}(x_\gamma, \chi) = \begin{cases} 2, & \chi \in \mathbb{R}_{>0}(\epsilon_1), \\ 0, & \chi \in \mathbb{R}_{>0}(-\epsilon_1). \end{cases}$$

Hence by the definition (4.2.6.1), for $\omega \in \Omega$ we have

$$n(\gamma, \omega B) = \begin{cases} 2, & \text{if } \wp(\omega(\lambda + \rho)) \in \mathbb{R}_{>0}(\epsilon_1), \\ 0, & \text{if } \wp(\omega(\lambda + \rho)) \in \mathbb{R}_{>0}(-\epsilon_1). \end{cases}$$

Now the term $\epsilon_R(\gamma)$ in (4.2.2.1) is -1. By the above computation and by (4.2.2.1), we obtain

$$\text{(4.5.2.1)} \quad \Phi^G_M(\gamma, \Theta) = 2(-1)^{q(G)+1}\delta_{P_2(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1} \cdot \sum_{\substack{\omega \in \Omega \\ \wp(\omega(\lambda+\rho)) \in \mathbb{R}_{>0}(\epsilon_1)}} \epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha \in \Phi(\omega)} \alpha^{-1}(\gamma).$$

Next we compute $(-1)^{q(G)+1}L_M(\gamma)$. By Proposition 2.2.4 and Lemma 4.3.1, we have

$$\text{(4.5.2.2)} \quad (-1)^{q(G)+1}L_M(\gamma) = 2(-1)^{q(G)+1}\delta_{P_2(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1} \cdot \sum_{\substack{\omega \in \Omega \\ \langle \omega(\lambda+\rho), \varpi_2 \rangle > 0}} \epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha \in \Phi(\omega)} \alpha^{-1}(\gamma).$$

Comparing (4.5.2.1) and (4.5.2.2), we see that the proof reduces to checking that for all $\omega \in \Omega$, we have

$$\langle \omega(\lambda + \rho), \varpi_2 \rangle > 0 \Longleftrightarrow \wp(\omega(\lambda + \rho)) \in \mathbb{R}_{>0}(\epsilon_1).$$

This is obvious. □

4.6. Kostant–Weyl terms and discrete series characters, case M_{12}

4.6.1. — We keep the notations in §4.2 and §4.3. We take $S = \{1, 2\}$ and $M = M_{12}$. (We drop the assumption that d is odd made in §4.5.) Recall from §4.1 that we have fixed an elliptic maximal torus $T_{12} = \mathbb{G}_m \times \mathbb{G}_m \times T_{W_2}$ in M. Consider a regular element $\gamma \in T_{12}(\mathbb{R})$. We write

$$\gamma = (a, b, \gamma_{W_2}),$$

with $a, b \in \mathbb{R}^\times$. Let $L_M(\gamma)$ be as in Definition 2.2.3. We fix an element $g_0 \in M_{2,l}(\mathbb{Q})^\sharp$, as in Definition 2.2.6.

Lemma 4.6.2. — *We have*

$$\begin{aligned} L_M(\gamma) = {} & \delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2} \operatorname{Tr}(\gamma \mid R\Gamma(\operatorname{Lie} N_{12}, \mathbb{V})_{>t_{12}}) \\ & + \delta_{P_{12}(\mathbb{R})}(g_0 \gamma g_0^{-1})^{1/2} \operatorname{Tr}(g_0\gamma g_0^{-1} \mid R\Gamma(\operatorname{Lie} N_{12}, \mathbb{V})_{>t_{12}}) \\ & - \left| D_M^{M_2}(\gamma) \right|_{\mathbb{R}}^{1/2} \delta_{P_2(\mathbb{R})}(\gamma)^{1/2} \operatorname{Tr}(\gamma \mid R\Gamma(\operatorname{Lie} N_2, \mathbb{V})_{>t_2}). \end{aligned} \tag{4.6.2.1}$$

Proof. — The lemma follows from Proposition 2.2.8, the fact that $\dim A_M/A_{M_2} = 1$, and the fact that $n_M^{M_2} = 2$. Here $n_M^{M_2}$ is clearly equal to the cardinality of $\mathcal{W}_{M_l}^{M_{2,l}}$, and we already showed in the proof of Proposition 2.2.8 that this group is $\mathbb{Z}/2\mathbb{Z}$. □

Definition 4.6.3. — When d is odd, let

$$\begin{aligned} \omega_0 &:= s_{\epsilon_2} \in \Omega, \\ \omega_1 &:= s_{\epsilon_1 - \epsilon_2} \in \Omega, \\ \omega_2 &:= s_{\epsilon_1} \in \Omega. \end{aligned}$$

When d is even, let

$$\omega_0 := s_{\epsilon_2+\epsilon_3} s_{\epsilon_2-\epsilon_3} \in \Omega.$$

Here s_α denotes the reflection in Ω corresponding to $\alpha \in \Phi(G_{\mathbb{C}}, T_{12,\mathbb{C}})$.

The following lemma is similar to an argument on p. 1702 of [**Mor11**].

Lemma 4.6.4. — *Let $s \in \{\omega_0, \omega_1, \omega_2\}$ if d is odd, and let $s = \omega_0$ if d is even.*

(1) *The automorphism of $T_{12,\mathbb{C}}$ induced by s is defined over $\mathbb{R}$.*

(2) *Let $\gamma \in T_{12}(\mathbb{R})$ be regular, and let $\gamma' := s(\gamma) \in T_{12}(\mathbb{R})$. For any $\omega \in \Omega$ we have*

$$\frac{\delta_{P_{12}(\mathbb{R})}(\gamma')^{1/2} \prod_{\alpha \in \Phi(\omega)} \alpha^{-1}(\gamma')}{\delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2} \prod_{\alpha \in \Phi(s\omega)} \alpha^{-1}(\gamma)} = \prod_{\alpha \in \Phi_M^+} \frac{|\alpha(\gamma')|^{-1/2}}{|\alpha(\gamma)|^{-1/2}} \cdot \prod_{\alpha \in \Phi(s)} |\alpha(\gamma)|^{-1} \alpha(\gamma). \tag{4.6.4.1}$$

Here $|\cdot|$ denotes the usual absolute value on $\mathbb{C}$, and as usual $\Phi(s)$ denotes $\Phi^+ \cap (-s\Phi^+)$.

Proof. — **(1)** The automorphism in question is given by

$$(x, y, z) \longmapsto (y, x, z), \quad \forall (x, y) \in \mathbb{G}_m^2,\ z \in T_{W_2,\mathbb{C}},$$

when $s = \omega_1$, and is given by

$$(x, y, z) \longmapsto (x^{-1}, y, z), \quad \forall (x, y) \in \mathbb{G}_m^2,\ z \in T_{W_2,\mathbb{C}},$$

when $s = \omega_2$. In these cases the claim is obvious. When $s = \omega_0$, the automorphism in question is of the form

$$(x, y, z) \longmapsto (x, y^{-1}, f(z)), \quad (x, y) \in \mathbb{G}_m^2,\ z \in T_{W_2,\mathbb{C}},$$

for some automorphism f of $T_{W_2,\mathbb{C}}$. Since $T_{W_2} \cong \mathrm{U}(1)^{m-2}$, every automorphism of $T_{W_2,\mathbb{C}}$ is defined over $\mathbb{R}$. This proves the claim.

(2) Note that Φ^+ is the disjoint union of Φ_M^+ and the set of roots of $T_{12,\mathbb{C}}$ acting on $\mathrm{Lie}(N_{12})_{\mathbb{C}}$. Hence

$$\delta_{P_{12}(\mathbb{R})}(\nu) = \prod_{\alpha\in\Phi^+} |\alpha(\nu)| \prod_{\alpha\in\Phi_M^+} |\alpha(\nu)|^{-1}, \quad \forall \nu \in T_{12}(\mathbb{R}).$$

For any $\omega \in \Omega$, we have

$$\begin{aligned}\delta_{P_{12}(\mathbb{R})}(\gamma')^{1/2} &\prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma') \\ &= \prod_{\alpha\in\Phi^+} |\alpha(\gamma')|^{1/2} \prod_{\alpha\in\Phi_M^+} |\alpha(\gamma')|^{-1/2} \prod_{\alpha\in\Phi^+\cap(-\omega\Phi^+)} \alpha^{-1}(\gamma') \\ &= \prod_{\alpha\in s\Phi^+} |\alpha(\gamma)|^{1/2} \prod_{\alpha\in\Phi_M^+} |\alpha(\gamma')|^{-1/2} \prod_{\alpha\in s\Phi^+\cap(-s\omega\Phi^+)} \alpha^{-1}(\gamma).\end{aligned}$$

Also we have

$$\delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2} \prod_{\alpha\in\Phi(s\omega)} \alpha^{-1}(\gamma) = \prod_{\alpha\in\Phi^+} |\alpha(\gamma)|^{1/2} \prod_{\alpha\in\Phi_M^+} |\alpha(\gamma)|^{-1/2} \prod_{\alpha\in\Phi^+\cap(-s\omega\Phi^+)} \alpha^{-1}(\gamma).$$

Hence

$$\begin{aligned}&\frac{\delta_{P_{12}(\mathbb{R})}(\gamma')^{1/2} \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma')}{\delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2} \prod_{\alpha\in\Phi(s\omega)} \alpha^{-1}(\gamma)} \\ &\qquad = \prod_{\alpha\in\Phi_M^+} \frac{|\alpha(\gamma')|^{-1/2}}{|\alpha(\gamma)|^{-1/2}} \cdot \frac{\prod_{\alpha\in s\Phi^+} |\alpha(\gamma)|^{1/2}}{\prod_{\alpha\in\Phi^+} |\alpha(\gamma)|^{1/2}} \cdot \frac{\prod_{\alpha\in s\Phi^+\cap(-s\omega\Phi^+)} \alpha^{-1}(\gamma)}{\prod_{\alpha\in\Phi^+\cap(-s\omega\Phi^+)} \alpha^{-1}(\gamma)}.\end{aligned}$$

To finish the proof, we note that

$$\frac{\prod_{\alpha\in s\Phi^+} |\alpha(\gamma)|^{1/2}}{\prod_{\alpha\in\Phi^+} |\alpha(\gamma)|^{1/2}} = \frac{\prod_{\alpha\in\Phi^+\cap s\Phi^+} |\alpha(\gamma)|^{1/2} \prod_{\alpha\in-\Phi(s)} |\alpha(\gamma)|^{1/2}}{\prod_{\alpha\in\Phi^+\cap s\Phi^+} |\alpha(\gamma)|^{1/2} \prod_{\alpha\in\Phi(s)} |\alpha(\gamma)|^{1/2}} = \prod_{\alpha\in\Phi(s)} |\alpha(\gamma)|^{-1},$$

and that

$$\frac{\prod_{\alpha\in s\Phi^+\cap(-s\omega\Phi^+)}\alpha^{-1}(\gamma)}{\prod_{\alpha\in\Phi^+\cap(-s\omega\Phi^+)}\alpha^{-1}(\gamma)} = \frac{\prod_{\alpha\in(-\Phi^+)\cap s\Phi^+\cap(-s\omega\Phi^+)}\alpha^{-1}(\gamma)}{\prod_{\alpha\in\Phi^+\cap(-s\Phi^+)\cap(-s\omega\Phi^+)}\alpha^{-1}(\gamma)}$$
$$= \frac{\prod_{\alpha\in\Phi(s)\cap(s\omega\Phi^+)}\alpha(\gamma)}{\prod_{\alpha\in\Phi(s)\cap(-s\omega\Phi^+)}\alpha^{-1}(\gamma)}$$
$$= \prod_{\alpha\in\Phi(s)}\alpha(\gamma).$$

The desired (4.6.4.1) follows. □

Lemma 4.6.5. — *For any $g_0 \in M_{2,l}(\mathbb{Q})^\sharp$ (see Definition 2.2.6), there exists $g \in \mathrm{SO}(W_2)(\mathbb{R}) \subset G(\mathbb{R})$ such that gg_0 normalizes T_{12} and the image of gg_0 in Ω is ω_0 as in Definition 4.6.3.*

Proof. — Recall that $T_{12} = \mathbb{G}_m^2 \times T_{W_2}$, where $\mathbb{G}_m^2 = \mathrm{GL}(V_1)\times\mathrm{GL}(V_2/V_1)$, and T_{W_2} is an elliptic maximal torus in $\mathrm{SO}(W_2)$. From the definition of $M_{2,l}(\mathbb{Q})^\sharp$, we know that g_0 normalizes $\mathbb{G}_m^2$, stabilizes $W_2 \subset V$, and restricts to an element $g_0|_{W_2} \in \mathrm{O}(W_2)(\mathbb{R}) - \mathrm{SO}(W_2)(\mathbb{R})$. Since all elliptic maximal tori in $\mathrm{SO}(W_2)$ over $\mathbb{R}$ are conjugate under $\mathrm{SO}(W_2)(\mathbb{R})$, there exists $g \in \mathrm{SO}(W_2)(\mathbb{R})$ such that gg_0 normalizes T_{12}. We let h denote $(gg_0)|_{W_2}$, which is an element of $\mathrm{O}(W_2)(\mathbb{R}) - \mathrm{SO}(W_2)(\mathbb{R})$ normalizing T_{W_2}.

If d is odd, we can take g to be $-\mathrm{id}_{W_2}\cdot(g_0|_{W_2})^{-1}$. Then gg_0 permutes $\{e_2, e_2'\}$ non-trivially, fixes e_1 and e_1', and acts as $-\mathrm{id}_{W_2}$ on W_2. It follows that the image of gg_0 in Ω is ω_0, as desired.

Assume that d is even. Then $m = d/2 \geq 3$. By our definition of the $\mathbb{Z}$-basis $\{\epsilon_1, \cdots, \epsilon_m\}$ of $X^*(T_{12})$, we know that $\{\epsilon_3, \cdots, \epsilon_m\}$ is a $\mathbb{Z}$-basis of $X^*(T_{W_2})$. Moreover,

$$\Phi(\mathrm{SO}(W_2)_{\mathbb{C}}, T_{W_2,\mathbb{C}}) = \{\pm\epsilon_i \pm \epsilon_j \mid 3 \leq i < j \leq m\}.$$

It is easy to check that there exists an element $h' \in \mathrm{O}(W_2)(\mathbb{C}) - \mathrm{SO}(W_2)(\mathbb{C})$ normalizing $T_{W_2,\mathbb{C}}$ such that the automorphism σ' of $X^*(T_{W_2})$ induced by h' satisfies $\sigma'(\epsilon_3) = -\epsilon_3$ and $\sigma'(\epsilon_i) = \epsilon_i$ for $4 \leq i \leq m$. Denote by σ the automorphism of $X^*(T_{W_2})$ induced by h. It suffices to show that

$$\sigma \in \Omega_{\mathbb{R}}(\mathrm{SO}(W_2), T_{W_2})\sigma' \subset \mathrm{Aut}(X^*(T_{W_2})).$$

Here $\Omega_{\mathbb{R}}(\mathrm{SO}(W_2), T_{W_2})$ is the real Weyl group $\mathrm{Nor}_{\mathrm{SO}(W_2)(\mathbb{R})}(T_{W_2})/T_{W_2}(\mathbb{R})$, viewed as a subgroup of $\mathrm{Aut}(X^*(T_{W_2}))$. Since h and h' differ by left-multiplication by an element of $\mathrm{SO}(W_2)(\mathbb{C})$ normalizing $T_{W_2,\mathbb{C}}$, we have $\sigma \in \Omega_{\mathbb{C}}(\mathrm{SO}(W_2), T_{W_2})\sigma'$. We finish the proof by noting that $\Omega_{\mathbb{C}}(\mathrm{SO}(W_2), T_{W_2}) = \Omega_{\mathbb{R}}(\mathrm{SO}(W_2), T_{W_2})$, since $\mathrm{SO}(W_2)$ is anisotropic over $\mathbb{R}$. □

Definition 4.6.6. — For $\omega \in \Omega$, define

$$N_1(\omega) := \begin{cases} 1, & \text{if } \langle \omega(\lambda+\rho), \varpi_i \rangle > 0 \text{ for } i = 1,2, \\ 0, & \text{otherwise.} \end{cases}$$

$$N_2(\omega) := \begin{cases} 1, & \text{if } \langle \omega(\lambda+\rho), \omega_0\varpi_i \rangle > 0 \text{ for } i = 1,2, \\ 0, & \text{otherwise.} \end{cases}$$

$$N_3(\omega) := \begin{cases} 1, & \text{if } \langle \omega(\lambda+\rho), \varpi_2 \rangle > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Here ω_0 is as in Definition 4.6.3.

Lemma 4.6.7. — *Let $\gamma = (a, b, \gamma_{W_2})$ be a regular element of $T_{12}(\mathbb{R})$. The quantity $\tilde{L}_M(\gamma) := L_M(\gamma) \cdot (\delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1})^{-1}$ can be computed as follows.*

(1) *If d is odd, then*

$$\tilde{L}_M(\gamma) = \sum_{\omega\in\Omega} \Big[N_1(\omega) - \operatorname{sgn}(b) N_2(\omega) - \operatorname{sgn}(1-b^{-1}) N_3(\omega) \Big] \epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma).$$

(2) *If d is even, then*

$$\tilde{L}_M(\gamma) = \sum_{\omega\in\Omega} \Big[N_1(\omega) + N_2(\omega) - N_3(\omega) \Big] \epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma).$$

Proof. — Our starting point is (4.6.2.1). Let $\gamma' = \omega_0(\gamma)$. By Lemma 4.6.5, we may replace $g_0\gamma g_0^{-1}$ in the second summand on the RHS of (4.6.2.1) by γ'. Now we would like to rewrite the third summand. Define

$$\eta_2(\gamma) := \prod_{\alpha\in\Phi^+_{M_2} - \Phi^+_M} \frac{\left|1-\alpha^{-1}(\gamma)\right|}{1-\alpha^{-1}(\gamma)}.$$

Then arguing as on p. 1701 of [**Mor11**], we have

$$\left| D^{M_2}_M(\gamma) \right|^{1/2} \delta_{P_2(\mathbb{R})}(\gamma)^{1/2} \Delta_{M_2}(\gamma)^{-1} = \eta_2(\gamma)\delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1}.$$

Hence we can rewrite (4.6.2.1) as follows:

$$\begin{aligned} \text{(4.6.7.1)}\quad L_M(\gamma) = {} & \delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2} \operatorname{Tr}(\gamma \mid R\Gamma(\operatorname{Lie} N_{12}, \mathbb{V})_{>t_{12}}) \\ & + \delta_{P_{12}(\mathbb{R})}(\gamma')^{1/2} \operatorname{Tr}(\gamma' \mid R\Gamma(\operatorname{Lie} N_{12}, \mathbb{V})_{>t_{12}}) \\ & - \delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1}\Delta_{M_2}(\gamma)\eta_2(\gamma) \operatorname{Tr}(\gamma \mid R\Gamma(\operatorname{Lie} N_2, \mathbb{V})_{>t_2}). \end{aligned}$$

Using Lemma 4.3.1 to compute the $\mathrm{Tr}(\cdots)$ terms in (4.6.7.1), we get

$$\begin{aligned}L_M(\gamma) = \delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1} &\sum_{\substack{\omega\in\Omega \\ \langle\omega(\lambda+\rho),\varpi_i\rangle>0,\ \forall i\in\{1,2\}}} \epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma) \\ + \delta_{P_{12}(\mathbb{R})}(\gamma')^{1/2}\Delta_M(\gamma')^{-1} &\sum_{\substack{\omega\in\Omega \\ \langle\omega(\lambda+\rho),\varpi_i\rangle>0,\ \forall i\in\{1,2\}}} \epsilon(\omega)(\omega\lambda)(\gamma') \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma') \\ - \delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1}\eta_2(\gamma) &\sum_{\substack{\omega\in\Omega \\ \langle\omega(\lambda+\rho),\varpi_2\rangle>0}} \epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma).\end{aligned}$$

By Lemma 4.6.4, the second summand in the above is equal to

$$\delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1}A(\gamma,\gamma') \sum_{\substack{\omega\in\Omega \\ \langle\omega(\lambda+\rho),\varpi_i\rangle>0,\ \forall i\in\{1,2\}}} \epsilon(\omega)(\omega\lambda)(\gamma') \prod_{\alpha\in\Phi(\omega_0\omega)} \alpha^{-1}(\gamma),$$

where

$$A(\gamma,\gamma') := \frac{\Delta_M(\gamma)}{\Delta_M(\gamma')} \prod_{\alpha\in\Phi_M^+} \frac{|\alpha(\gamma')|^{-1/2}}{|\alpha(\gamma)|^{-1/2}} \prod_{\alpha\in\Phi(\omega_0)} \frac{\alpha(\gamma)}{|\alpha(\gamma)|}.$$

Therefore we have

$$\begin{aligned}\tilde{L}_M(\gamma) = &\sum_{\substack{\omega\in\Omega \\ \langle\omega(\lambda+\rho),\varpi_i\rangle>0,\ \forall i\in\{1,2\}}} \epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma) \\ &+ A(\gamma,\gamma') \sum_{\substack{\omega\in\Omega \\ \langle\omega(\lambda+\rho),\varpi_i\rangle>0,\forall i\in\{1,2\}}} \epsilon(\omega)(\omega\lambda)(\gamma') \prod_{\alpha\in\Phi(\omega_0\omega)} \alpha^{-1}(\gamma) \\ &- \eta_2(\gamma) \sum_{\substack{\omega\in\Omega \\ \langle\omega(\lambda+\rho),\varpi_2\rangle>0}} \epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma).\end{aligned}$$

Making the substitution $\omega \mapsto \omega_0\omega$ in the second summation and using the following obvious relations:

$$\begin{gathered}\omega_0^2 = 1, \\ (\omega_0\omega\lambda)(\gamma') = (\omega\lambda)(\gamma), \\ \epsilon(\omega_0\omega) = \epsilon(\omega_0)\epsilon(\omega), \\ \langle\omega_0\omega(\lambda+\rho),\varpi_i\rangle = \langle\omega(\lambda+\rho),\omega_0\varpi_i\rangle,\end{gathered}$$

we obtain

$$\begin{aligned}
\tilde{L}_M(\gamma) = & \sum_{\substack{\omega\in\Omega \\ \langle\omega(\lambda+\rho),\varpi_i\rangle>0,\ \forall i\in\{1,2\}}} \epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma) \\
& + \epsilon(\omega_0)A(\gamma,\gamma') \sum_{\substack{\omega\in\Omega \\ \langle\omega(\lambda+\rho),\omega_0\varpi_i\rangle>0,\forall i\in\{1,2\}}} \epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma) \\
& - \eta_2(\gamma) \sum_{\substack{\omega\in\Omega \\ \langle\omega(\lambda+\rho),\varpi_2\rangle>0}} \epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma)
\end{aligned}$$

$$(4.6.7.2) \qquad = \sum_{\omega\in\Omega} \Big[N_1(\omega) + \epsilon(\omega_0)A(\gamma,\gamma')N_2(\omega) - \eta_2(\gamma)N_3(\omega)\Big] \cdot \Big[\epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma)\Big].$$

To finish the proof it remains to compute the quantities $\epsilon(\omega_0)$, $A(\gamma,\gamma')$, and $\eta_2(\gamma)$, which we carry out separately in the odd and even cases.

First assume that d is odd. Then

$$(4.6.7.3) \qquad \epsilon(\omega_0) = -1.$$

To compute $A(\gamma,\gamma')$, first note that $\Delta_M(\gamma)/\Delta_M(\gamma')$ and

$$\prod_{\alpha\in\Phi_M^+} |\alpha(\gamma')|^{\frac{-1}{2}} |\alpha(\gamma)|^{\frac{1}{2}}$$

are both 1, since $\gamma^{-1}\gamma'$ lies in the center of M. To compute

$$\prod_{\alpha\in\Phi(\omega_0)} |\alpha(\gamma)|^{-1} \alpha(\gamma),$$

we have $\Phi(\omega_0) = \{\epsilon_2\} \cup \{\epsilon_2 \pm \epsilon_j \mid j \geq 3\}$, and we know that $\epsilon_2 + \epsilon_j$ is the complex conjugate of $\epsilon_2 - \epsilon_j$ for $j \geq 3$, with respect to the real structure of T_{12}. (In fact, the complex conjugation acts on $X^*(T_{W_2}) = \mathrm{span}_{\mathbb{Z}} \{\epsilon_3, \cdots, \epsilon_m\}$ as -1.) Hence we have

$$(4.6.7.4) \qquad A(\gamma,\gamma') = \prod_{\alpha\in\Phi(\omega_0)} |\alpha(\gamma)|^{-1} \alpha(\gamma) = |\epsilon_2(\gamma)|^{-1} \epsilon_2(\gamma) = \mathrm{sgn}(b).$$

We are left to compute $\eta_2(\gamma)$. We have $\Phi_{M_2}^+ - \Phi_M^+ = \{\epsilon_2\} \cup \{\epsilon_2 \pm \epsilon_j \mid j \geq 3\}$. Since $\epsilon_2 + \epsilon_j$ is the complex conjugate of $\epsilon_2 - \epsilon_j$ for $j \geq 3$, we have

$$(4.6.7.5) \qquad \eta_2(\gamma) = \frac{\left|1 - \epsilon_2^{-1}(\gamma)\right|}{1 - \epsilon_2^{-1}(\gamma)} = \mathrm{sgn}(1 - b^{-1}).$$

The proof is finished by combining (4.6.7.2), (4.6.7.3), (4.6.7.4), and (4.6.7.5).

Now assume that d is even. Then $\epsilon(\omega_0) = 1$. To finish the proof it suffices to check that $A(\gamma,\gamma') = \eta_2(\gamma) = 1$.

We compute $A(\gamma,\gamma')$. Let $x_j := \epsilon_j(\gamma), 1 \le j \le m$. We have

$$\text{(4.6.7.6)}\qquad \frac{\Delta_M(\gamma)}{\Delta_M(\gamma')} = \prod_{\alpha\in\Phi_M^+} \frac{1-\alpha^{-1}(\gamma)}{1-\alpha^{-1}(\gamma')} = \prod_{\alpha\in\{\epsilon_3\pm\epsilon_j|j\ge 4\}} \frac{1-\alpha^{-1}(\gamma)}{1-\alpha^{-1}(\gamma')}$$

$$= \prod_{j\ge 4} \frac{1-x_3^{-1}x_j^{-1}}{1-x_3x_j^{-1}} \frac{1-x_3^{-1}x_j}{1-x_3x_j} = \prod_{j\ge 4} x_3^{-2}.$$

Also

(4.6.7.7)

$$\prod_{\alpha\in\Phi_M^+} \frac{|\alpha(\gamma')|^{-1/2}}{|\alpha(\gamma)|^{-1/2}} = \prod_{\alpha\in\{\epsilon_3\pm\epsilon_j|j\ge 4\}} \frac{|\alpha(\gamma')|^{-1/2}}{|\alpha(\gamma)|^{-1/2}} = \prod_{j\ge 4} \left| \frac{x_3^{-1}x_j}{x_3x_j} \frac{x_3^{-1}x_j^{-1}}{x_3x_j^{-1}} \right|^{-1/2} = \prod_{j\ge 4} |x_3|^2 .$$

To compute

$$\prod_{\alpha\in\Phi(\omega_0)} \alpha(\gamma)\, |\alpha(\gamma)|^{-1},$$

we have

$$\Phi(\omega_0) = \{\epsilon_2 \pm \epsilon_j \mid j \ge 3\} \cup \{\epsilon_3 \pm \epsilon_j \mid j \ge 4\}.$$

Note that $\epsilon_2 + \epsilon_j$ is the complex conjugate of $\epsilon_2 - \epsilon_j$, for $j \ge 3$. Hence we have

$$\text{(4.6.7.8)}\qquad \prod_{\alpha\in\Phi(\omega_0)} \frac{\alpha(\gamma)}{|\alpha(\gamma)|} = \prod_{\alpha\in\{\epsilon_3\pm\epsilon_j|j\ge 4\}} \frac{\alpha(\gamma)}{|\alpha(\gamma)|} = \prod_{j\ge 4} \frac{x_3x_jx_3x_j^{-1}}{|x_3x_jx_3x_j^{-1}|} = \prod_{j\ge 4} \frac{x_3^2}{|x_3|^2}.$$

Combining (4.6.7.6) (4.6.7.7) (4.6.7.8), we conclude that $A(\gamma,\gamma') = 1$, as desired.

We are left to check that $\eta_2(\gamma) = 1$. We have $\Phi_{M_2}^+ - \Phi_M^+ = \{\epsilon_2 \pm \epsilon_j \mid j \ge 3\}$. As we observed before, $\epsilon_2 + \epsilon_j$ is the complex conjugate of $\epsilon_2 - \epsilon_j$ for all $j \ge 3$. Hence $\eta_2(\gamma) = 1$ as desired. □

4.6.8. — Keep the setting of §4.6.1. In the following we compare $L_M(\gamma)$ with $\Phi_M^G(\gamma,\Theta)$. We will also introduce and study a variant of $\Phi_M^G(\gamma,\Theta)$, denoted by $\Phi_M^G(\gamma,\Theta)_{\mathrm{eds}}$.

We have $A_M = M^{\mathrm{GL}} = \mathbb{G}_m \times \mathbb{G}_m$, and $T_{12}(\mathbb{R})_1 = \{\pm 1\} \times \{\pm 1\} \times T_{W_2}(\mathbb{R})$. The projection of γ in $A_M(\mathbb{R})^0 \cong \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ is $(|a|, |b|)$, and

$$x_\gamma = (\log|a|, \log|b|) \in \mathbb{R}^2 \cong \mathrm{Lie}(A_M) = X_*(A_M)_{\mathbb{R}}.$$

Let $\wp$ be the natural restriction map $X^*(T_{12})_{\mathbb{R}} \to X^*(A_M)_{\mathbb{R}}$. We identify $X^*(A_M)_{\mathbb{R}}$ with $\mathbb{R}^2$, and let $\mathbb{R}^2_{\mathrm{odd}}, \mathbb{R}^2_{\mathrm{eds}}, \mathbb{R}^2_{\mathrm{even}}$ be the subsets of $\mathbb{R}^2$ defined in §4.2.9. Note that when d is odd (resp. even), we have $\wp(\omega(\lambda+\rho)) \in \mathbb{R}^2_{\mathrm{odd}}$ (resp. $\in \mathbb{R}^2_{\mathrm{even}}$) for all $\omega \in \Omega$. Suppose f is a function $\mathbb{R}^2_{\mathrm{odd}} \to \mathbb{C}$ (resp. $\mathbb{R}^2_{\mathrm{even}} \to \mathbb{C}$) when d is odd (resp. even). We write $[\![f]\!]$ for the function

$$[\![f]\!] : \Omega \longrightarrow \mathbb{C}$$
$$\omega \longmapsto f(\wp(\omega(\lambda+\rho))).$$

Recall from §4.2.9 that $(\mathcal{I}), (\mathcal{II}), \cdots, (\mathcal{VIII}), (\mathscr{A})$ denote the characteristic functions of some open cones in $\mathbb{R}^2$.

Lemma 4.6.9. — *When d is odd, we have the following identities between functions on Ω:*

$$\begin{aligned} N_1(\cdot) &= [\![(\mathcal{I}) + (\mathcal{II}) + (\mathcal{VIII})]\!], \\ N_2(\cdot) &= [\![(\mathcal{I}) + (\mathcal{VII}) + (\mathcal{VIII})]\!], \\ N_3(\cdot) &= [\![(\mathcal{I}) + (\mathcal{II}) + (\mathcal{VII}) + (\mathcal{VIII})]\!]. \end{aligned}$$

When d is even, we have the following identities between functions on Ω:

$$\begin{aligned} N_1(\cdot) &= [\![(\mathscr{A}) + (\mathcal{II})]\!], \\ N_2(\cdot) &= [\![(\mathscr{A}) + (\mathcal{VII})]\!], \\ N_3(\cdot) &= [\![(\mathscr{A}) + (\mathcal{II}) + (\mathcal{VII})]\!]. \end{aligned}$$

Proof. — This follows immediately from Definition 4.6.6. □

4.6.10. — Recall from §4.2 that $\Phi_M^G(\gamma, \Theta)$ can be computed by (4.2.2.1). Using the notation $[\![f]\!]$ introduced in §4.6.8, we recall the definition of $n(\gamma, \omega B)$ appearing in (4.2.2.1) as follows:

$$n(\gamma, \omega B) := \begin{cases} [\![\bar{c}_{R_\gamma}(x_\gamma, \cdot)]\!](\omega), & \text{if } \gamma \in G(\mathbb{R})^0, \\ 0, & \text{if } \gamma \notin G(\mathbb{R})^0. \end{cases}$$

Let $R_{\mathrm{eds}} := \{\pm\epsilon_1, \pm\epsilon_2\} \subset X^*(A_M)_{\mathbb{R}}$. Under the identification $X^*(A_M)_{\mathbb{R}} \cong \mathbb{R}^2$, the subset R_{eds} is identified with the root system U_{eds} considered in §4.2.9. In particular, the Weyl group of R_{eds} contains -1, and the function $\bar{c}_{R_{\mathrm{eds}}}$ associated to R_{eds} is identified with the function $\bar{c}_{U_{\mathrm{eds}}} : \mathbb{R}^2_{\mathrm{eds}} \times \mathbb{R}^2_{\mathrm{eds}} \to \mathbb{Z}$ considered in §4.2.9.

When d is odd, we define

$$n_{\mathrm{eds}}(\gamma, \omega B) := \begin{cases} [\![\bar{c}_{R_{\mathrm{eds}}}(x_\gamma, \cdot)]\!](\omega), & \text{if } a, b > 0, \\ 0, & \text{otherwise}, \end{cases} \tag{4.6.10.1}$$

for $\omega \in \Omega$. Here $[\![\bar{c}_{R_{\mathrm{eds}}}(x_\gamma, \cdot)]\!]$ is well defined because $\mathbb{R}^2_{\mathrm{odd}} \subset \mathbb{R}^2_{\mathrm{eds}}$.

Analogous to (4.2.2.1), we define, *when d is odd*,

$$\begin{aligned} \Phi_M^G(\gamma, \Theta)_{\mathrm{eds}} := (-1)^{q(G)} \epsilon_R(\gamma) \delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2} \Delta_M(\gamma)^{-1} \\ \cdot \sum_{\omega \in \Omega} \epsilon(\omega) n_{\mathrm{eds}}(\gamma, \omega B)(\omega\lambda)(\gamma) \prod_{\alpha \in \Phi(\omega)} \alpha^{-1}(\gamma). \end{aligned} \tag{4.6.10.2}$$

Lemma 4.6.11. — *For both parity of d, let $\nu = (s, t, u) \in T_{12}(\mathbb{R}) = \mathbb{R}^\times \times \mathbb{R}^\times \times T_{W_2}(\mathbb{R})$ be an element with $s, t < 0$. Then $\nu \in G(\mathbb{R})^0$.*

Proof. — Since $T_{W_2}(\mathbb{R})$ is connected (being a product of copies of U(1)), we know that ν is in the same connected component of $G(\mathbb{R})$ as

$$\nu_1 := (-1,-1,1) \in T_{12}(\mathbb{R}).$$

It remains to show that $\nu_1 \in G(\mathbb{R})^0$. We know that ν_1 acts as -1 on $\mathbb{R}X_1 + \mathbb{R}X_2$ and on $\mathbb{R}Y_1 + \mathbb{R}Y_2$, where $X_i = e_i + e_i'$ and $Y_i = e_i - e_i'$. Now $\mathbb{R}X_1 + \mathbb{R}X_2$ is a positive definite plane and $\mathbb{R}Y_1 + \mathbb{R}Y_2$ is a negative definite plane, and ν_1 acts on both of them with determinant 1. Also ν_1 acts as the identity on the orthogonal complement of these two planes. This implies that $\nu_1 \in G(\mathbb{R})^0$, by the standard description of the connected components of indefinite special orthogonal groups (see [**Kna02**, I.17]). □

Proposition 4.6.12. — *Assume that d is odd. Let $\gamma = (a, b, \gamma_{W_2}) \in T_{12}(\mathbb{R})$ be a regular element. Let $(x_1, x_2) = (\log|a|, \log|b|)$. When $ab < 0$, we have*

$$\Phi_M^G(\gamma, \Theta) = \Phi_M^G(\gamma, \Theta)_{\mathrm{eds}} = 0.$$

When $ab > 0$, assume that $x_1 < -|x_2|$. Then we have

$$4(-1)^{q(G)} L_M(\gamma) = \Phi_M^G(\gamma, \Theta) + \Phi_M^G(\gamma, \Theta)_{\mathrm{eds}}.$$

Proof. — We first treat the case $ab < 0$. Then $\Phi_M^G(\gamma, \Theta)_{\mathrm{eds}} = 0$ since all $n_{\mathrm{eds}}(\gamma, \omega B)$ vanish by definition. To show $\Phi_M^G(\gamma, \Theta) = 0$, note that $R_\gamma = \{\pm\epsilon_1\}$ or $\{\pm\epsilon_2\}$. Thus the Weyl group of R_γ (as a root system in $X^*(A_M)_{\mathbb{R}} = \mathbb{R}^2$) does not contain -1. By Corollary 4.2.7, we have $\Phi_M^G(\gamma, \Theta) = 0$.

We now treat the case $ab > 0$. First assume that a and b are both positive. Under our assumption that $x_1 < -|x_2|$, there are two cases to consider, namely $0 < a < b < 1$ or $0 < ab < 1 < b$. (Here $b \neq 1$ since γ is regular.) We have

$$\epsilon_R(\gamma) = \begin{cases} 1, & \text{if } 0 < a < b < 1, \\ -1, & \text{if } 0 < ab < 1 < b. \end{cases}$$

Comparing Lemma 4.6.7 with (4.2.2.1 and (4.6.10.2), we see that the current proposition reduces to the following two statements:

– When $0 < a < b < 1$, we have

$$\frac{1}{4}(n(\gamma, \omega B) + n_{\mathrm{eds}}(\gamma, \omega B)) = N_1(\omega) - N_2(\omega) + N_3(\omega), \quad \forall \omega \in \Omega. \tag{4.6.12.1}$$

– When $0 < ab < 1 < b$, we have

$$\frac{1}{4}(n(\gamma, \omega B) + n_{\mathrm{eds}}(\gamma, \omega B)) = -N_1(\omega) + N_2(\omega) + N_3(\omega), \quad \forall \omega \in \Omega. \tag{4.6.12.2}$$

Since obviously $\gamma \in T_{12}(\mathbb{R})^0 \subset G(\mathbb{R})^0$, we have

$$n(\gamma, \omega B) = [\![\bar{c}_{R_\gamma}(x_\gamma, \cdot)]\!](\omega), \quad \forall \omega \in \Omega,$$

by definition. Since $R_\gamma = \{\pm\epsilon_1, \pm\epsilon_2, \pm\epsilon_1 \pm \epsilon_2\} = U_{\mathrm{odd}}$, we have $\bar{c}_{R_\gamma}(x_\gamma, \cdot) = \mathbf{f}_{\mathrm{odd},x_\gamma}$. (See §4.2.9 for the notation.) In other words we have

$$n(\gamma, \omega B) = [\![\mathbf{f}_{\mathrm{odd},x_\gamma}]\!](\omega), \quad \forall \omega \in \Omega. \tag{4.6.12.3}$$

Similarly we have

$$n_{\mathrm{eds}}(\gamma, \omega B) = [\![\mathbf{f}_{\mathrm{eds},x_\gamma}]\!](\omega), \quad \forall \omega \in \Omega. \tag{4.6.12.4}$$

When $0 < a < b < 1$, we have $x_\gamma \in (\mathcal{V})$. By (4.2.10.1), (4.2.10.4), (4.6.12.3), and (4.6.12.4), we have

$$\frac{1}{4} n(\gamma, \omega B) = [\![(\mathcal{II}) + (\mathcal{VIII})]\!](\omega),$$

$$\frac{1}{4} n_{\mathrm{eds}}(\gamma, \omega B) = [\![(\mathcal{I}) + (\mathcal{II})]\!](\omega).$$

Thus the LHS of (4.6.12.1) is equal to $[\![(\mathcal{I}) + 2(\mathcal{II}) + (\mathcal{VIII})]\!](\omega)$. On the other hand, by Lemma 4.6.9, the RHS of (4.6.12.1) is also equal to $[\![(\mathcal{I}) + 2(\mathcal{II}) + (\mathcal{VIII})]\!](\omega)$. Hence (4.6.12.1) holds, as desired.

When $0 < ab < 1 < b$, we have $x_\gamma \in (\mathcal{IV})$. By (4.2.10.2) and (4.2.10.5), we have

$$\frac{1}{4} n(\gamma, \omega B) = [\![(\mathcal{I}) + (\mathcal{VII})]\!](\omega),$$

$$\frac{1}{4} n_{\mathrm{eds}}(\gamma, \omega B) = [\![(\mathcal{VII}) + (\mathcal{VIII})]\!](\omega).$$

Thus the LHS of (4.6.12.2) is equal to $[\![(\mathcal{I}) + 2(\mathcal{VII}) + (\mathcal{VIII})]\!](\omega)$. By Lemma 4.6.9, the RHS of (4.6.12.2) is also equal to $[\![(\mathcal{I}) + 2(\mathcal{VII}) + (\mathcal{VIII})]\!](\omega)$. Hence (4.6.12.2) holds, as desired.

We now assume that a and b are both negative. In this case $\Phi_M^G(\gamma, \Theta)_{\mathrm{eds}} = 0$ by definition. We have $\epsilon_R(\gamma) = 1$. Comparing Lemma 4.6.7 with (4.2.2.1), we see that the current proposition reduces to the following identity:

$$\frac{1}{4} n(\gamma, \omega B) = N_1(\omega) + N_2(\omega) - N_3(\omega), \quad \forall \omega \in \Omega. \tag{4.6.12.5}$$

By Lemma 4.6.11, we have $\gamma \in G(\mathbb{R})^0$, and so

$$n(\gamma, \omega B) = [\![\bar{c}_{R_\gamma}(x_\gamma, \cdot)]\!](\omega), \quad \forall \omega \in \Omega,$$

by definition. Since $R_\gamma = \{\pm\epsilon_1 \pm \epsilon_2\} = U_{\mathrm{even}}$, we have $\bar{c}_{R_\gamma}(x_\gamma, \cdot) = \mathbf{f}_{\mathrm{even},x_\gamma}$. (See §4.2.9 for the notation). Thus

$$n(\gamma, \omega B) = [\![\mathbf{f}_{\mathrm{even},x_\gamma}]\!](\omega), \quad \forall \omega \in \Omega. \tag{4.6.12.6}$$

Since $x_1 < -|x_2| < 0$, we have $x_\gamma \in (\mathcal{IV}) \cup (\mathcal{V}) \subset (\mathscr{C})$. Hence by (4.2.10.6) and (4.6.12.6), we have

$$\frac{1}{4} n(\gamma, \omega B) = [\![(\mathscr{A})]\!](\omega) = [\![(\mathcal{I}) + (\mathcal{VIII})]\!](\omega).$$

By Lemma 4.6.9, the RHS of (4.6.12.5) is also equal to $[\![(\mathcal{I}) + (\mathcal{VIII})]\!](\omega)$. Hence (4.6.12.5) holds, as desired. □

The following proposition will also be needed in §8.12 below.

Proposition 4.6.13. — *Assume that d is odd. Let $\gamma = (a, b, \gamma_{W_2}) \in T_{12}(\mathbb{R})$ be a regular element, with $ab > 0$. Let ω_1, ω_2 be as in Definition 4.6.3, and let*

$$\gamma' := \omega_1(\gamma) = (b, a, \gamma_{W_2}) \in T_{12}(\mathbb{R}),$$
$$\gamma'' := \omega_2(\gamma) = (a^{-1}, b, \gamma_{W_2}) \in T_{12}(\mathbb{R}).$$

Then we have

$$\Phi_M^G(\gamma, \Theta) = \Phi_M^G(\gamma', \Theta) = \Phi_M^G(\gamma'', \Theta), \tag{4.6.13.1}$$

$$\epsilon_R(\gamma)\epsilon_{R_{\text{eds}}}(\gamma)\Phi_M^G(\gamma, \Theta)_{\text{eds}} = -\epsilon_R(\gamma')\epsilon_{R_{\text{eds}}}(\gamma')\Phi_M^G(\gamma', \Theta)_{\text{eds}}, \tag{4.6.13.2}$$

$$\epsilon_R(\gamma)\epsilon_{R_{\text{eds}}}(\gamma)\Phi_M^G(\gamma, \Theta)_{\text{eds}} = \epsilon_R(\gamma'')\epsilon_{R_{\text{eds}}}(\gamma'')\Phi_M^G(\gamma'', \Theta)_{\text{eds}}. \tag{4.6.13.3}$$

Here $\epsilon_{R_{\text{eds}}}(\gamma)$ is defined to be

$$(-1)^{\#\left\{\alpha \in \Phi^+ \cap R_{\text{eds}} \mid 0 < \alpha(\gamma) < 1\right\}},$$

and similarly for $\epsilon_{R_{\text{eds}}}(\gamma')$ and $\epsilon_{R_{\text{eds}}}(\gamma'')$.

Proof. — The equalities in (4.6.13.1) hold because ω_1 and ω_2 can be represented by elements of $(\mathrm{Nor}_G\, M)(\mathbb{R})$, and $\Phi_M^G(\cdot, \Theta)$ is invariant under $(\mathrm{Nor}_G\, M)(\mathbb{R})$.

We now prove (4.6.13.2). We have $\Delta_M(\gamma) = \Delta_M(\gamma')$ because $\gamma^{-1}\gamma'$ lies in the center of M. Also $\epsilon_{R_{\text{eds}}}(\gamma) = \epsilon_{R_{\text{eds}}}(\gamma')$. Hence we have reduced the proof to showing that

(4.6.13.4)
$$\delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2} \sum_{\omega} \epsilon(\omega) n_{\text{eds}}(\gamma, \omega B)(\omega\lambda)(\gamma) \prod_{\alpha \in \Phi(\omega)} \alpha^{-1}(\gamma)$$
$$= -\delta_{P_{12}(\mathbb{R})}(\gamma')^{1/2} \sum_{\omega} \epsilon(\omega) n_{\text{eds}}(\gamma', \omega B)(\omega\lambda)(\gamma') \prod_{\alpha \in \Phi(\omega)} \alpha^{-1}(\gamma').$$

We claim that for all $\omega \in \Omega$, we have $n_{\text{eds}}(\gamma', \omega B) = n_{\text{eds}}(\gamma, \omega_1 \omega B)$. Indeed, if a and b are both negative, then both sides are by definition zero. If a and b are both positive, then our claim follows from the following property:

$$\bar{c}_{R_{\text{eds}}}(y, y') = \bar{c}_{R_{\text{eds}}}(\omega_1 y, \omega_1 y'), \quad \forall y, y' \in \mathbb{R}^2_{\text{eds}},$$

which is a direct consequence of (4.2.10.3).

By the claim and Lemma 4.6.4, the RHS of (4.6.13.4) is equal to

$$- \delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2} \sum_{\omega} \epsilon(\omega) n_{\text{eds}}(\gamma, \omega_1 \omega B)(\omega\lambda)(\gamma')$$
$$\cdot \prod_{\alpha \in \Phi(\omega_1 \omega)} \alpha^{-1}(\gamma) \prod_{\alpha \in \Phi_M^+} \frac{|\alpha(\gamma')|^{-\frac{1}{2}}}{|\alpha(\gamma)|^{-\frac{1}{2}}} \prod_{\alpha \in \Phi(\omega_1)} \frac{\alpha(\gamma)}{|\alpha(\gamma)|}.$$

Under the substitution $\omega \mapsto \omega_1\omega$ in the summation, the above becomes

$$\delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\sum_{\omega}\epsilon(\omega)n_{\mathrm{eds}}(\gamma,\omega B)(\omega\lambda)(\gamma) \cdot \prod_{\alpha\in\Phi(\omega)}\alpha^{-1}(\gamma)\prod_{\alpha\in\Phi_M^+}\frac{|\alpha(\gamma')|^{-\frac{1}{2}}}{|\alpha(\gamma)|^{-\frac{1}{2}}}\prod_{\alpha\in\Phi(\omega_1)}\frac{\alpha(\gamma)}{|\alpha(\gamma)|}.$$

To finish the proof of (4.6.13.4) it suffices to check

$$\prod_{\alpha\in\Phi_M^+}\frac{|\alpha(\gamma')|^{-\frac{1}{2}}}{|\alpha(\gamma)|^{-\frac{1}{2}}}\prod_{\alpha\in\Phi(\omega_1)}\frac{\alpha(\gamma)}{|\alpha(\gamma)|}=1.$$

Since $\gamma^{-1}\gamma'$ lies in the center of M, the first product in the above is equal to 1. The second product is also equal to 1, because $\Phi(\omega_1)=\{\epsilon_1-\epsilon_2\}$, and $(\epsilon_1-\epsilon_2)(\gamma)=a/b>0$. We have thus proved (4.6.13.4). As we have already seen, this implies (4.6.13.2).

We now prove (4.6.13.3) in a completely analogous way. We have $\Delta_M(\gamma)=\Delta_M(\gamma')$, and $\epsilon_{R_{\mathrm{eds}}}(\gamma)=-\operatorname{sgn}(a)\epsilon_{R_{\mathrm{eds}}}(\gamma')$, so we need to check

$$\begin{aligned}(4.6.13.5)\quad &\delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\sum_{\omega}\epsilon(\omega)n_{\mathrm{eds}}(\gamma,\omega B)(\omega\lambda)(\gamma)\prod_{\alpha\in\Phi(\omega)}\alpha^{-1}(\gamma)\\ &=-\operatorname{sgn}(a)\delta_{P_{12}(\mathbb{R})}(\gamma'')^{1/2}\sum_{\omega}\epsilon(\omega)n_{\mathrm{eds}}(\gamma'',\omega B)(\omega\lambda)(\gamma'')\prod_{\alpha\in\Phi(\omega)}\alpha^{-1}(\gamma'').\end{aligned}$$

Again it easily follows from the definition of n_{eds} and (4.2.10.3) that $n_{\mathrm{eds}}(\gamma'',\omega B)=n_{\mathrm{eds}}(\gamma,\omega_2\omega B)$, for all $\omega\in\Omega$. By this fact and Lemma 4.6.4, the RHS of (4.6.13.5) is equal to

$$-\operatorname{sgn}(a)\delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\sum_{\omega}\epsilon(\omega)n_{\mathrm{eds}}(\gamma,\omega_2\omega B)(\omega\lambda)(\gamma'') \cdot \prod_{\alpha\in\Phi(\omega_2\omega)}\alpha^{-1}(\gamma)\prod_{\alpha\in\Phi_M^+}\frac{|\alpha(\gamma'')|^{-\frac{1}{2}}}{|\alpha(\gamma)|^{-\frac{1}{2}}}\prod_{\alpha\in\Phi(\omega_2)}\frac{\alpha(\gamma)}{|\alpha(\gamma)|}.$$

Under the substitution $\omega\mapsto\omega_2\omega$ in the summation the above becomes

$$\operatorname{sgn}(a)\delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\sum_{\omega}\epsilon(\omega)n_{\mathrm{eds}}(\gamma,\omega B)(\omega\lambda)(\gamma) \cdot \prod_{\alpha\in\Phi(\omega)}\alpha^{-1}(\gamma)\prod_{\alpha\in\Phi_M^+}\frac{|\alpha(\gamma'')|^{-\frac{1}{2}}}{|\alpha(\gamma)|^{-\frac{1}{2}}}\prod_{\alpha\in\Phi(\omega_2)}\frac{\alpha(\gamma)}{|\alpha(\gamma)|}.$$

To finish the proof of (4.6.13.5), it suffices to check

$$\prod_{\alpha\in\Phi_M^+}\frac{|\alpha(\gamma'')|^{-\frac{1}{2}}}{|\alpha(\gamma)|^{-\frac{1}{2}}}\prod_{\alpha\in\Phi(\omega_2)}\frac{\alpha(\gamma)}{|\alpha(\gamma)|}=\operatorname{sgn}(a).$$

Again the first product in the above is equal to 1, so we need to check that the second product is equal to $\mathrm{sgn}(a)$. For this, we may replace the product over all $\alpha \in \Phi(\omega_2)$ by the product over those $\alpha \in \Phi(\omega_2)$ that are real. This is because $\Phi(\omega_2)$ is stable under complex conjugation, and we obviously have

$$\frac{\alpha(\gamma)}{|\alpha(\gamma)|}\frac{\bar{\alpha}(\gamma)}{|\bar{\alpha}(\gamma)|} = 1$$

for any $\alpha, \bar{\alpha} \in \Phi(\omega_2)$ that are complex conjugate to each other. Now the real roots in $\Phi(\omega_2)$ are $\epsilon_1, \epsilon_1 + \epsilon_2, \epsilon_1 - \epsilon_2$. Hence

$$\prod_{\alpha \in \Phi(\omega_2)} \frac{\alpha(\gamma)}{|\alpha(\gamma)|} = \prod_{\alpha \in \{\epsilon_1, \epsilon_1+\epsilon_2, \epsilon_1-\epsilon_2\}} \frac{\alpha(\gamma)}{|\alpha(\gamma)|} = \frac{a}{|a|}\frac{ab}{|ab|}\frac{ab^{-1}}{|ab^{-1}|} = \frac{a}{|a|} = \mathrm{sgn}(a),$$

as desired. We have thus proved (4.6.13.5). As we have already seen, this implies (4.6.13.3). □

The following proposition is the counterpart of Proposition 4.6.12 in the even case.

Proposition 4.6.14. — *Assume that d is even. Let $\gamma = (a, b, \gamma_{W_2}) \in T_{12}(\mathbb{R})$ be a regular element. Let $(x_1, x_2) = (\log|a|, \log|b|)$. When $ab < 0$, we have*

$$\Phi_M^G(\gamma, \Theta) = 0.$$

When $ab > 0$, assume that $x_1 < -|x_2|$. Then we have

$$4(-1)^{q(G)} L_M(\gamma) = \Phi_M^G(\gamma, \Theta).$$

Proof. — When $ab < 0$, we have $R_\gamma = \emptyset$. Thus $\Phi_M^G(\gamma, \Theta) = 0$ by Corollary 4.2.7.

Assume that $ab > 0$. Under our assumption that $x_1 < -|x_2|$, we have $\epsilon_R(\gamma) = 1$. In view of Lemma 4.6.7, to prove the current proposition it suffices to prove

$$\frac{1}{4} n(\gamma, \omega B) = N_1(\omega) + N_2(\omega) - N_3(\omega), \quad \forall \omega \in \Omega. \tag{4.6.14.1}$$

By Lemma 4.6.11, we have $\gamma \in G(\mathbb{R})^0$, and so

$$n(\gamma, \omega B) = [\![\bar{c}_{R_\gamma}(x_\gamma, \cdot)]\!](\omega), \quad \forall \omega \in \Omega,$$

by definition. Since $R_\gamma = \{\pm\epsilon_1 \pm \epsilon_2\} = U_{\mathrm{even}}$, we have $\bar{c}_{R_\gamma}(x_\gamma, \cdot) = \mathbf{f}_{\mathrm{even}, x_\gamma}$. (See §4.2.9 for the notation). Thus

$$n(\gamma, \omega B) = [\![\mathbf{f}_{\mathrm{even}, x_\gamma}]\!](\omega), \quad \forall \omega \in \Omega. \tag{4.6.14.2}$$

Since $x_1 < -|x_2|$, we have $x_\gamma \in (\mathscr{C})$. By (4.2.10.6) and (4.6.14.2), we have

$$\frac{1}{4} n(\gamma, \omega B) = [\![(\mathscr{A})]\!](\omega).$$

Now by Lemma 4.6.9, the RHS of (4.6.14.1) is also equal to $[\![(\mathscr{A})]\!](\omega)$. Hence (4.6.14.1) holds, as desired. □

CHAPTER 5

ENDOSCOPIC DATA FOR SPECIAL ORTHOGONAL GROUPS

In this chapter, let F be a local or global field of characteristic zero. Let $V = (V, q)$ be a quadratic space over F of dimension d and discriminant δ (see §1.2). Let $G = \mathrm{SO}(V)$. Let $m = \lfloor d/2 \rfloor$, which is the absolute rank of G. As usual, we refer to "the odd case" and "the even case" according to the parity of d.

5.1. The quasi-split inner form

We need to explicitly fix an inner twisting between G and a quasi-split inner form. For this, let $\underline{V} = (\underline{V}, \underline{q})$ be the unique (up to isomorphism) quasi-split quadratic space over F of dimension d and discriminant δ.

Definition 5.1.1. — We fix an isomorphism of quadratic spaces over $\overline{F}$:

$$\phi_V : (V, q) \otimes_F \overline{F} \xrightarrow{\sim} (\underline{V}, \underline{q}) \otimes_F \overline{F}.$$

If $F = \mathbb{R}$, we may and shall assume that ϕ_V satisfies the following condition: Let (a, b) be the signature of (V, q). If $a > b$ (resp. $a \leq b$), then there exists an orthogonal basis $\{v_1, \cdots, v_d\}$ of V, and an orthogonal basis $\{\underline{v}_1, \cdots, \underline{v}_d\}$ of $\underline{V}$, such that for each $1 \leq j \leq d$, we have $q(v_j), \underline{q}(\underline{v}_j) \in \{\pm 1\}$, and $\phi_V(v_j) = \underline{v}_j \otimes \lambda_j$ for some $\lambda_j \in \left\{1, \sqrt{-1}\right\}$, with $\lambda_j = \sqrt{-1}$ only if $q(v_j) = 1$ (resp. only if $q(v_j) = -1$).

5.1.2. — Let $G^* := \mathrm{SO}(\underline{V}, \underline{q})$, which is quasi-split over F by Proposition 1.2.8. Using ϕ_V as in Definition 5.1.1, we define the isomorphism

$$\begin{aligned} \psi_V : G_{\overline{F}} &\xrightarrow{\sim} G^*_{\overline{F}} \\ g &\longmapsto \phi_V g \phi_V^{-1}. \end{aligned}$$

Define the function

$$\begin{aligned} u_V : \Gamma_F &\longrightarrow \mathrm{GL}(\underline{V} \otimes_F \overline{F}) \\ \rho &\longmapsto {}^{\rho}\phi_V \phi_V^{-1}. \end{aligned}$$

Clearly the image of u_V is contained in $\mathrm{O}(\underline{V})(\overline{F})$. If we fix F-bases of V and $\underline{V}$, then since q and $\underline{q}$ have the same discriminant, the square of the determinant of the matrix of ϕ_V lies in $F^{\times,2}$, which implies that the determinant of the matrix of ϕ_V lies in F. Hence $u_V(\rho)$ has determinant 1 for each $\rho \in \Gamma_F$. Thus the image of u_V is contained in $G^*(\overline{F})$. Note that we have

$$ {}^{\rho}\psi_V \psi_V^{-1} = \mathrm{Int}(u_V(\rho)) \in \mathrm{Aut}(G^*_{\overline{F}}), \quad \forall \rho \in \Gamma_F. \tag{5.1.2.1} $$

It follows that ψ_V is an inner twisting.

Remark 5.1.3. — If we view $\mathrm{SO}(V)$ and $\mathrm{SO}(\underline{V})$ as abstract reductive groups over F, then in the odd case there is a unique $\mathrm{SO}(\underline{V})(\overline{F})$-conjugacy class of inner twistings $\mathrm{SO}(V)_{\overline{F}} \xrightarrow{\sim} \mathrm{SO}(\underline{V})_{\overline{F}}$, whereas in the even case there are two such conjugacy classes, interchanged under the conjugation by any element of $\mathrm{O}(\underline{V})(\overline{F}) - \mathrm{SO}(\underline{V})(\overline{F})$. If we change the choice of ϕ_V to some ϕ'_V, then $\phi'_V = g \circ \phi_V$ for some $g \in \mathrm{O}(\underline{V})(\overline{F})$. The inner twisting ψ'_V arising from ϕ'_V stays in the same $\mathrm{SO}(\underline{V})(\overline{F})$-conjugacy class as ψ_V if and only if $g \in \mathrm{SO}(\underline{V})(\overline{F})$. Thus for the purpose of realizing $G^* = \mathrm{SO}(\underline{V})$ as an inner form of G, it suffices to remember ϕ_V up to replacing it by $g \circ \phi_V$ for $g \in \mathrm{SO}(\underline{V})(\overline{F})$.

Remark 5.1.4. — The pair (ψ_V, u_V) realizes G as a *pure inner form* of G^* in the sense of Vogan [**Vog93**]; cf. the introduction of [**Kal16**]. The pair (ψ_V, u_V) itself is called a *pure inner twist*; cf. [**Kal11**, §2]. Fixing such a pure inner twist (or rather its $G^*(\overline{F})$-conjugacy class, see below) is more refined than just fixing G^* as an inner form of G, and it plays an essential role in normalizing transfer factors when F is a local field. Specifically, suppose $(H, {}^L H, s, \eta)$ is an elliptic endoscopic datum for G, and suppose we have fixed a normalization of transfer factors between H and G^*. Then the datum (ψ_V, u_V) allows one to "transport" that normalization to a normalization of transfer factors between H and G, as observed by Kottwitz and explained in [**Kal11**, §2.2]. For this purpose, it actually suffices to just remember ϕ_V up to replacing it by $g \circ \phi_V$ for $g \in G^*(\overline{F}) = \mathrm{SO}(\underline{V})(\overline{F})$, which will result in (ψ_V, u_V) being replaced by $(\mathrm{Int}(g) \circ \psi_V, \rho \mapsto {}^{\rho}g u_V(\rho) g^{-1})$ and will not change the transported normalization between H and G. By contrast, if one abstractly modifies (ψ_V, u_V) by keeping ψ_V unchanged and replacing u_V by $\rho \mapsto {}^{\rho}g u_V(\rho) g^{-1}$ for some $g \in G^*(\overline{F})$, the resulting normalization of transfer factors between H and G can change, as observed in [**Wal10**, §1.11 (4)].

Definition 5.1.5. — When d is even and δ is trivial, we fix an $\mathrm{SO}(\underline{V})(F)$-orbit of hyperbolic bases (Definition 1.2.2) of $\underline{V}$ once and for all, denoted by $[\mathbb{B}_{\underline{V}}]$. When d is even and δ is non-trivial, we fix $\alpha \in \overline{F}$ such that $x = \alpha^2 \in F^{\times}$ is a lift of δ, and we fix an $\mathrm{SO}(\underline{V})(F)$-orbit $[\mathbb{B}_{\underline{V}}]$ of near-hyperbolic bases of $\underline{V}$ such that all members of this orbit have discriminant x (see Definition 1.2.2). If $F = \mathbb{R}$, we identify $\overline{F}$ with $\mathbb{C}$ and take $\alpha = \sqrt{-1}$.

5.2. Some matrix groups over $\mathbb{C}$

5.2.1. — We define some algebraic groups over $\mathbb{C}$, which we also identify with their $\mathbb{C}$-points. Let N be a positive even integer. Let $\{\hat{e}_k \mid 1 \le k \le N\}$ be the standard basis of $\mathbb{C}^N$. Define two $N \times N$ matrices

$$I_N^- := \begin{pmatrix} & & & & -1 \\ & & & 1 & \\ & & -1 & & \\ & \iddots & & & \\ 1 & & & & \end{pmatrix}, \qquad I_N^+ := \begin{pmatrix} I_{N/2} & \\ & -I_{N/2} \end{pmatrix} I_N^-.$$

Thus I_N^+ and I_N^- define a quadratic form and a symplectic form on $\mathbb{C}^N$ respectively. We use these forms to define the groups $\mathrm{O}_N(\mathbb{C})$, $\mathrm{SO}_N(\mathbb{C})$, and $\mathrm{Sp}_N(\mathbb{C})$, as subgroups of $\mathrm{GL}_N(\mathbb{C})$. By convention, $\mathrm{SO}_0(\mathbb{C}) = \mathrm{Sp}_0(\mathbb{C}) = \mathrm{GL}_0(\mathbb{C}) = \{1\}$.

We introduce a short-hand notation in order to conveniently denote certain diagonal matrices. For $x_1, \cdots, x_n \in \mathbb{C}^\times$, we write $\mathrm{symdiag}(x_1, \cdots, x_n)$ for the $2n \times 2n$ diagonal matrix $\mathrm{diag}(x_1, \cdots, x_n, x_n^{-1}, \cdots, x_1^{-1})$.

Definition 5.2.2. — Let $m = d/2$. In the reductive group $\mathrm{Sp}_N(\mathbb{C})$ (resp. $\mathrm{SO}_N(\mathbb{C})$), we fix once and for all a Borel pair $(\mathcal{T}, \mathcal{B})$, together with an isomorphism $(\mathbb{C}^\times)^m \xrightarrow{\sim} \mathcal{T}$, as follows. Let $\mathcal{T}$ be the intersection of $\mathrm{Sp}_N(\mathbb{C})$ (resp. $\mathrm{SO}_N(\mathbb{C})$) with the diagonal torus in $\mathrm{GL}_N(\mathbb{C})$, and define the isomorphism $(\mathbb{C}^\times)^m \xrightarrow{\sim} \mathcal{T}$ by

$$(t_1, \cdots, t_m) \longmapsto \mathrm{symdiag}(t_1, \cdots, t_m).$$

Using this isomorphism we identify $X^*(\mathcal{T})$ and $X_*(\mathcal{T})$ with $\mathbb{Z}^m$. The root datum of $\mathrm{Sp}_N(\mathbb{C})$ (resp. $\mathrm{SO}_N(\mathbb{C})$) on $(X^*(\mathcal{T}), X_*(\mathcal{T}))$ is dual to the standard root datum $\mathrm{RD}(\mathsf{B}_m)$ (resp. $\mathrm{RD}(\mathsf{D}_m)$) as in §1.2.5. We define $\mathcal{B}$ by the condition that the based root datum $\mathrm{BRD}(\mathcal{T}, \mathcal{B})$ is dual to the standard based root datum $\mathrm{BRD}(\mathsf{B}_m)$ (resp. $\mathrm{BRD}(\mathsf{D}_m)$) as in §1.2.5. We call $(\mathcal{T}, \mathcal{B})$ the *standard Borel pair*.

5.3. Fixing the L-group

5.3.1. — Let $\mathrm{BRD}(G)$ be the *canonical based root datum* of $G_{\overline{F}}$, namely the projective limit

$$\mathrm{BRD}(G) = \varprojlim_{(T,B)} \mathrm{BRD}(T, B),$$

where (T, B) runs through the Borel pairs in $G_{\overline{F}}$, and the transition maps are the canonical isomorphisms induced by inner automorphisms of $G_{\overline{F}}$. Since G is defined over F, there is a canonical action of Γ_F on $\mathrm{BRD}(G)$; see [**Bor79**, §1.3]. Recall that the *L-group of* G consists of the following data (cf. [**Bor79**, §2], [**KS99**, §1.2]):

(1) a reductive group $\widehat{G}$ over $\mathbb{C}$.
(2) a Borel pair $(\mathcal{T}, \mathcal{B})$ in $\widehat{G}$.

(3) an action of Γ_F on $\widehat{G}$ via algebraic automorphisms such that there exists a Γ_F-stable splitting extending $(\mathcal{T}, \mathcal{B})$. In particular, Γ_F acts on the based root datum $\mathrm{BRD}(\mathcal{T}, \mathcal{B})$.

(4) a Γ_F-equivariant isomorphism

$$\mathfrak{v} : \mathrm{BRD}(G) \xrightarrow{\sim} \mathrm{BRD}(\mathcal{T}, \mathcal{B})^\vee, \tag{5.3.1.1}$$

where $\mathrm{BRD}(\mathcal{T}, \mathcal{B})^\vee$ denotes the dual of $\mathrm{BRD}(\mathcal{T}, \mathcal{B})$.

Given the above data, one defines

$$^L G := \widehat{G} \rtimes \Gamma',$$

where Γ' is taken to be one of the following groups depending on the context: If F is a number field, we typically take Γ' to be Γ_F or a sufficiently large finite quotient of it. When $F = \mathbb{R}$, we typically take Γ' to be the Weil group $W_\mathbb{R}$, which acts on $\widehat{G}$ through the map $W_\mathbb{R} \to \Gamma_\mathbb{R}$. When $F = \mathbb{C}$ we take Γ' to be trivial. (This case will never be considered in the paper.) When F is a non-archimedean local field of characteristic zero, we typically take Γ' to be the Weil group W_F acting on $\widehat{G}$ through $W_F \hookrightarrow \Gamma_F$, or a sufficiently large (finite or infinite) quotient of W_F. Here "sufficiently large" always means that Γ' should admit a quotient $\mathrm{Gal}(E/F)$, where E/F is a Galois extension sufficiently large such that the Γ_F-action on $\widehat{G}$ in (3) above factors through $\mathrm{Gal}(E/F)$. As a result, Γ' acts on $\widehat{G}$. For our specific G, this means that when d is even and δ is non-trivial, Γ' should admit $\mathrm{Gal}(F(\alpha)/F)$ as a natural quotient, where α is as in Definition 5.1.5.

We have a canonical Γ_F-equivariant isomorphism between $\mathrm{BRD}(G)$ and $\mathrm{BRD}(G^*)$ (coming from the fixed $G^*(\overline{F})$-conjugacy class of inner twistings $G_{\overline{F}} \xrightarrow{\sim} G^*_{\overline{F}}$ represented by ψ_V). Thus if $\widehat{G}$ and $(\mathcal{T}, \mathcal{B})$ are as in (1), (2), (3) above, then specifying $\mathfrak{v}$ as in (4) is equivalent to specifying a Γ_F-equivariant isomorphism

$$\mathfrak{v}^* : \mathrm{BRD}(G^*) \xrightarrow{\sim} \mathrm{BRD}(\mathcal{T}, \mathcal{B})^\vee. \tag{5.3.1.2}$$

In other words, fixing an L-group of G is equivalent to fixing an L-group of G^*.

5.3.2. — We now explicitly present the L-group of G. We take $\widehat{G}$ to be $\mathrm{Sp}_{d-1}(\mathbb{C})$ (resp. $\mathrm{SO}_d(\mathbb{C})$) as in §5.2 if d is odd (resp. even). Define the action of Γ_F on $\widehat{G}$ as follows. The action is trivial unless d is even and δ is non-trivial. In the latter case, we define the action to factor through $\Gamma_F \to \mathrm{Gal}(F(\alpha)/F)$ (see Definition 5.1.5 for α), and let the non-trivial element of $\mathrm{Gal}(F(\alpha)/F)$ act on $\widehat{G} = \mathrm{SO}_d(\mathbb{C})$ by conjugation by the permutation matrix on $\mathbb{C}^d$ that switches $\hat{e}_m$ and $\hat{e}_{m+1}$ and fixes all the other $\hat{e}_i$'s.

We take $(\mathcal{T}, \mathcal{B})$ to be the standard Borel pair fixed in Definition 5.2.2. Then it is easy to check that the condition in (3) in §5.3.1 is indeed satisfied. To complete the presentation of the L-group, we have yet to specify (5.3.1.1). As we have already noted, this is equivalent to specifying (5.3.1.2).

Under the isomorphism $(\mathbb{C}^\times)^m \xrightarrow{\sim} \mathcal{T}$ specified in Definition 5.2.2, the based root datum $\mathrm{BRD}(\mathcal{T}, \mathcal{B})^\vee$ is identified with the standard based root datum $\mathrm{BRD}(\mathsf{B}_m)$ (resp. $\mathrm{BRD}(\mathsf{D}_m)$) in the odd (resp. even) case. Moreover the Γ_F-action on $\mathrm{BRD}(\mathsf{B}_m)$ or $\mathrm{BRD}(\mathsf{D}_m)$ induced by the Γ_F-action on $\widehat{G}$ fixed above is the trivial action unless d is even and δ is non-trivial, in which case it is given by the unique non-trivial action of $\mathrm{Gal}(F(\alpha)/F) = \mathbb{Z}/2\mathbb{Z}$ on $\mathrm{BRD}(\mathsf{D}_m)$. Hence to specify (5.3.1.2), it suffices to specify a Γ_F-equivariant isomorphism $\mathfrak{v}^{*\prime} : \mathrm{BRD}(G^*) \xrightarrow{\sim} \mathrm{BRD}(\mathsf{B}_m)$ or $\mathfrak{v}^{*\prime} : \mathrm{BRD}(G^*) \xrightarrow{\sim} \mathrm{BRD}(\mathsf{D}_m)$, where Γ_F acts on the right hand sides in the way just described.

In the odd case, there is a unique choice of $\mathfrak{v}^{*\prime}$. In the even case, remember that when δ is trivial (resp. non-trivial), we have fixed $[\mathbb{B}_{\underline{V}}]$ (resp. α and $[\mathbb{B}_{\underline{V}}]$) in Definition 5.1.5. Any member $\mathbb{B}_{\underline{V}}$ of $[\mathbb{B}_{\underline{V}}]$ gives rise to a Borel pair (T, B) in G^*, and it together with α gives rise to an isomorphism $\mathrm{BRD}(T, B) \xrightarrow{\sim} \mathrm{BRD}(\mathsf{D}_m)$, as in §1.2.7. We thus obtain an isomorphism $\mathrm{BRD}(G^*) \xrightarrow{\sim} \mathrm{BRD}(\mathsf{D}_m)$, which we easily check is Γ_F-equivariant, and depends on $\mathbb{B}_{\underline{V}}$ only via $[\mathbb{B}_{\underline{V}}]$. This specifies $\mathfrak{v}^{*\prime}$.

The presentation of the L-group of G is complete.

5.3.3. — Suppose $F = \mathbb{Q}$, and let v be a place of $\mathbb{Q}$. Fix a field embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_v$. Then our above presentation of the L-group of G naturally gives rise to a presentation of the L-group of $G_{\mathbb{Q}_v}$. On the other hand, if $(\underline{V}, \underline{q})$ is the quasi-split quadratic space over $\mathbb{Q}$ fixed in §5.1, then $\underline{V}_{\mathbb{Q}_v} = (\underline{V}, \underline{q}) \otimes_{\mathbb{Q}} \mathbb{Q}_v$ is up to isomorphism the unique quasi-split quadratic space over $\mathbb{Q}_v$ of dimension d and discriminant δ. Thus one could choose the data as in Definitions 5.1.1, and 5.1.5 with respect to the base field $\mathbb{Q}_v$ and with V and $\underline{V}$ replaced by $V_{\mathbb{Q}_v}$ and $\underline{V}_{\mathbb{Q}_v}$, say $\phi_{V_{\mathbb{Q}_v}}$, $[\mathbb{B}_{\underline{V}_{\mathbb{Q}_v}}]$, and α_v, and obtain from these data a presentation of the L-group of $G_{\mathbb{Q}_v} = \mathrm{SO}(V_{\mathbb{Q}_v})$ by going through the above constructions again. These two presentations of the L-group of $G_{\mathbb{Q}_v}$ are identical in the odd case, and in the even case they are identical as long as the following conditions are satisfied:

(1) We have $\phi_{V_{\mathbb{Q}_v}} = g_v \circ (\phi_V \otimes_{\overline{\mathbb{Q}}} \mathrm{id}_{\overline{\mathbb{Q}}_v})$ for some $g_v \in G^*(\overline{\mathbb{Q}}_v)$.

(2) The isomorphism $\mathrm{BRD}(G^*) \xrightarrow{\sim} \mathrm{BRD}(\mathsf{D}_m)$ arising from $[\mathbb{B}_{\underline{V}}]$ and α is compatible with the isomorphism $\mathrm{BRD}(G^*_{\mathbb{Q}_v}) = \mathrm{BRD}(\mathrm{SO}(\underline{V}_{\mathbb{Q}_v})) \xrightarrow{\sim} \mathrm{BRD}(\mathsf{D}_m)$ arising from $[\mathbb{B}_{\underline{V}_{\mathbb{Q}_v}}]$ and α_v.

In the rest of the paper these compatibility conditions are always implicitly assumed when we simultaneously deal with $\mathbb{Q}$ and its localizations; note that when ϕ_V is given, there indeed exists $\phi_{V_{\mathbb{R}}}$ satisfying simultaneously (1) in the above and the extra condition in Definition 5.1.1. By contrast, we will not assume that the local data $\phi_{V_{\mathbb{Q}_v}}, [\mathbb{B}_{\underline{V}_{\mathbb{Q}_v}}], \alpha_v$ are induced by the global data $\phi_V, [\mathbb{B}_{\underline{V}}], \alpha$ on the nose. Under condition (1) we also know that the inner class of the inner twisting $\psi_V : G_{\overline{\mathbb{Q}}} \xrightarrow{\sim} G^*_{\overline{\mathbb{Q}}}$ (arising from ϕ_V) induces the inner class of the inner twisting $\psi_{V_{\mathbb{Q}_v}} : G_{\overline{\mathbb{Q}}_v} \xrightarrow{\sim} G^*_{\overline{\mathbb{Q}}_v}$ (arising from $\phi_{V_{\mathbb{Q}_v}}$) via base change.

5.4. The elliptic endoscopic data

5.4.1. — Keep the setting of §5.3. Denote by $\mathscr{E}(G)$ the set of isomorphism classes of elliptic endoscopic data for G, in the sense of [**KS99**, §2.1]. In the following we construct explicit representatives of $\mathscr{E}(G)$, following [**Wal10**]. Recall from [**KS99**, §2.1] that in general, the category of elliptic endoscopic data for G is a full subcategory of the category of endoscopic data for G, and the latter is a full subcategory of the groupoid category described as follows:

– The objects are tuples $(H, \mathcal{H}, s, \eta)$, where H is a quasi-split reductive group over F, $\mathcal{H}$ is a group containing $\widehat{H}$ as a subgroup, s is an element of $\widehat{G}$, and η is an injective group homomorphism $\mathcal{H} \to {}^L G$.

– An isomorphism from $(H, \mathcal{H}, s, \eta)$ to $(H', \mathcal{H}', s', \eta')$ is an element $g \in \widehat{G}$ such that $g \operatorname{im}(\eta) g^{-1} = \operatorname{im}(\eta')$ and $gsg^{-1} \equiv s' \mod Z(\widehat{G})$.

We do not recall here the conditions characterizing the subcategories of endoscopic data and elliptic endoscopic data.

In the following, all our explicit representatives of $\mathscr{E}(G)$ will be of the form $(H, {}^L H, s, \eta)$. Thus in the terminology of [**Kal11**], we represent each isomorphism class of elliptic endoscopic data by an *extended endoscopic triple*. The advantage of doing so is that we could avoid introducing z-extensions, which is in general a necessity for the theory of endoscopy when G^{der} is not simply connected; cf. [**Taï19**, §2.3].

We first define a set of numerical parameters that will be used.

Definition 5.4.2. — Let V be a quadratic space over F of dimension d and discriminant δ. Define a set $\mathscr{P}_V$ as follows.

(1) When d is odd, we let $\mathscr{P}_V$ be the set of pairs (d^+, d^-) of positive odd integers such that $d^+ + d^- = d + 1$. We define an involution `sw` on $\mathscr{P}_V$ (called *swapping*) by sending (d^+, d^-) to (d^-, d^+).

(2) When d is even, we let $\mathscr{P}_V$ be the set of quadruples $(d^+, \delta^+, d^-, \delta^-)$, where:

- d^+ and d^- are non-negative even integers such that $d^+ + d^- = d$.
- δ^+ and δ^- are elements of $F^\times / F^{\times,2}$ such that $\delta^+ \delta^- = \delta$.
- Neither of (d^+, δ^+) and (d^-, δ^-) is equal to $(0, x)$ for any non-trivial $x \in F^\times / F^{\times,2}$. If $d \geq 4$, then neither of (d^+, δ^+) and (d^-, δ^-) is equal to $(2, 1)$.

We define an involution `sw` on $\mathscr{P}_V$ by sending $(d^+, \delta^+, d^-, \delta^-)$ to $(d^-, \delta^-, d^+, \delta^+)$.

When d is odd, we sometimes write elements of $\mathscr{P}_V$ also as $(d^+, \delta^+, d^-, \delta^-)$, understanding that $\delta^+ = \delta^- = 1$.

5.4.3. — Fix an element $(d^+, \delta^+, d^-, \delta^-) \in \mathscr{P}_V$. We shall construct an elliptic endoscopic datum for $G = \mathrm{SO}(V)$ associated to this parameter. The endoscopic datum will be of the form $(H, \mathcal{H}, s, \eta)$, where

– H is given as $H = H^+ \times H^-$, with $H^\pm = \mathrm{SO}(V^\pm)$, where $V^\pm$ are the unique (up to isomorphism) quasi-split quadratic spaces over F of dimension $d^\pm$ and discriminant $\delta^\pm$; remember that $\delta^\pm$ are understood to be trivial in the odd case.
– $\mathcal{H} = {}^L H$ is the L-group of H; cf. the discussion in §5.4.1.
– s is a semi-simple element of $\widehat{G}$.
– η is an L-embedding ${}^L H \to {}^L G$.

To be more precise, in the even case, we fix similar choices as in Definition 5.1.5 for $V^\pm$, which we shall denote by $\alpha^\pm$ and $[\mathbb{B}_{V^\pm}]$. (Here α^+ is only needed when δ^+ is non-trivial, and similarly for α^-.) We then use these choices to specify the analogues of (5.3.1.2) for $H^\pm$ in both the odd and even cases, and present the L-groups ${}^L H^\pm$ as $\widehat{H^\pm} \rtimes \Gamma'$ as in §5.3, where $\widehat{H^\pm}$ are the matrix groups $\mathrm{Sp}_{d^\pm - 1}(\mathbb{C})$ (resp. $\mathrm{SO}_{d^\pm}(\mathbb{C})$) in the odd (resp. even) case. In the even case, Γ' needs to be large enough so as to admit $\mathrm{Gal}(F(\alpha^+)/F)$ (resp. $\mathrm{Gal}(F(\alpha^-)/F)$, resp. $\mathrm{Gal}(F(\alpha^+, \alpha^-)/F)$) as a quotient when δ^+ is non-trivial (resp. δ^- is nontrivial, resp. δ^+ and δ^- are both non-trivial).

We present the L-group ${}^L H$ of H as the fiber product of ${}^L H^+$ and ${}^L H^-$ over Γ'. Thus ${}^L H$ is a semi-direct product

$$(\widehat{H^+} \times \widehat{H^-}) \rtimes \Gamma',$$

and $\widehat{H} = \widehat{H^+} \times \widehat{H^-}$ is equipped with the standard Borel pair

$$(\mathcal{T}_{\widehat{H}}, \mathcal{B}_{\widehat{H}}) = (\mathcal{T}_{V^+} \times \mathcal{T}_{V^-}, \mathcal{B}_{V^+} \times \mathcal{B}_{V^-}).$$

Here $(\mathcal{T}_{V^\pm}, \mathcal{B}_{V^\pm})$ are as in Definition 5.2.2 for the matrix groups $\widehat{H^\pm}$.

We now specify the components s and η. The element $s \in \widehat{G}$ will always be a diagonal matrix, with ± 1's on the diagonal. We write $s = \mathrm{diag}(s_1, \cdots, s_{d-1})$ or $\mathrm{diag}(s_1, \cdots, s_d)$, when d is odd or even respectively.

For $w \in \Gamma'$, we write[(1)]

$$\eta(w) = (\rho(w), w) \in {}^L G = \widehat{G} \rtimes \Gamma'.$$

To specify the map $\eta : {}^L H \to {}^L G$ it suffices to specify the map $\eta|_{\widehat{H}} : \widehat{H} \to \widehat{G}$ and the map $\rho : \Gamma' \to \widehat{G}$.

We now specify the numbers $s_i \in \mathbb{C}^\times$, and the maps $\eta|_{\widehat{H}}$ and ρ.

[(1)] In practice, it can happen that ${}^L H$ is presented as $\widehat{H} \rtimes \Gamma'_H$ whereas ${}^L G$ is presented as $\widehat{G} \rtimes \Gamma'_G$, for different quotient groups Γ'_H and Γ'_G of the absolute Galois (or Weil) group of F. For instance, in the even case, when δ is trivial and δ^+ and δ^- are both non-trivial, we can take Γ'_H to be $\mathrm{Gal}(F(\alpha^+, \alpha^-)/F)$ and take Γ'_G to be trivial. In all cases, we may and shall assume that Γ'_G is always a quotient of Γ'_H. Then the formula $\eta(w) = (\rho(w), w)$ is understood as $\eta(w) = (\rho(w), \pi(w))$, where π is the quotient map $\Gamma'_H \to \Gamma'_G$. In the text we slightly abuse notation to write Γ' for both Γ'_H and Γ'_G.

5.4.3.1. *The odd case.* — Write $m^{\pm} := \lfloor d^{\pm}/2 \rfloor$. Define

$$s_k := \begin{cases} 1, & \text{if } m^- + 1 \le k \le d - m^- - 1, \\ -1, & \text{otherwise.} \end{cases}$$

Define the map

$$\eta|_{\widehat{H}} : \widehat{H} = \widehat{H^+} \times \widehat{H^-} = \mathrm{Sp}_{d^+-1}(\mathbb{C}) \times \mathrm{Sp}_{d^--1}(\mathbb{C}) \longrightarrow \widehat{G} = \mathrm{Sp}_{d-1}(\mathbb{C})$$

to be the restriction of the map $\mathrm{GL}_{d^+-1}(\mathbb{C}) \times \mathrm{GL}_{d^--1}(\mathbb{C}) \to \mathrm{GL}_{d-1}(\mathbb{C})$ given by the identification

$$\begin{aligned} \mathbb{C}^{d^+-1} \times \mathbb{C}^{d^--1} &\xrightarrow{\sim} \mathbb{C}^{d-1} \\ (\hat{e}_k, 0) &\longmapsto \hat{e}_{k+m^-} \\ (0, \hat{e}_k) &\longmapsto \begin{cases} \hat{e}_k, & \text{if } k \le m^-, \\ \hat{e}_{k+d^+-1}, & \text{if } m^- + 1 \le k \le d^- - 1. \end{cases} \end{aligned}$$

Finally, define ρ to be trivial.

5.4.3.2. *The even case.* — Write $m^{\pm} := d^{\pm}/2$. Define

$$s_k := \begin{cases} 1, & \text{if } m^- + 1 \le k \le d - m^-, \\ -1, & \text{otherwise.} \end{cases}$$

Define the map

$$\eta|_{\widehat{H}} : \widehat{H} = \widehat{H^+} \times \widehat{H^-} = \mathrm{SO}_{d^+}(\mathbb{C}) \times \mathrm{SO}_{d^-}(\mathbb{C}) \longrightarrow \widehat{G} = \mathrm{SO}_d(\mathbb{C})$$

to be the restriction of the map $\mathrm{GL}_{d^+}(\mathbb{C}) \times \mathrm{GL}_{d^-}(\mathbb{C}) \to \mathrm{GL}_d(\mathbb{C})$ given by the identification

$$\begin{aligned} \mathbb{C}^{d^+} \times \mathbb{C}^{d^-} &\xrightarrow{\sim} \mathbb{C}^{d} \\ (\hat{e}_k, 0) &\longmapsto \hat{e}_{k+m^-}, \\ (0, \hat{e}_k) &\longmapsto \begin{cases} \hat{e}_k, & \text{if } k \le m^-, \\ \hat{e}_{k+d^+}, & \text{if } m^- + 1 \le k \le d^-. \end{cases} \end{aligned} \tag{5.4.3.1}$$

We define $\rho : \Gamma' \to \widehat{G}$ as follows. First we define a matrix $S \in \mathrm{GL}_d(\mathbb{C})$. If $d^+ \neq 0$, we take S to be the permutation matrix that switches $\hat{e}_{m^-}$ and $\hat{e}_{d-m^-+1}$, switches $\hat{e}_m$ and $\hat{e}_{m+1}$, and leaves all the other $\hat{e}_i$'s fixed. If $d^+ = 0$, we take S to be I_d. Thus in all cases we have $S \in \widehat{G}$. We then let $\rho : \Gamma' \to \widehat{G}$ be the map

$$w \longmapsto \begin{cases} 1, & \text{if } w|_{F(\alpha^-)} = \mathrm{id}, \\ S, & \text{otherwise.} \end{cases} \tag{5.4.3.2}$$

Here remember that α^- is a fixed square root in $\overline{F}$ of a fixed lift of δ^- in $F^\times$ when δ^- is non-trivial. If δ^- is trivial, we understand $F(\alpha^-)$ as F. The above formula

(5.4.3.2) makes sense because when δ^- is non-trivial we have assumed that Γ' admits $\mathrm{Gal}(F(\alpha^-)/F)$ as a quotient.

5.4.4. — In both the odd and even cases, the construction in §5.4.3 associates to each parameter $\mathfrak{p} \in \mathscr{P}_V$ an elliptic endoscopic datum

$$\mathfrak{e}_{\mathfrak{p}} = \mathfrak{e}_{\mathfrak{p}}(V) = (H, {}^L H, s, \eta)$$

for G. Moreover, the construction $\mathfrak{p} \mapsto \mathfrak{e}_{\mathfrak{p}}$ induces a bijection

$$\mathscr{P}_V/\mathtt{sw} \xrightarrow{\sim} \mathscr{E}(G).$$

These facts are well known (see [**Wal10**, §1.8] or [**Taï19**, §2.3]) and can be proved similarly as [**Mor11**, Prop. 2.1.1].

5.4.5. — Let $\mathfrak{p} \in \mathscr{P}_V$. The *outer automorphism group* $\mathrm{Out}(\mathfrak{e}_{\mathfrak{p}})$ of the endoscopic datum $\mathfrak{e}_{\mathfrak{p}} = (H, {}^L H, s, \eta)$ is defined in [**KS99**, §2.1]. Note that the group $\bar{Z}$ discussed on p. 19 of [**KS99**] is trivial, as $Z(\widehat{G})$ is contained in $\eta(\widehat{H})$. Hence $\mathrm{Out}(\mathfrak{e}_{\mathfrak{p}})$ is isomorphic to $\mathrm{Aut}(\mathfrak{e}_{\mathfrak{p}})/\widehat{H}$ (where $\mathrm{Aut}(\mathfrak{e}_{\mathfrak{p}})$ denotes the automorphism group of $\mathfrak{e}_{\mathfrak{p}}$ in the category of endoscopic data), and can be naturally viewed as a subgroup of $\mathrm{Out}_F(H) := \mathrm{Aut}_F(H)/H^{\mathrm{ad}}(F)$; see *loc. cit.* for details.

In the odd case, $\mathrm{Out}(\mathfrak{e}_{\mathfrak{p}})$ is trivial unless $\mathfrak{p} = \mathtt{sw}(\mathfrak{p})$, in which case we have $\mathrm{Out}(\mathfrak{e}_{\mathfrak{p}}) \cong \mathbb{Z}/2\mathbb{Z}$, with the non-trivial element acting by swapping H^+ and H^-.

In the even case, write $\mathfrak{p} = (d^+, \delta^+, d^-, \delta^-)$. Then $\mathrm{Out}(\mathfrak{e}_{\mathfrak{p}})$ is trivial when $d^+ d^- = 0$. When $d^+ = d^- = d/2$ and $\delta = 1$, we have $\mathrm{Out}(\mathfrak{e}_{\mathfrak{p}}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where the non-trivial element of the first $\mathbb{Z}/2\mathbb{Z}$ induces simultaneously non-trivial outer automorphisms on H^+ and H^-, and the non-trivial element of the second $\mathbb{Z}/2\mathbb{Z}$ acts by swapping H^+ and H^-. In the remaining cases, we have $\mathrm{Out}(\mathfrak{e}_{\mathfrak{p}}) \cong \mathbb{Z}/2\mathbb{Z}$, with the non-trivial element acting by the simultaneously non-trivial outer automorphisms on H^+ and H^-.

5.5. The endoscopic G-data for Levi subgroups

5.5.1. — Let M be a Levi subgroup of G. The notion of an *endoscopic G-triple for M* is introduced by Kottwitz in his unpublished notes, and recalled in [**Mor10b**, §2.4]. (For $G = M$, this is the usual notion of an endoscopic triple for M, as in [**Kot84b**, §7.4].) Given an endoscopic datum $(M', \mathcal{M}', s_M, \eta_M)$ for M, we shall say that it is an *endoscopic G-datum for M*, if $(M', s_M, \eta_M|_{\widehat{M'}})$ is an endoscopic G-triple for M in the sense of [**Mor10b**, Def. 2.4.1]. By an *isomorphism* between two endoscopic G-data $(M'_1, \mathcal{M}'_1, s_{M,1}, \eta_{M,1})$ and $(M'_2, \mathcal{M}'_2, s_{M,2}, \eta_{M,2})$ for M, we mean an element $g \in \widehat{M}$ such that $g \,\mathrm{im}(\eta_{M,1}) g^{-1} = \mathrm{im}(\eta_{M,2})$ and $g s_{M,1} g^{-1} \equiv s_{M,2} \mod Z(\widehat{G})$. Here $Z(\widehat{G})$ is canonically embedded in $Z(\widehat{M})$.

We mention that the category of endoscopic G-data for M (where the morphisms are the isomorphisms) is in fact equivalent to the category of *endoscopic G-pairs for M* in Kottwitz's unpublished notes.

It is easy to see that the association $(M', \mathcal{M}', s_M, \eta_M) \mapsto (M', s_M, \eta_M|_{\widehat{M}'})$ defines a bijection

$$\{\text{endoscopic } G\text{-data for } M\}/\text{isom} \xrightarrow{\sim} \{\text{endoscopic } G\text{-triples for } M\}/\text{isom}.$$

We also have a similar bijection

$$\{\text{endoscopic data for } G\}/\text{isom} \xrightarrow{\sim} \{\text{endoscopic triples for } G\}/\text{isom}.$$

As recalled in [**Mor10b**, §2.4], Kottwitz constructs a map

$$\{\text{endoscopic } G\text{-triples for } M\}/\text{isom} \longrightarrow \{\text{endoscopic triples for } G\}/\text{isom}.$$

We thus obtain a map

$$\{\text{endoscopic } G\text{-data for } M\}/\text{isom} \longrightarrow \{\text{endoscopic data for } G\}/\text{isom}.$$

We say that an endoscopic G-datum for M is *bi-elliptic*, if both the underlying endoscopic datum for M and the associated endoscopic datum for G (well-defined up to isomorphism) are elliptic. We denote by $\mathscr{E}_G(M)$ the set of isomorphism classes of bi-elliptic endoscopic G-data for M. Thus we have natural maps $\mathscr{E}_G(M) \to \mathscr{E}(G)$ and $\mathscr{E}_G(M) \to \mathscr{E}(M)$.

In the following we construct explicit representatives of $\mathscr{E}_G(M)$. For later purposes, it suffices to consider only certain Levi subgroups M specified as follows.

5.5.2. — Consider a subspace W of V such that the quadratic form on V is non-degenerate on W and such that the orthogonal complement $W^\perp$ of W in V is even-dimensional and split as a quadratic space. We write d_W for the dimension of W, and let $n = \lfloor d_W/2 \rfloor$. Recall that V has dimension d and discriminant δ, and as always m denotes $\lfloor d/2 \rfloor$. Clearly the discriminant of W equals δ, and d_W has the same parity as d.

Fix $r, t \in \mathbb{Z}_{\geq 0}$ such that $m = n + r + 2t$. Thus $\dim W^\perp = 2(r + 2t)$. We fix a hyperbolic basis (Definition 1.2.2)

$$\mathbb{B}_{W^\perp} = \left\{ f_1, \cdots, f_{2(r+2t)} \right\}$$

of $W^\perp$, which exists since $W^\perp$ is split. Using this basis, we identify $\mathrm{SO}(W^\perp)$ as a subgroup of $\mathrm{GL}_{2(r+2t)}$, and define an embedding

$$\mathbb{G}_m^r \times \mathrm{GL}_2^t \longrightarrow \mathrm{SO}(W^\perp) \tag{5.5.2.1}$$

by sending $(z_1, \cdots, z_r, w_1, \cdots, w_t)$ to the block diagonal matrix

$$\mathrm{diag}(z_1, \cdots, z_r, w_1, \cdots, w_t, J_2(w_t^\mathsf{T})^{-1} J_2, \cdots, J_2(w_1^\mathsf{T})^{-1} J_2, z_r^{-1}, \cdots, z_1^{-1}),$$

where

$$J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We denote the image of (5.5.2.1) by M^{GL}, and define M to be $M^{\mathrm{GL}} \times \mathrm{SO}(W)$, viewed as a subgroup of G. Then M is a Levi subgroup of G. We also write M^{SO} for $\mathrm{SO}(W)$.

5.5.3. — We proceed similarly as before to fix the quasi-split inner form of $\mathrm{SO}(W)$, present the L-group of $\mathrm{SO}(W)$, and fix explicit representatives of the isomorphism classes of the elliptic endoscopic data for $\mathrm{SO}(W)$. We need to fix notation and impose some compatibility conditions. Since d_W has the same parity as d, in the following we shall still refer to the "odd case" and the "even case" unambiguously. As in §5.1, we fix the unique (up to isomorphism) quasi-split quadratic space $\underline{W}$ over F of dimension d_W and discriminant δ (which is the common discriminant of V and W) and fix an isomorphism

$$\phi_W : W \otimes_F \overline{F} \xrightarrow{\sim} \underline{W} \otimes_F \overline{F}$$

of quadratic spaces over $\overline{F}$, from which we get the inner twisting

$$\begin{aligned} \psi_W : \mathrm{SO}(W)_{\overline{F}} &\xrightarrow{\sim} \mathrm{SO}(\underline{W})_{\overline{F}} \\ g &\longmapsto \phi_W g \phi_W^{-1}. \end{aligned}$$

Note that as quadratic spaces over F, $\underline{V}$ is isomorphic to the orthogonal direct sum of $W^{\perp}$ and $\underline{W}$. We fix such an isomorphism

$$\phi_W^V : W^{\perp} \oplus \underline{W} \xrightarrow{\sim} \underline{V},$$

and use it to obtain an embedding

$$M^{\mathrm{GL}} \times \mathrm{SO}(\underline{W}) \hookrightarrow G^* \tag{5.5.3.1}$$

whose image is a Levi subgroup.

We remind the reader that when $F = \mathbb{R}$ we require both ϕ_V and ϕ_W to satisfy the extra condition in Definition 5.1.1. In general, we assume the following compatibility condition, which can obviously be arranged by adjusting ϕ_W.

(1) The diagram

$$\begin{array}{ccc} (W^{\perp} \oplus W) \otimes \overline{F} & = \!= \!= \!= & V \otimes \overline{F} \\ \Big\downarrow{\scriptstyle \mathrm{id} \oplus \phi_W} & & \Big\downarrow{\scriptstyle \phi_V} \\ (W^{\perp} \oplus \underline{W}) \otimes \overline{F} & \xrightarrow{\phi_W^V} & \underline{V} \otimes \overline{F} \end{array}$$

commutes up to an element of $G^*(\overline{F}) = \mathrm{SO}(\underline{V})(\overline{F})$.

As a consequence of this condition, we know that the diagram

$$\begin{array}{ccc} M_{\overline{F}} = (M^{\mathrm{GL}} \times \mathrm{SO}(W))_{\overline{F}} & \xrightarrow{\text{inclusion}} & G_{\overline{F}} \\ \Big\downarrow{\scriptstyle (\mathrm{id}, \psi_W)} & & \Big\downarrow{\scriptstyle \psi_V} \\ (M^{\mathrm{GL}} \times \mathrm{SO}(\underline{W}))_{\overline{F}} & \xrightarrow{(5.5.3.1)} & G^*_{\overline{F}} \end{array}$$

commutes up to an inner automorphism of $G^*_{\overline{F}}$.

In the odd case, we present the L-group ${}^L\,\mathrm{SO}(W)$ as in §5.3. In the even case, we make similar choices as in Definition 5.1.5 for $\underline{W}$, to be denoted by $\alpha_{\underline{W}}$ (needed only when δ is non-trivial) and $[\mathbb{B}_{\underline{W}}]$, and use them to present the L-group ${}^L\,\mathrm{SO}(W)$ as in §5.3. We may and shall assume the following compatibility conditions:

(2) There is a member $\mathbb{B}_{\underline{W}} \in [\mathbb{B}_{\underline{W}}]$ such that ϕ^V_W sends the ordered basis $(\mathbb{B}_{W^\perp}, \mathbb{B}_{\underline{W}})$ to a member of $[\mathbb{B}_{\underline{V}}]$.

(3) When δ is non-trivial, the choices $\alpha_{\underline{W}}$ and α are equal.

Note that the above two conditions are consistent: if (2) is already arranged then we have $\alpha^2 = \alpha^2_{\underline{W}}$ when δ is non-trivial, and so we can arrange (3).

In both the odd and even cases, we canonically identify M^{GL} with $\mathbb{G}_m^r \times \mathrm{GL}_2^t$ via (5.5.2.1), and canonically present $\widehat{M^{\mathrm{GL}}}$ as $(\mathbb{C}^\times)^r \times \mathrm{GL}_2(\mathbb{C})^t$. We now present the L-group of M as

$${}^L M = \widehat{M^{\mathrm{GL}}} \times {}^L\,\mathrm{SO}(W).$$

The above compatibility conditions (1)–(3) ensure that the canonical $\widehat{G}$-conjugacy class of maps ${}^L M \to {}^L G$ arising from the fact that M is a Levi subgroup of G is represented by the following map:

$$\begin{aligned} (5.5.3.2) \quad {}^L M = (\mathbb{C}^\times)^r \times \mathrm{GL}_2(\mathbb{C})^t \times \widehat{\mathrm{SO}(W)} \rtimes \Gamma' \ni (g_1, \cdots, g_r, h_1, \cdots, h_t, k) \rtimes \tau \\ \longmapsto \mathrm{diag}(g_1, \cdots, g_r, h_1, \cdots, h_t, k, h_t^\dagger, \cdots, h_1^\dagger, g_r^{-1}, \cdots, g_1^{-1}) \rtimes \tau \\ \in {}^L G = \widehat{G} \rtimes \Gamma', \end{aligned}$$

where we define

$$(5.5.3.3) \qquad h^\dagger := \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} (h^\intercal)^{-1} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad \forall h \in \mathrm{GL}_2(\mathbb{C}),$$

(i.e., $h^\dagger$ is the adjoint of h^{-1} with respect to the symplectic form defined by $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$).

We now construct explicit representatives of $\mathscr{E}_G(M)$.

Definition 5.5.4. — Let $\mathscr{P}_W$ be as in Definition 5.4.2 with respect to the quadratic space W, and for each positive integer x we write $[x]$ for the set $\{1, 2, \cdots, x\}$. Also set $[0] = \emptyset$. We define the following objects.

(1) Let $\mathscr{P}_{r,t}$ be the set of pairs (A, B), where A is a subset of $[r]$ and B is a subset of $[t]$. For $(A, B) \in \mathscr{P}_{r,t}$, we write A^c for the complement of A in $[r]$ and write B^c for the complement of B in $[t]$.

(2) Let $\mathscr{P}_{r,t} \times' \mathscr{P}_W$ be the subset of $\mathscr{P}_{r,t} \times \mathscr{P}_W$ consisting of those

$$(A, B, d^+, \delta^+, d^-, \delta^-) \in \mathscr{P}_{r,t} \times \mathscr{P}_W$$

such that the quadruple

$$(d^+ + 2\,|A| + 4\,|B|\,, \delta^+, d^- + 2\,|A^c| + 4\,|B^c|\,, \delta^-)$$

belongs to $\mathscr{P}_V$. (In the odd case, we understand that $\delta^+ = \delta^- = 1$, and note that $\mathscr{P}_{r,t} \times' \mathscr{P}_W = \mathscr{P}_{r,t} \times \mathscr{P}_W$.)

(3) Note that $(A, B, \mathfrak{p}) \mapsto (A^c, B^c, \mathtt{sw}(\mathfrak{p}))$ is an involution on the set $\mathscr{P}_{r,t} \times' \mathscr{P}_W$. We denote this involution still by $\mathtt{sw}$.

Definition 5.5.5. — Let A be a subset of $\mathbb{Z}_{\geq 1}$. For each $i \in \mathbb{Z}_{\geq 1}$, we define

$$\nabla_i(A) := \begin{cases} 1, & \text{if } i \in A, \\ -1, & \text{if } i \notin A. \end{cases}$$

5.5.6. — Fix an element $(A, B, \mathfrak{p}) \in \mathscr{P}_{r,t} \times' \mathscr{P}_W$. In the following we construct an endoscopic G-datum for M associated to this parameter, denoted by $\mathfrak{e}_{A,B,\mathfrak{p}}$. From $\mathfrak{p} \in \mathscr{P}_W$ we obtain the endoscopic datum $\mathfrak{e}_{\mathfrak{p}}(W)$ for $\mathrm{SO}(W)$ as in §5.4, which we write as

$$(M'^{,\mathrm{SO}}, {}^L M'^{,\mathrm{SO}}, s^{\mathrm{SO}}, \eta^{\mathrm{SO}} : {}^L M'^{,\mathrm{SO}} \to {}^L \mathrm{SO}(W)).$$

We then set

$$\mathfrak{e}_{A,B,\mathfrak{p}} := (M', {}^L M', s_M, \eta_M : {}^L M' \to {}^L M)$$

with components given as follows. Let

$$M' := M^{\mathrm{GL}} \times M'^{,\mathrm{SO}}.$$

Let s_M be the element of $\widehat{M} = \widehat{M^{\mathrm{GL}}} \times \widehat{M^{\mathrm{SO}}}$ whose component in $\widehat{M^{\mathrm{SO}}}$ is s^{SO} and whose component in $\widehat{M^{\mathrm{GL}}} = (\mathbb{C}^\times)^r \times \mathrm{GL}_2(\mathbb{C})^t$ is

$$s^{\mathrm{GL}} = (\nabla_1(A), \cdots, \nabla_r(A), \nabla_1(B) I_2, \cdots, \nabla_t(B) I_2). \tag{5.5.6.1}$$

We present the L-group ${}^L M'$ of M' as

$${}^L M' = \widehat{M^{\mathrm{GL}}} \times {}^L M'^{,\mathrm{SO}},$$

and define η_M to be the map

$$\eta_M = (\mathrm{id}, \eta^{\mathrm{SO}}) : {}^L M' = \widehat{M^{\mathrm{GL}}} \times {}^L M'^{,\mathrm{SO}} \to {}^L M = \widehat{M^{\mathrm{GL}}} \times {}^L M^{\mathrm{SO}}.$$

For each $\mathfrak{p} \in \mathscr{P}_W$, we also set

$$\mathfrak{e}_{\mathfrak{p}}(M) := (M', {}^L M', s'_M, \eta_M),$$

where $M', {}^LM', \eta_M$ are as above, and s'_M is the element of $\widehat{M} = \widehat{M^{\mathrm{GL}}} \times \widehat{M^{\mathrm{SO}}}$ whose component in $\widehat{M^{\mathrm{SO}}}$ is s^{SO} and whose component in $\widehat{M^{\mathrm{GL}}}$ is the trivial element. Then $\mathfrak{e}_{\mathfrak{p}}(M)$ is an elliptic endoscopic datum for M.

Proposition 5.5.7. — *For each $(A, B, \mathfrak{p}) \in \mathscr{P}_{r,t} \times' \mathscr{P}_W$, the tuple $\mathfrak{e}_{A,B,\mathfrak{p}}$ is a bi-elliptic endoscopic G-datum for M whose underlying endoscopic datum for M is isomorphic to $\mathfrak{e}_{\mathfrak{p}}(M)$. The construction $(A, B, \mathfrak{p}) \mapsto \mathfrak{e}_{A,B,\mathfrak{p}}$ induces a bijection*

$$\left(\mathscr{P}_{r,t} \times' \mathscr{P}_W\right)/\mathtt{sw} \xrightarrow{\sim} \mathscr{E}_G(M).$$

Moreover, for $(A, B, d^+, \delta^+, d^-, \delta^-) \in \mathscr{P}_{r,t} \times' \mathscr{P}_W$, the image of $\mathfrak{e}_{A,B,d^+,\delta^+,d^-,\delta^-}$ under the map $\mathscr{E}_G(M) \to \mathscr{E}(G)$ is represented by

$$\mathfrak{e}_{d^+ + 2|A| + 4|B|, \delta^+, d^- + 2|A^c| + 4|B^c|, \delta^-}.$$

(Remember that if the common parity of d_W and d is odd, then we keep the convention that $\delta^{\pm} = 1$ as in Definition 5.4.2.)

Proof. — This can be checked in a similar way as the proof of [**Mor11**, Lem. 2.3.3]. The key point is that M^{GL} is a product of copies of $\mathbb{G}_m$ and GL_2, and these groups do not have any non-trivial elliptic endoscopic data. □

5.5.8. — Let $(A, B, \mathfrak{p}) \in \mathscr{P}_{r,t} \times' \mathscr{P}_W$. Write $\mathfrak{e}_{A,B,\mathfrak{p}}$ as $(M', {}^LM', s_M, \eta_M)$. We define the *outer G-automorphism group* of $\mathfrak{e}_{A,B,\mathfrak{p}}$ to be

$$\mathrm{Out}_G(\mathfrak{e}_{A,B,\mathfrak{p}}) := \mathrm{Aut}_G(\mathfrak{e}_{A,B,\mathfrak{p}})/\widehat{M}',$$

where $\mathrm{Aut}_G(\mathfrak{e}_{A,B,\mathfrak{p}})$ denotes the automorphism group of $\mathfrak{e}_{A,B,\mathfrak{p}}$ in the category of endoscopic G-data for M (see §5.5.1). We make two remarks on this definition. Firstly, $\mathrm{Out}_G(\mathfrak{e}_{A,B,\mathfrak{p}})$ is naturally isomorphic to the outer automorphism group of the endoscopic G-triple $(M', s_M, \eta_M|_{\widehat{M}'})$ defined in [**Mor10b**, §2.4]. (This is explained in Kottwitz's unpublished notes.) Secondly, $\mathrm{Out}_G(\mathfrak{e}_{A,B,\mathfrak{p}})$ is naturally isomorphic to a subgroup of the outer automorphism group $\mathrm{Out}(\mathfrak{e}_{A,B,\mathfrak{p}})$ of the underlying endoscopic datum for M. (See §5.4.5 for the latter.)

We now explicitly determine $\mathrm{Out}_G(\mathfrak{e}_{A,B,\mathfrak{p}})$. In the odd case, we always have $\mathrm{Out}_G(\mathfrak{e}_{A,B,\mathfrak{p}}) = \{1\}$. In the even case, write $\mathfrak{p} = (d^+, \delta^+, d^-, \delta^-)$. Then $\mathrm{Out}_G(\mathfrak{e}_{A,B,\mathfrak{p}})$ is trivial if $d^+d^- = 0$. In the remaining cases, we have $\mathrm{Out}_G(\mathfrak{e}_{A,B,\mathfrak{p}}) \cong \mathbb{Z}/2\mathbb{Z}$, where the non-trivial element acts via the non-trivial outer automorphism on M^{GL}, and via the simultaneously non-trivial outer automorphisms on the two special orthogonal groups constituting $M'^{,\mathrm{SO}}$.

5.5.9. — Let $(A, B, d^+, \delta^+, d^-, \delta^-) \in \mathscr{P}_{r,t} \times' \mathscr{P}_W$. We have the endoscopic G-datum

$$\mathfrak{e}_{A,B,d^+,\delta^+,d^-,\delta^-} = (M', {}^LM', s_M, \eta_M : {}^LM' \to {}^LM)$$

for M, and the endoscopic datum

$$\mathfrak{e}_{d^+ +2|A|+4|B|,\delta^+,d^- +2|A^c|+4|B^c|,\delta^-} = (H, {}^L H, s, \eta : {}^L H \to {}^L G)$$

for G. Thus the isomorphism class of $(H, {}^L H, s, \eta)$ in $\mathscr{E}(G)$ is equal to the image of the isomorphism class of $(M', {}^L M', s_M, \eta_M)$ in $\mathscr{E}_G(M)$, by Proposition 5.5.7. As explained on p. 43 of [**Mor10b**], the endoscopic G-datum $(M', {}^L M', s_M, \eta_M)$ for M determines an $H(F)$-conjugacy class of Levi subgroups of H, all of which are isomorphic to M'. In the following we upgrade this construction to an $H(F)$-conjugacy class of F-embeddings $M' \hookrightarrow H$ with images Levi subgroups. (This depends on our explicit presentation of the groups.) As a result we obtain a $\widehat{H}$-conjugacy class of embeddings ${}^L M' \to {}^L H$. Our construction will be such that the following diagram commutes up to $\widehat{G}$-conjugation:

(5.5.9.1)
$$\begin{array}{ccc} {}^L H & \xrightarrow{\ \eta\ } & {}^L G \\ \uparrow & & \uparrow \\ {}^L M' & \xrightarrow{\ \eta_M\ } & {}^L M \end{array}$$

Here the vertical arrow on the right is canonical up to $\widehat{G}$-conjugation, arising from the fact that M is a Levi subgroup of G (cf. (5.5.3.2)).

Recall from §5.5.2 that M^{GL} is a subgroup of $\mathrm{SO}(W^\perp)$, and that $W^\perp$ is equipped with a hyperbolic basis $\{f_1, \cdots, f_{2(r+2t)}\}$. Let

$$(W^\perp)_{A,B} := \mathrm{span}\{f_i, f_{2(r+2t)+1-i} \mid i \in A \text{ or } \lceil \frac{i-r}{2} \rceil \in B\},$$
$$(W^\perp)_{A^c,B^c} := \mathrm{span}\{f_i, f_{2(r+2t)+1-i} \mid i \in A^c \text{ or } \lceil \frac{i-r}{2} \rceil \in B^c\}.$$

The natural action of M^{GL} on $W^\perp$ stabilizes $(W^\perp)_{A,B}$ and $(W^\perp)_{A^c,B^c}$. Let $M^{\mathrm{GL}}_{A,B}$ (resp. $M^{\mathrm{GL}}_{A^c,B^c}$) be the maximal quotient of M^{GL} acting faithfully on $(W^\perp)_{A,B}$ (resp. $(W^\perp)_{A^c,B^c}$). Concretely, if we write $A = \{i_1, \cdots, i_u\}$ and $B = \{j_1, \cdots, j_v\}$ where $i_1 < i_2 < \cdots < i_u$ and $j_1 < j_2 < \cdots < j_v$, then $M^{\mathrm{GL}}_{A,B}$ is identified with $\mathbb{G}_m^u \times \mathrm{GL}_2^v$, and the quotient map

$$M^{\mathrm{GL}} \cong \mathbb{G}_m^r \times \mathrm{GL}_2^t \longrightarrow M^{\mathrm{GL}}_{A,B} \cong \mathbb{G}_m^u \times \mathrm{GL}_2^v$$

is given by

$$(z_1, \cdots, z_r, w_1, \cdots, w_t) \longmapsto (z_{i_1}, \cdots, z_{i_u}, w_{j_1}, \cdots, w_{j_v}).$$

Similarly we have a concrete description of the quotient map $M^{\mathrm{GL}} \to M^{\mathrm{GL}}_{A^c,B^c}$.

We now specify the $H(F)$-conjugacy class of embeddings $M' \hookrightarrow H$. First consider the odd case. Choose an isometric isomorphism $\mathbf{f}^+$ from the orthogonal direct sum of $(W^\perp)_{A,B}$ and W^+ to V^+. (Such $\mathbf{f}^+$ indeed exists since both quadratic spaces are split and have dimension $d^+ + 2|A| + 4|B|$.) This choice, together with the natural action of $M^{\mathrm{GL}}_{A,B}$ on $(W^\perp)_{A,B}$ and the natural action of $\mathrm{SO}(W^+)$ on W^+, determines an embedding

$$\mathbf{f}_*^+ : M_{A,B}^{\mathrm{GL}} \times \mathrm{SO}(W^+) \longrightarrow \mathrm{SO}(V^+).$$

We claim that the $\mathrm{SO}(V^+)(F)$-conjugacy class of $\mathbf{f}_*^+$ is independent of the choice of $\mathbf{f}^+$. Indeed, the $\mathrm{O}(V^+)(F)$-conjugacy class of $\mathbf{f}_*^+$ is clearly independent of the choice of $\mathbf{f}^+$. The element of $\mathrm{O}(V^+)(F)$ acting as 1 on $\mathbf{f}^+((W^\perp)_{A,B})$ and as -1 on $\mathbf{f}^+(W^+)$ has determinant -1 and centralizes $\mathbf{f}_*^+$. Hence the $\mathrm{O}(V^+)(F)$-conjugacy class of $\mathbf{f}_*^+$ is in fact equal to the $\mathrm{SO}(V^+)(F)$-conjugacy class of $\mathbf{f}_*^+$. Our claim follows.

Similarly, we choose an isometric isomorphism $\mathbf{f}^-$ from the orthogonal direct sum of $(W^\perp)_{A^c,B^c}$ and W^- to V^-. We then obtain an embedding

$$\mathbf{f}_*^- : M_{A^c,B^c}^{\mathrm{GL}} \times \mathrm{SO}(W^-) \longrightarrow \mathrm{SO}(V^-),$$

whose $\mathrm{SO}(V^-)(F)$-conjugacy class is independent of the choice of $\mathbf{f}^-$. Taking the direct product of $\mathbf{f}_*^+$ and $\mathbf{f}_*^-$, we obtain the desired embedding $M' \to H$ which is canonical up to $H(F)$-conjugacy.

We now consider the even case. Since the orthogonal direct sum of $(W^\perp)_{A,B}$ and W^+ is a quasi-split quadratic space of the same dimension and discriminant as V^+, we can choose an isometric isomorphism $\mathbf{f}^+$ between them just as in the odd case. We then obtain the embedding

$$\mathbf{f}_*^+ : M_{A,B}^{\mathrm{GL}} \times \mathrm{SO}(W^+) \longrightarrow \mathrm{SO}(V^+).$$

At this point, only the $\mathrm{O}(V^+)(F)$-conjugacy of $\mathbf{f}_*^+$ is well defined. We explain how to narrow this down to an $\mathrm{SO}(V^+)(F)$-conjugacy class. As before we canonically identify $M_{A,B}^{\mathrm{GL}}$ with $\mathbb{G}_m^u \times \mathrm{GL}_2^v$ (where $u = |A|$ and $v = |B|$). Consider the canonical embedding $\iota_{A,B}^{\mathrm{GL}} : \mathbb{G}_m^{u+2v} \to M_{A,B}^{\mathrm{GL}}$ given by

$$(z_1, \cdots, z_{u+2v}) \longmapsto (z_1, \cdots, z_u, \begin{pmatrix} z_{u+1} & \\ & z_{u+2} \end{pmatrix}, \cdots, \begin{pmatrix} z_{u+2v-1} & \\ & z_{u+2v} \end{pmatrix}).$$

We divide our discussion into the cases where δ^+ is trivial and non-trivial.

Suppose δ^+ is trivial. As in §5.4.3, W^+ is equipped with an $\mathrm{SO}(W^+)(F)$-orbit $[\mathbb{B}_{W^+}]$ of hyperbolic bases, and V^+ is equipped with an $\mathrm{SO}(V^+)(F)$-orbit $[\mathbb{B}_{V^+}]$ of hyperbolic bases. They determine an $\mathrm{SO}(W^+)(F)$-conjugacy class of embeddings

$$\iota_{W^+} = \iota_{\mathbb{B}_{W^+}} : \mathbb{G}_m^{d^+/2} \longrightarrow \mathrm{SO}(W^+)$$

and an $\mathrm{SO}(V^+)(F)$-conjugacy class of embeddings

$$\iota_{V^+} = \iota_{\mathbb{B}_{V^+}} : \mathbb{G}_m^{d^+/2+u+2v} \longrightarrow \mathrm{SO}(V^+)$$

(cf. §1.2.7). We impose the condition that the embedding

$$\mathbf{f}_*^+ \circ (\iota_{A,B}^{\mathrm{GL}} \times \iota_{W^+}) : \mathbb{G}_m^{u+2v} \times \mathbb{G}_m^{d^+/2} \longrightarrow \mathrm{SO}(V^+)$$

should be $\mathrm{SO}(V^+)(F)$-conjugate to ι_{V^+} under the obvious identification

$$\mathbb{G}_m^{u+2v} \times \mathbb{G}_m^{d^+/2} \xrightarrow{\sim} \mathbb{G}_m^{d^+/2+u+2v}$$
$$((z_1, z_2, \cdots), (w_1, w_2, \cdots)) \longmapsto (z_1, z_2, \cdots, w_1, w_2, \cdots).$$

This extra condition narrows the $\mathrm{O}(V^+)(F)$-conjugacy class of $\mathbf{f}_*^+$ to an $\mathrm{SO}(V^+)(F)$-conjugacy class.

Now suppose δ^+ is non-trivial. In this case, W^+ is equipped with an $\mathrm{SO}(W^+)(F)$-orbit $[\mathbb{B}_{W^+}]$ of near-hyperbolic bases, and we have fixed a square root $\alpha^{+,\prime} \in \overline{F}$ of the common discriminant of members of $[\mathbb{B}_{W^+}]$. Similarly, V^+ is equipped with an $\mathrm{SO}(V^+)(F)$-orbit $[\mathbb{B}_{V^+}]$ of near-hyperbolic bases, and we have fixed a square root $\alpha^+ \in \overline{F}$ of the common discriminant of members of $[\mathbb{B}_{V^+}]$. (Here α^+ may not be equal to $\alpha^{+,\prime}$.) These extra data determine an $\mathrm{SO}(W^+)(F)$-conjugacy class of embeddings

$$\iota_{W^+} = \iota_{\alpha^{+,\prime},\mathbb{B}_{W^+}} : \mathbb{G}_m^{d^+/2-1} \times \mathrm{U}(1)_{\alpha^{+,\prime}} \longrightarrow \mathrm{SO}(W^+),$$

and an $\mathrm{SO}(V^+)(F)$-conjugacy class of embeddings

$$\iota_{V^+} = \iota_{\alpha^+,\mathbb{B}_{V^+}} : \mathbb{G}_m^{d^+/2+u+2v-1} \times \mathrm{U}(1)_{\alpha^+} \longrightarrow \mathrm{SO}(V^+)$$

(cf. §1.2.7). Note that $\mathrm{U}(1)_{\alpha^{+,\prime}}$ is canonically identified with $\mathrm{U}(1)_{\alpha^+}$, since the fields $F(\alpha^{+,\prime})$ and $F(\alpha^+)$ are the same. We impose the condition that

$$\mathbf{f}_*^+ \circ (\iota_{A,B}^{\mathrm{GL}} \times \iota_{W^+}) : \mathbb{G}_m^{u+2v} \times \mathbb{G}_m^{d^+/2-1} \times \mathrm{U}(1)_{\alpha^{+,\prime}} \longrightarrow \mathrm{SO}(V^+)$$

should be $\mathrm{SO}(V^+)(F)$-conjugate to ι_{V^+} under the obvious identification

$$\mathbb{G}_m^{u+2v} \times \mathbb{G}_m^{d^+/2-1} \times \mathrm{U}(1)_{\alpha^{+,\prime}} \xrightarrow{\sim} \mathbb{G}_m^{d^+/2+u+2v-1} \times \mathrm{U}(1)_{\alpha^+}$$
$$((z_1, z_2, \cdots), (w_1, w_2, \cdots), y) \longmapsto (z_1, z_2, \cdots, w_1, w_2, \cdots, y).$$

This extra condition narrows the $\mathrm{O}(V^+)(F)$-conjugacy class of $\mathbf{f}_*^+$ to an $\mathrm{SO}(V^+)(F)$-conjugacy class.

We have specified an $\mathrm{SO}(V^+)(F)$-conjugacy class of embeddings $M_{A,B}^{\mathrm{GL}} \times \mathrm{SO}(W^+) \to \mathrm{SO}(V^+)$. Similarly, we specify an $\mathrm{SO}(V^-)(F)$-conjugacy class of embeddings $M_{A^c,B^c}^{\mathrm{GL}} \times \mathrm{SO}(W^-) \to \mathrm{SO}(V^-)$. Taking the direct product we obtain the desired embedding $M' \to H$ which is canonical up to $H(F)$-conjugacy.

Write $A = \{i_1, \cdots, i_u\}$, $B = \{j_1, \cdots, j_v\}$, $A^c = \{p_1, \cdots, p_{r-u}\}$, and $B^c = \{q_1, \cdots, q_{t-v}\}$ with increasing ordering (i.e., $i_1 < \cdots < i_u$ etc.). In both the odd and even cases, the $\widehat{H}$-conjugacy class of embeddings ${}^LM' \to {}^LH$ arising from our construction is represented by the map

$$
\begin{aligned}
(5.5.9.2)\quad {}^L M' = (\mathbb{C}^\times)^r \times \mathrm{GL}_2(\mathbb{C})^t \times \widehat{\mathrm{SO}(W^+)} \times \widehat{\mathrm{SO}(W^-)} \rtimes \Gamma' \\
\ni (g_1, \cdots, g_r, h_1, \cdots, h_t, k^+, k^-) \rtimes \tau \longmapsto \\
\mathrm{diag}(g_{i_1}, \cdots, g_{i_u}, h_{j_1}, \cdots, h_{j_v}, k^+, h_{j_v}^\dagger, \cdots, h_{j_1}^\dagger, g_{i_u}^{-1}, \cdots, g_{i_1}^{-1}) \\
\times \mathrm{diag}(g_{p_1}, \cdots, g_{p_{r-u}}, h_{q_1}, \cdots, h_{q_{t-v}}, k^-, h_{q_{t-v}}^\dagger, \cdots, h_{q_1}^\dagger, g_{p_{r-u}}^{-1}, \cdots, g_{p_1}^{-1}) \\
\rtimes \tau \\
\in {}^L H = \widehat{H^+} \times \widehat{H^-} \rtimes \Gamma',
\end{aligned}
$$

where the notation $\dagger$ is as in (5.5.3.3). Using the formulas (5.5.3.2) and (5.5.9.2), one sees that the diagram (5.5.9.1) indeed commutes up to $\widehat{G}$-conjugation.

5.6. Admissible isomorphisms and embeddings

5.6.1. — Keep the setting of §5.4. For any torus T over F, we denote by $\widehat{T}$ the dual torus over $\mathbb{C}$, whose group of characters is canonically identified with $X_*(T)$. If $f : T_1 \to T_2$ is a homomorphism of tori over F, we denote by $\hat{f}$ the dual homomorphism $\widehat{T_2} \to \widehat{T_1}$.

For any Borel pair (T, B) in $G_{\overline{F}}$ and any Borel pair $(\underline{T}, \underline{B})$ in $G^*_{\overline{F}}$, the fixed isomorphisms (5.3.1.1) and (5.3.1.2) give rise to isomorphisms $\widehat{T} \xrightarrow{\sim} \mathcal{T}$ and $\widehat{\underline{T}} \xrightarrow{\sim} \mathcal{T}$ of tori over $\mathbb{C}$. We denote these isomorphisms by $\mathfrak{d}_{B,\mathcal{B}}$ and $\mathfrak{d}_{\underline{B},\mathcal{B}}$ respectively.

Now consider an elliptic endoscopic datum $(H, {}^L H, s, \eta)$ for G as in §5.4.3. Given a Borel pair (T_H, B_H) in $H_{\overline{F}}$, there is a similar isomorphism

$$\mathfrak{d}_{B_H, \mathcal{B}_{\widehat{H}}} : \widehat{T_H} \xrightarrow{\sim} \mathcal{T}_{\widehat{H}}.$$

Here $(\mathcal{T}_{\widehat{H}}, \mathcal{B}_{\widehat{H}})$ is the standard Borel pair in $\widehat{H}$ as in §5.4.3. Note that $\eta : {}^L H \to {}^L G$ maps $\mathcal{T}_{\widehat{H}}$ isomorphically onto $\mathcal{T}$. Hence we obtain isomorphisms

$$\mathfrak{d}_{B,\mathcal{B}}^{-1} \circ \eta \circ \mathfrak{d}_{B_H, \mathcal{B}_{\widehat{H}}} : \widehat{T_H} \xrightarrow{\sim} \widehat{T},$$
$$\mathfrak{d}_{\underline{B},\mathcal{B}}^{-1} \circ \eta \circ \mathfrak{d}_{B_H, \mathcal{B}_{\widehat{H}}} : \widehat{T_H} \xrightarrow{\sim} \widehat{\underline{T}},$$

or equivalently, isomorphisms

$$j : T_H \xrightarrow{\sim} T \subset G_{\overline{F}},$$
$$\underline{j} : T_H \xrightarrow{\sim} \underline{T} \subset G^*_{\overline{F}}.$$

We call j an *admissible isomorphism* between T_H and T, and an *admissible embedding* of T_H into $G_{\overline{F}}$; cf. [**LS87**, §1.3]. Similar terminology applies to $\underline{j}$. We shall also say that j is associated to the Borel pairs (T_H, B_H) and (T, B), and say that $\underline{j}$ is associated to the Borel pairs (T_H, B_H) and $(\underline{T}, \underline{B})$.

The following facts are well known and straightforward to verify.

Lemma 5.6.2. — *Fix maximal tori* $T_H \subset H_{\overline{F}}$, $T \subset G_{\overline{F}}$, *and* $\underline{T} \subset G^*_{\overline{F}}$.

(1) *The set of admissible isomorphisms between* T_H *and* T *(resp. between* T_H *and* $\underline{T}$*) is a torsor under the Weyl group of* $(G_{\overline{F}}, T)$ *(resp. the Weyl group of* $(G^*_{\overline{F}}, \underline{T})$*).*

(2) *The set of admissible embeddings of* T_H *into* $G_{\overline{F}}$ *(resp. into* $G^*_{\overline{F}}$*) is a single orbit under* $G(\overline{F})$*-conjugation (resp.* $G^*(\overline{F})$*-conjugation).*

(3) *Let* $j : T_H \to G_{\overline{F}}$ *and* $\underline{j} : T_H \to G^*_{\overline{F}}$ *be arbitrary embeddings such that*

$$j = \mathrm{Int}(g) \circ \psi_V^{-1} \circ \underline{j}$$

for some $g \in G(\overline{F})$*. (Here* ψ_V *is the fixed inner twisting between* G *and* G^**; see §5.1.) Then* $\underline{j}$ *is admissible if and only if* j *is admissible.*

□

CHAPTER 6

TRANSFER FACTORS FOR REAL SPECIAL ORTHOGONAL GROUPS

6.1. Cuspidality and transfer of elliptic tori

6.1.1. — We keep the notation in §5, specialized to $F = \mathbb{R}$. Thus (V, q) is a quadratic space over $\mathbb{R}$ of dimension d and discriminant δ, and $G = \mathrm{SO}(V, q)$ is a reductive group over $\mathbb{R}$. We are interested in the case where G contains anisotropic maximal tori. When d is odd, this is always the case. When d is even, this is the case if and only if $\delta = (-1)^{d/2} \in \mathbb{R}^\times/\mathbb{R}^{\times,2}$. (Note that if $(d, \delta) = (2, 1)$, then $G \cong \mathbb{G}_m$ contains elliptic maximal tori but not anisotropic maximal tori.) In the following we assume that G contains anisotropic maximal tori. We discuss a systematic way of parameterizing anisotropic maximal tori in G. As usual, we let $m := \lfloor d/2 \rfloor$. Our assumption clearly implies that V admits an *elliptic decomposition* defined as follows.

Definition 6.1.2. — By an *elliptic decomposition* of V, we mean an ordered tuple $(V_j, o_j)_{1 \leq j \leq m}$, where $V_1, \cdots, V_m$ are mutually orthogonal definite planes in V, and o_j is an orientation on V_j. Thus the orthogonal direct sum of $V_1, \cdots, V_m$ is a hyperplane in V (resp. equal to V) when d is odd (resp. even). We denote by $\mathrm{ED}(V)$ the set of all elliptic decompositions of V. By abuse of notation, we often write $(V_j)_j$ for an element of $\mathrm{ED}(V)$, understanding that each V_j is equipped with an orientation.

Definition 6.1.3. — By a *parameterized anisotropic maximal torus* in G, we mean an anisotropic maximal torus T_G in G together with an isomorphism $\mathrm{U}(1)^m \xrightarrow{\sim} T_G$.

Definition 6.1.4. — By a *fundamental pair* in G, we mean a pair (T_G, B), where T_G is an anisotropic maximal torus in G, and B is a Borel subgroup of $G_{\mathbb{C}}$ containing $T_{G,\mathbb{C}}$.

Remark 6.1.5. — Since any two anisotropic maximal tori in G are conjugate under $G(\mathbb{R})$, the number of $G(\mathbb{R})$-orbits of fundamental pairs in G is equal to the cardinality of $\Omega_{\mathbb{C}}(G, T_G)/\Omega_{\mathbb{R}}(G, T_G)$, where T_G is an arbitrary anisotropic maximal torus in G.

6.1.6. — Given $\mathcal{D} = (V_j)_j \in \mathrm{ED}(V)$, we obtain a parameterized anisotropic maximal torus from the embedding

$$f_{\mathcal{D}} : \mathrm{U}(1)^m \xrightarrow{\sim} T_{\mathcal{D}} \subset G,$$

where the j-th copy of U(1) acts by rotation on the oriented definite plane V_j. The (absolute) root datum of G on

$$(X^*(T_{\mathcal{D}}), X_*(T_{\mathcal{D}})) \xrightarrow[f_{\mathcal{D}}]{\sim} (X^*(\mathrm{U}(1)^m), X_*(\mathrm{U}(1)^m)) = (\mathbb{Z}^m, \mathbb{Z}^m)$$

is the standard root datum $\mathrm{RD}(\mathsf{B}_m)$ or $\mathrm{RD}(\mathsf{D}_m)$ when d is odd or even. Hence the standard based root datum $\mathrm{BRD}(\mathsf{B}_m)$ or $\mathrm{BRD}(\mathsf{D}_m)$ gives rise to a Borel subgroup $B_{\mathcal{D}}$ of $G_{\mathbb{C}}$ containing $T_{\mathcal{D},\mathbb{C}}$. Thus we obtain a fundamental pair $(T_{\mathcal{D}}, B_{\mathcal{D}})$ from $\mathcal{D} \in \mathrm{ED}(V)$.

6.1.7. — Recall from §5.1 that we have fixed a quasi-split quadratic space $(\underline{V}, \underline{q})$ and fixed an isomorphism $\phi_V : V \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} \underline{V} \otimes_{\mathbb{R}} \mathbb{C}$ of quadratic spaces over $\mathbb{C}$. By definition we have $G^* = \mathrm{SO}(\underline{V})$. We have the obvious analogues of Definitions 6.1.2, 6.1.3, 6.1.4, and the constructions in §6.1.6, with V and G replaced by $\underline{V}$ and G^*. Note that our assumption that G contains anisotropic maximal tori implies that G^* also contains anisotropic maximal tori, since these conditions both boil down to the numerical condition that either d is odd or d is even and $\delta = (-1)^{d/2}$. In particular $\mathrm{ED}(\underline{V}) \neq \emptyset$.

Recall from Definition 5.1.5 that when d is even and when δ is trivial (resp. non-trivial), we have fixed a $G^*(\mathbb{R})$-orbit $[\mathbb{B}_{\underline{V}}]$ of hyperbolic bases (resp. near-hyperbolic bases) of $\underline{V}$. Note that all members of $[\mathbb{B}_{\underline{V}}]$ induce the same orientation on $\underline{V}$. We denote this orientation by $o_{\underline{V}}$. Still under the assumption that d is even, we define an orientation o_V on V as follows. Let (a, b) be the signature of V. Since $\delta = (-1)^{d/2}$, both a and b are even. Also $\underline{V}$ has signature

$$(a^*, b^*) = (2\lceil d/4 \rceil, 2\lfloor d/4 \rfloor).$$

We define o_V to be $(-1)^{(b-b^*)/2}$ times the pull-back of $o_{\underline{V}}$ along the $\mathbb{R}$-linear isomorphism

$$\wedge^d \phi_V : \wedge^d V \xrightarrow{\sim} \wedge^d \underline{V}.$$

Here $\wedge^d \phi_V$ is indeed defined over $\mathbb{R}$ because V and $\underline{V}$ have the same discriminant.

When d is even, every elliptic decomposition of $\underline{V}$ (resp. V) gives rise to an orientation on $\underline{V}$ (resp. V). We define $\mathrm{ED}(\underline{V}, o_{\underline{V}})$ to be the set of elliptic decompositions of $\underline{V}$ that induce the orientation $o_{\underline{V}}$, and similarly define $\mathrm{ED}(V, o_V)$. In order to have uniform notation in the odd and even cases, we set

$$\mathrm{ED}(\underline{V})^o := \begin{cases} \mathrm{ED}(\underline{V}), & \text{if } d \text{ is odd}, \\ \mathrm{ED}(\underline{V}, o_{\underline{V}}), & \text{if } d \text{ is even}, \end{cases}$$

and

$$\mathrm{ED}(V)^o := \begin{cases} \mathrm{ED}(V), & \text{if } d \text{ is odd,} \\ \mathrm{ED}(V, o_V), & \text{if } d \text{ is even.} \end{cases}$$

Lemma 6.1.8. — *Assume that d is even. Let $\{v_1, \cdots, v_d\}$ be an orthogonal basis of V and $\{\underline{v}_1, \cdots, \underline{v}_d\}$ an orthogonal basis of $\underline{V}$ satisfying the condition in Definition 5.1.1. Then $\{v_1, \cdots, v_d\}$ induces the orientation o_V if and only if $\{\underline{v}_1, \cdots, \underline{v}_d\}$ induces the orientation $o_{\underline{V}}$.*

Proof. — Comparing the signatures we see that the cardinality of the set

$$\{j \mid 1 \leq j \leq d, \phi(v_j) = \underline{v}_j \otimes \sqrt{-1}\}$$

is congruent to $b - b^*$ (mod 2). Hence the determinant of the matrix of ϕ_V under the given bases is equal to $(-1)^{(b-b^*)/2}$. The lemma follows. □

6.1.9. — Consider an elliptic endoscopic datum $(H, {}^L H, s, \eta)$ for G. We assume that it is one of the explicit representatives constructed in §5.4. Recall that $H = \mathrm{SO}(V^+) \times \mathrm{SO}(V^-)$, where $V^\pm$ are quasi-split quadratic spaces over $\mathbb{R}$. In the even case, we denote by $o_{V^\pm}$ the orientation on $V^\pm$ determined by $[\mathbb{B}_{V^\pm}]$. (See §5.4.3 for $[\mathbb{B}_{V^\pm}]$.) We assume that H contains anisotropic maximal tori, or equivalently, that both $\mathrm{SO}(V^+)$ and $\mathrm{SO}(V^-)$ contain anisotropic maximal tori. In particular, $\mathrm{ED}(V^\pm)$ are non-empty. Similarly as in §6.1.7, we set

$$\mathrm{ED}(V^\pm)^o := \begin{cases} \mathrm{ED}(V^\pm), & \text{if } d^\pm \text{ is odd,} \\ \mathrm{ED}(V^\pm, o_{V^\pm}), & \text{if } d^\pm \text{ is even.} \end{cases}$$

Let $m^\pm := \lfloor d^\pm/2 \rfloor$. We fix an element

$$\mathcal{D}_H = (\mathcal{D}_H^+, \mathcal{D}_H^-) \in \mathrm{ED}(V^+)^o \times \mathrm{ED}(V^-)^o.$$

Then we get a parameterized anisotropic maximal torus

$$f_{\mathcal{D}_H} : \mathrm{U}(1)^{m^+} \times \mathrm{U}(1)^{m^-} \xrightarrow{\sim} T_{\mathcal{D}_H^+} \times T_{\mathcal{D}_H^-} = T_{\mathcal{D}_H} \subset \mathrm{SO}(V^+) \times \mathrm{SO}(V^-) = H,$$

and a fundamental pair $(T_{\mathcal{D}_H}, B_{\mathcal{D}_H})$ in H, by the obvious generalization of Definitions 6.1.3 and 6.1.4. We also fix $\underline{\mathcal{D}} \in \mathrm{ED}(\underline{V})^o$ and $\mathcal{D} \in \mathrm{ED}(V)^o$. Let

$$f_{\underline{\mathcal{D}}} : \mathrm{U}(1)^m \xrightarrow{\sim} T_{\underline{\mathcal{D}}} \subset G^*,$$
$$f_{\mathcal{D}} : \mathrm{U}(1)^m \xrightarrow{\sim} T_{\mathcal{D}} \subset G$$

be the associated parameterized anisotropic maximal tori, and let $(T_{\underline{\mathcal{D}}}, B_{\underline{\mathcal{D}}}), (T_{\mathcal{D}}, B_{\mathcal{D}})$ be the associated fundamental pairs in G^* and in G. We define the following composite maps with Convention 6.1.10 below in force:

$$j_{\mathcal{D}_H,\mathcal{D}} : T_{\mathcal{D}_H} \xrightarrow{f_{\mathcal{D}_H}^{-1}} \mathrm{U}(1)^{m^+} \times \mathrm{U}(1)^{m^-} \cong \mathrm{U}(1)^m \xrightarrow{f_{\mathcal{D}}} T_{\mathcal{D}},$$

$$j_{\mathcal{D}_H,\underline{\mathcal{D}}} : T_{\mathcal{D}_H} \xrightarrow{f_{\mathcal{D}_H}^{-1}} \mathrm{U}(1)^{m^+} \times \mathrm{U}(1)^{m^-} \cong \mathrm{U}(1)^m \xrightarrow{f_{\underline{\mathcal{D}}}} T_{\underline{\mathcal{D}}}.$$

Convention 6.1.10. — *We identify* $\mathrm{U}(1)^{m^+} \times \mathrm{U}(1)^{m^-}$ *with* $\mathrm{U}(1)^m$ *by the isomorphism*

$$((g_1, \cdots, g_{m^+}), (h_1, \cdots, h_{m^-})) \longmapsto (h_1, \cdots, h_{m^-}, g_1, \cdots, g_{m^+}).$$

Our next goal is to show that $j_{\mathcal{D}_H,\mathcal{D}}$ and $j_{\mathcal{D}_H,\underline{\mathcal{D}}}$ are admissible isomorphisms, in the sense of §5.6.

Lemma 6.1.11. — *In the setting of §6.1.9, the following diagram commutes:*

$$\begin{array}{ccc} \widehat{\mathrm{U}(1)}^m & \xleftarrow{\widehat{f_{\underline{\mathcal{D}}}}} & \widehat{T_{\underline{\mathcal{D}}}} \\ \Big\| & & \Big\downarrow {\scriptstyle \mathfrak{d}_{B_{\underline{\mathcal{D}}},\mathcal{B}}} \\ (\mathbb{C}^\times)^m & \longrightarrow & \mathcal{T} \end{array}$$

Here the bottom horizontal map is the isomorphism fixed in Definition 5.2.2, and $\mathfrak{d}_{B_{\underline{\mathcal{D}}},\mathcal{B}}$ *is as in §5.6.*

Proof. — In the odd case, $\underline{V}$ is split, so we can fix a hyperbolic basis $\mathbb{B}$ of $\underline{V}$. In the even case, we fix a member $\mathbb{B}$ of the $G^*(\mathbb{R})$-orbit $[\mathbb{B}_{\underline{V}}]$ of bases of $\underline{V}$ in Definition 5.1.5. When $\underline{V}$ is split (i.e., when either d is odd or d is even and δ is trivial), $\mathbb{B}$ is a hyperbolic basis, and we let $\iota_{\mathbb{B}} : \mathbb{G}_m^m \hookrightarrow G^*$ be the associated embedding as in §1.2.7. When $\underline{V}$ is not split (i.e., when d is even and δ is non-trivial), $\mathbb{B}$ is a near-hyperbolic basis, and we let $\iota_{\mathbb{B}} : \mathbb{G}_m^{m-1} \times \mathrm{U}(1) \hookrightarrow G^*$ be the associated embedding as in §1.2.7. In all cases we write $T'_{\mathbb{B}}$ for the image of $\iota_{\mathbb{B}}$. We view the base change of $\iota_{\mathbb{B}}$ to $\mathbb{C}$ as an isomorphism $\iota_{\mathbb{B},\mathbb{C}} : \mathbb{G}_{m,\mathbb{C}}^m \xrightarrow{\sim} T'_{\mathbb{B},\mathbb{C}}$ (as we canonically identify $\mathrm{U}(1)_{\mathbb{C}}$ with $\mathbb{G}_{m,\mathbb{C}}$). Now we claim that there exists $g \in G^*(\mathbb{C})$ such that the diagram

(6.1.11.1)
$$\begin{array}{ccc} \mathrm{U}(1)_{\mathbb{C}}^m & \xrightarrow{f_{\underline{\mathcal{D}},\mathbb{C}}} & T_{\underline{\mathcal{D}},\mathbb{C}} \\ \Big\downarrow & & \Big\uparrow {\scriptstyle \mathrm{Int}(g)} \\ \mathbb{G}_{m,\mathbb{C}}^m & \xrightarrow{\iota_{\mathbb{B},\mathbb{C}}} & T'_{\mathbb{B},\mathbb{C}} \end{array}$$

commutes. Here the left vertical arrow is the canonical isomorphism.

To prove the claim, first we observe that the truth of the claim does not depend on the choices of $\mathbb{B}$ and $\underline{\mathcal{D}}$ (as long as they both induce the correct orientation $o_{\underline{V}}$ in the even case). Using this observation, we easily reduce the claim for both the odd and even cases to the even case where $\underline{V}$ has signature $(2, 2)$. In this case, take a basis $\{u_1, u_2, u_3, u_4\}$ of $\underline{V}$ under which the quadratic form has matrix $\mathrm{diag}(1, 1, -1, -1)$.

Without loss of generality we assume that this basis induces the orientation $o_{\underline{V}}$. Let V_1 be the oriented plane spanned by $\{u_1, u_2\}$, and let V_2 be the oriented plane spanned by $\{u_3, u_4\}$. Then $(V_1, V_2) \in \mathrm{ED}(\underline{V}, o_{\underline{V}})$. Define

$$x_1 = \frac{1}{2}(u_1 + u_3), \qquad y_1 = u_1 - u_3, \qquad x_2 = \frac{1}{2}(u_2 - u_4), \qquad y_2 = u_2 + u_4.$$

Then $\{x_1, x_2, y_2, y_1\}$ is a hyperbolic basis of $\underline{V}$, and it induces the orientation $o_{\underline{V}}$. We may and shall assume that $\underline{\mathcal{D}} = (V_1, V_2)$ and $\mathbb{B} = \{x_1, x_2, y_2, y_1\}$. Let $g \in \mathrm{End}(\underline{V} \otimes \mathbb{C})$ be given by

$$x_1 \longmapsto \frac{1}{2}(u_1 - iu_2), \quad y_1 \longmapsto u_1 + iu_2, \quad x_2 \longmapsto -\frac{1}{2}(u_3 - iu_4), \quad y_2 \longmapsto u_3 + iu_4.$$

Then $g \in \mathrm{O}(\underline{V})(\mathbb{C})$, and the diagram (6.1.11.1) commutes. We have $\det g = 1$ by direct computation, which proves the claim.

Now by the definition of $B_{\underline{\mathcal{D}}}$, we know that $f_{\underline{\mathcal{D}}}$ pulls back the based root datum $\mathrm{BRD}(T_{\underline{\mathcal{D}},\mathbb{C}}, B_{\underline{\mathcal{D}}})$ on $(X^*(T_{\underline{\mathcal{D}}}), X_*(T_{\underline{\mathcal{D}}}))$ to the standard based root datum $\mathrm{BRD}(\mathsf{B}_m)$ or $\mathrm{BRD}(\mathsf{D}_m)$ on $(\mathbb{Z}^m, \mathbb{Z}^m)$. By the commutative diagram (6.1.11.1), we know that the isomorphism $\mathrm{BRD}(T_{\underline{\mathcal{D}},\mathbb{C}}, B_{\underline{\mathcal{D}}}) \xrightarrow{\sim} \mathrm{BRD}(\mathsf{B}_m \text{ or } \mathsf{D}_m)$ induced by $f_{\underline{\mathcal{D}}}$ is equal to the isomorphism $\mathfrak{v}^{*\prime}$ fixed in §5.3. The lemma then follows from the definition of $\mathfrak{d}_{B_{\underline{\mathcal{D}}},\mathcal{B}}$. □

Lemma 6.1.12. — *In the setting of §6.1.9, $j_{\mathcal{D}_H,\underline{\mathcal{D}}}$ is admissible, and it is associated to the Borel pairs $(T_{\mathcal{D}_H,\mathbb{C}}, B_{\mathcal{D}_H})$ and $(T_{\underline{\mathcal{D}},\mathbb{C}}, B_{\underline{\mathcal{D}}})$.*

Proof. — The map $\eta : {}^L H \to {}^L G$ restricts to an isomorphism $\mathcal{T}_{\widehat{H}} = \mathcal{T}_{V^+} \times \mathcal{T}_{V^-} \xrightarrow{\sim} \mathcal{T}$. This isomorphism is given by

$$\begin{aligned} (\mathbb{C}^\times)^{m^+} \times (\mathbb{C}^\times)^{m^-} &\longrightarrow (\mathbb{C}^\times)^m \\ ((g_1, \cdots, g_{m^+}), (h_1, \cdots, h_{m^-})) &\longmapsto (h_1, \cdots, h_{m^-}, g_1, \cdots, g_{m^+}), \end{aligned}$$

under the identifications $\mathcal{T}_{V^+} \cong (\mathbb{C}^\times)^{m^+}, \mathcal{T}_{V^-} \cong (\mathbb{C}^\times)^{m^-}, \mathcal{T} \cong (\mathbb{C}^\times)^m$ as in Definition 5.2.2. This fact, together with Lemma 6.1.11 (applied to $\underline{V}$ and $V^{\pm}$), implies the current lemma. □

Lemma 6.1.13. — *In the setting of §6.1.9, $j_{\mathcal{D}_H,\mathcal{D}}$ is admissible.*

Proof. — Since $\mathrm{ED}(V)^o$ is a single $G(\mathbb{R})$-orbit, the truth of the lemma does not depend on the choice of $\mathcal{D} \in \mathrm{ED}(V)^o$. Let $\{v_1, \cdots, v_d\}$ be a basis of V and $\{\underline{v}_1, \cdots, \underline{v}_d\}$ a basis of $\underline{V}$, satisfying the condition in Definition 5.1.1. Let $m = \lfloor d/2 \rfloor$. Up to reordering, we may assume that $q(v_j) = q(v_{j+1})$ for all $j \in \{1, 3, \cdots, 2m-1\}$. When d is even, we may further assume that $\{v_1, \cdots, v_d\}$ induces the orientation o_V (because we may switch the order of v_1 and v_2 without changing the other conditions). For each $1 \leq j \leq m$, let V_j be the oriented plane spanned by $\{v_{2j-1}, v_{2j}\}$, and let $\underline{V}_j$ be the oriented plane spanned by $\{\underline{v}_{2j-1}, \underline{v}_{2j}\}$. Then $(V_j)_j \in \mathrm{ED}(V)^o$ and $(\underline{V}_j)_j \in \mathrm{ED}(\underline{V})$. By Lemma 6.1.8, we have $(\underline{V}_j)_j \in \mathrm{ED}(\underline{V})^o$. In §6.1.9, we can take $\mathcal{D}$ to be $(V_j)_j$, and take $\underline{\mathcal{D}}$ to be $(\underline{V}_j)_j$. By Lemma 6.1.12, we know that $j_{\mathcal{D}_H,\underline{\mathcal{D}}}$ is admissible. In view of Lemma 5.6.2 (3), we complete the proof by noting that $j_{\mathcal{D}_H,\mathcal{D}} = \psi_V^{-1} \circ j_{\mathcal{D}_H,\underline{\mathcal{D}}}$. □

6.2. Transfer factors, when d is not divisible by 4

6.2.1. — We keep the setting of §6.1, and in particular keep the assumption that G and G^* contain anisotropic maximal tori. By an *equivalence class of Whittaker data for* G^*, we mean a $G^*(\mathbb{R})$-conjugacy class of pairs (B, λ) consisting of a Borel subgroup B of G^* defined over $\mathbb{R}$ and a generic character $\lambda : N_B(\mathbb{R}) \to \mathbb{C}^\times$, where N_B denotes the unipotent radical of B. See [**KS99**, §5.3] for more details. It is a standard result that the set of equivalence classes of Whittaker data for G^* is a torsor under the finite abelian group $G^{*,\mathrm{ad}}(\mathbb{R})/G^*(\mathbb{R})$.

Assume that d is not divisible by 4. Then the map $G^*(\mathbb{R}) \to G^{*,\mathrm{ad}}(\mathbb{R})$ is surjective, which can be seen by noting that $\ker(\mathbf{H}^1(\mathbb{R}, Z_{G^*}) \to \mathbf{H}^1(\mathbb{R}, G^*))$ is trivial. Hence G^* has a unique equivalence class of Whittaker data. As in §6.1.9, we fix an elliptic endoscopic datum $(H, {}^LH, s, \eta)$ for G, assumed to be one of the explicit representatives constructed in §5.4. Thus we have $H = \mathrm{SO}(V^+) \times \mathrm{SO}(V^-)$, where $V^\pm$ is quasi-split and has dimension $d^\pm$ and discriminant $\delta^\pm$. As usual, both $V^\pm$ are split in the odd case. We write $m = \lfloor d/2 \rfloor, m^\pm = \lfloor d^\pm/2 \rfloor$. We assume that H contains anisotropic tori.

In this paper, unless otherwise stated, "transfer factor" always means "absolute geometric transfer factor".

The *Whittaker normalization* of the transfer factors between H and G^* was defined by Kottwitz–Shelstad in [**KS99**, §5.3] (in the general setting of twisted endoscopy), and a correction was later made in [**KS12**]. In this paper, we always use the classical normalization of local class field theory as opposed to Deligne's normalization; see [**KS12**, §§4.1, 4.2]. Thus among the four $\Delta, \Delta', \Delta_D, \Delta'_D$ discussed at the end of [**KS12**, §5.1], we only consider Δ and Δ'. Moreover, since we always have $s^2 = 1$, we have $\Delta = \Delta'$. We shall call the transfer factors $\Delta'_\lambda(\cdot,\cdot)$ given in [**KS12**, (5.5.2)] the *Whittaker-normalized transfer factors*. By the discussion at the end of [**KS12**, §5.5] and by $s^2 = 1$, we have

$$\Delta'_\lambda = \epsilon_L(V, \psi)\Delta'_0 = \epsilon_L(V, \psi)\Delta_0,$$

where Δ_0 is the Langlands–Shelstad normalization defined on p. 248 of [**LS87**].

In the following we denote the Whittaker-normalized transfer factors between H and G^* by $\underline{\Delta}_{\mathrm{Wh}}(\cdot,\cdot)$. Also, having fixed $\psi_V : G \to G^*$ and $u_V : \Gamma_\infty \to G^*(\mathbb{C})$ as in §5.1, we can derive from $\underline{\Delta}_{\mathrm{Wh}}(\cdot,\cdot)$ a normalization of the transfer factors between H and G as in Remark 5.1.4, to be denoted by $\Delta_{\mathrm{Wh}}(\cdot,\cdot)$. (See §6.2.7 below for more details.)

In [**Kot90**, §7], another normalization $\Delta_{j,B}(\cdot,\cdot)$ of the transfer factors between H and G is considered, which is associated to a certain datum (j,B). The goal of this section is to compare the two normalizations Δ_{Wh} and $\Delta_{j,B}$.

In the following we assume that V is of signature (p,q) with $p>q$, and that $d=p+q$ is not divisible by 4.

Transfer factors between H and G^*

Definition 6.2.2. — We define a subset $\mathrm{ED}(\underline{V})^o_{\mathrm{Wh}}$ of $\mathrm{ED}(\underline{V})^o$ (see §6.1.7) as follows. When d is odd, let $\mathrm{ED}(\underline{V})^o_{\mathrm{Wh}}$ consist of those $(\underline{V}_j)_{1\le j\le m}\in\mathrm{ED}(\underline{V})^o=\mathrm{ED}(\underline{V})$ such that $\underline{V}_j$ is $(-1)^{j+1}\operatorname{sgn}(\delta)$-definite for each $1\le j\le m$. When d is even (but not divisible by 4), let $\mathrm{ED}(\underline{V})^o_{\mathrm{Wh}}$ consist of those $(\underline{V}_j)_{1\le j\le m}\in\mathrm{ED}(\underline{V})^o$ such that $\underline{V}_j$ is $(-1)^{j+1}$-definite for each $1\le j\le m$.

Remark 6.2.3. — Let $(\underline{V}_j)_j$ be an arbitrary element of $\mathrm{ED}(\underline{V})^o$. Recall that $\underline{V}$ has discriminant δ and determinant $(-1)^m\delta$. In the odd case, $\underline{V}$ has signature $(m+1,m)$ when $\delta>0$, and signature $(m,m+1)$ when $\delta<0$. Therefore there are precisely $\lceil m/2\rceil$ (resp. $\lfloor m/2\rfloor$) positive definite planes among the $\underline{V}_j$'s when $\delta>0$ (resp. when $\delta<0$). It follows that there exists $\sigma\in\mathfrak{S}_m$ such that $(\underline{V}_{\sigma(j)})_j\in\mathrm{ED}(\underline{V})^o_{\mathrm{Wh}}$. In the even case, there are precisely $\lceil m/2\rceil$ positive definite planes among the $\underline{V}_j$'s no matter what δ is, so again there exists $\sigma\in\mathfrak{S}_m$ such that $(\underline{V}_{\sigma(j)})_j\in\mathrm{ED}(\underline{V})^o_{\mathrm{Wh}}$. (Here $(\underline{V}_{\sigma(j)})_j$ automatically induces the same orientation on $\underline{V}$ as $(\underline{V}_j)_j$ does.) Moreover, in both cases $\mathrm{ED}(\underline{V})^o_{\mathrm{Wh}}$ is a single $G^*(\mathbb{R})$-orbit with respect to the natural $G^*(\mathbb{R})$-action on $\mathrm{ED}(\underline{V})^o$.

Lemma 6.2.4. — *Let $\underline{\mathcal{D}}\in\mathrm{ED}(\underline{V})^o_{\mathrm{Wh}}$. Let $(T_{\underline{\mathcal{D}}},B_{\underline{\mathcal{D}}})$ be the associated fundamental pair in G^*, as in §6.1.6. Then every $B_{\underline{\mathcal{D}}}$-simple root in $X^*(T_{\underline{\mathcal{D}}})$ is (imaginary) non-compact. In other words, $(T_{\underline{\mathcal{D}}},B_{\underline{\mathcal{D}}})$ is a* fundamental pair of Whittaker type *in the terminology of* [**She15**].

Proof. — Let $\{\epsilon_1^\vee,\cdots,\epsilon_m^\vee\}$ be the standard basis of $X_*(\mathrm{U}(1)^m)$, and let $\{\epsilon_1,\cdots,\epsilon_m\}$ be the standard basis of $X^*(\mathrm{U}(1)^m)$. Let $f_{\underline{\mathcal{D}}}:\mathrm{U}(1)^m\xrightarrow{\sim}T_{\underline{\mathcal{D}}}$ be the parameterized anisotropic maximal torus associated to $\underline{\mathcal{D}}$, as in §6.1.6. We identify $X^*(T_{\underline{\mathcal{D}}})$ with $X^*(\mathrm{U}(1)^m)$ via $f_{\underline{\mathcal{D}}}$. Then the $B_{\underline{\mathcal{D}}}$-simple roots are $\alpha_j=\epsilon_j-\epsilon_{j+1},1\le j\le m-1$, and $\alpha_m=\epsilon_m$ (resp. $\alpha_m=\epsilon_{m-1}+\epsilon_m$) in the odd (resp. even) case. Denote the complex conjugation by τ. It suffices to check that for each $1\le j\le m$ and for one (and hence any) root vector E_j of α_j, we have

$$[E_j,\tau E_j]=C(E_j)H_j\in\operatorname{Lie}G^*$$

for some $C(E_j)\in\mathbb{R}_{>0}$. Here H_j is the coroot $\alpha_j^\vee$ viewed as an element of $\operatorname{Lie}G^*$.

Write $\underline{\mathcal{D}}=(\underline{V}_j)_j$. Since $\underline{\mathcal{D}}\in\mathrm{ED}(\underline{V})^o_{\mathrm{Wh}}$, there exists an integer r such that $\underline{V}_j$ is $(-1)^{r+j}$-definite for each $1\le j\le m$. Moreover, we have $(-1)^r=-\operatorname{sgn}(\delta)$ when

d is odd, and $(-1)^r = -1$ when d is even. For each $1 \leq j \leq m$, let $\{f_j, f'_j\}$ be an orthogonal basis of $\underline{V}_j$ inducing the given orientation on $\underline{V}_j$ such that

$$\underline{q}(f_j) = \underline{q}(f'_j) = (-1)^{r+j}.$$

Let

$$e_j := f_j \otimes 1 - f'_j \otimes i \in \underline{V} \otimes \mathbb{C},$$
$$e'_j := (-1)^{r+j} \frac{1}{2} \tau(e_j) \in \underline{V} \otimes \mathbb{C}.$$

In the odd case we also fix a non-zero vector $l \in \underline{V}$ which is orthogonal to each $\underline{V}_j$, and satisfies $\underline{q}(l) \in \{\pm 1\}$. Thus $\underline{q}(l)$ is the sign of the determinant of the quadratic space $\underline{V}$, which is $(-1)^m \operatorname{sgn}(\delta) = (-1)^{r+m+1}$.

Now $\{e_1, \cdots, e_m, e'_1, \cdots, e'_m, l\}$ (resp. $\{e_1, \cdots, e_m, e'_1, \cdots, e'_m\}$) is a $\mathbb{C}$-basis of $\underline{V} \otimes \mathbb{C}$ in the odd (resp. even) case, and we have

$$[e_j, e_k] = [e'_j, e'_k] = [e_j, l] = [e'_j, l] = 0, \qquad [e_j, e'_k] = \delta_{j,k}.$$

Note that for each $1 \leq j \leq m$, the cocharacter $f_{\underline{\mathcal{D}}} \circ \epsilon_j^\vee$ of G^* acts on $\underline{V}$ with weight 1 on e_j, weight -1 on e'_j, and weight 0 on e_k, e'_k for all $k \neq j$. In the odd case, it also acts with weight 0 on l.

For $1 \leq j \leq m-1$, we define $E_j \in \operatorname{End}(\underline{V} \otimes \mathbb{C})$ by

$$\begin{aligned} &e_j \longmapsto 0, && e_{j+1} \longmapsto e_j, && e'_j \longmapsto -e'_{j+1}, \\ &e'_{j+1} \longmapsto 0, && e_k, e'_k \longmapsto 0 \text{ for } k \notin \{j, j+1\}, \\ &l \longmapsto 0 \text{ (if } d \text{ is odd).} \end{aligned}$$

It is easy to see that $E_j \in \operatorname{Lie} G^*_{\mathbb{C}}$ and that it is indeed a root vector of α_j. We compute that τE_j is given by

$$\begin{aligned} &e_j \longmapsto e_{j+1}, && e_{j+1} \longmapsto 0, && e'_j \longmapsto 0, \\ &e'_{j+1} \longmapsto -e'_j, && e_k, e'_k \longmapsto 0 \text{ for } k \notin \{j, j+1\}, \\ &l \longmapsto 0 \text{ (if } d \text{ is odd).} \end{aligned}$$

Then $[E_j, \tau E_j]$ is given by

$$\begin{aligned} &e_j \longmapsto e_j, && e_{j+1} \longmapsto -e_{j+1}, && e'_j \longmapsto -e'_j, \\ &e'_{j+1} \longmapsto e'_{j+1} && e_k, e'_k, \longmapsto 0 \text{ for } k \notin \{j, j+1\}, \\ &l \longmapsto 0 \text{ (if } d \text{ is odd).} \end{aligned}$$

Thus $[E_j, \tau E_j] = H_j$, as desired.

In the odd case, we define $E_m \in \operatorname{End}(\underline{V} \otimes \mathbb{C})$ by

$$\begin{aligned} &l \longmapsto e_m, && e'_m \longmapsto -\underline{q}(l)^{-1} l = (-1)^{r+m} l, \\ &e_k \longmapsto 0 \text{ for } 1 \leq k \leq m, && e'_k \longmapsto 0 \text{ for } 1 \leq k \leq m-1. \end{aligned}$$

Then $E_m \in \operatorname{Lie} G^*_{\mathbb{C}}$ and it is a root vector of α_m. We compute that τE_m is given by

$$l \longmapsto (-1)^{m+r} 2e'_m, \qquad e_m \longmapsto 2l,$$
$$e_k \longmapsto 0 \text{ for } 1 \leq k \leq m-1, \qquad e'_k \longmapsto 0 \text{ for } 1 \leq k \leq m.$$

Then $[E_m, \tau E_m]$ is given by

$$l \longmapsto 0, \qquad e_m \longmapsto 2e_m,$$
$$e'_m \longmapsto -2e'_m, \qquad e_k, e'_k \longmapsto 0 \text{ for } 1 \leq k \leq m-1.$$

Thus $[E_m, \tau E_m] = H_m$, as desired.

In the even case, we define $E_m \in \operatorname{End}(\underline{V} \otimes \mathbb{C})$ by

$$e'_m \longmapsto e_{m-1}, \qquad e'_{m-1} \longmapsto -e_m,$$
$$e_k \longmapsto 0 \text{ for } 1 \leq k \leq m, \qquad e'_k \longmapsto 0 \text{ for } 1 \leq k \leq m-2.$$

Then $E_m \in \operatorname{Lie} G^*_{\mathbb{C}}$ and it is a root vector of α_m. We compute that τE_m is given by

$$e_m \longmapsto -e'_{m-1}, \qquad e_{m-1} \longmapsto e'_m,$$
$$e_k \longmapsto 0 \text{ for } 1 \leq k \leq m-2, \qquad e'_k \longmapsto 0 \text{ for } 1 \leq k \leq m.$$

Then $[E_m, \tau E_m]$ is given by

$$e_m \longmapsto e_m, \qquad e_{m-1} \longmapsto e_{m-1},$$
$$e'_m \longmapsto -e'_m, \qquad e'_{m-1} \longmapsto -e'_{m-1},$$
$$e_k, e'_k \longmapsto 0 \text{ for } 1 \leq k \leq m-2.$$

Thus $[E_m, \tau E_m] = H_m$, as desired. □

6.2.5. — As before, we fix the standard Borel pair $(\mathcal{T}, \mathcal{B})$ in $\widehat{G}$, and the standard Borel pair $(\mathcal{T}_{V^+} \times \mathcal{T}_{V^-}, \mathcal{B}_{V^+} \times \mathcal{B}_{V^-}) = (\mathcal{T}_{\widehat{H}}, \mathcal{B}_{\widehat{H}})$ in $\widehat{H} = \widehat{\mathrm{SO}(V^+)} \times \widehat{\mathrm{SO}(V^-)}$; see Definition 5.2.2 and §5.4.3. We extend $(\mathcal{T}, \mathcal{B})$ to a Γ_∞-stable splitting $\mathbf{spl}_{\widehat{G}}$, and extend $(\mathcal{T}_{\widehat{H}}, \mathcal{B}_{\widehat{H}})$ to a Γ_∞-stable splitting $\mathbf{spl}_{\widehat{H}}$.

Note that $\eta : {}^L H \to {}^L G$ maps $(\mathcal{T}_{\widehat{H}}, \mathcal{B}_{\widehat{H}})$ into $(\mathcal{T}, \mathcal{B})$. Given this property, and given the choices $\mathbf{spl}_{\widehat{G}}$ and $\mathbf{spl}_{\widehat{H}}$, we have the following constructions (see [**She19**, §§7.3, 8.1], [**She10b**, §7], [**She10a**, §3]):

– Inside each equivalence class $\boldsymbol{\varphi}$ of discrete Langlands parameters for G^*, there is a canonical $\mathcal{T}$-conjugacy class of parameters, whose elements we shall call *almost canonical representatives*. Similarly, inside each equivalence class of discrete Langlands parameters for H, there are almost canonical representatives.

– Let $\boldsymbol{\varphi}$ be as above. Consider the set of equivalence classes of discrete Langlands parameters for H that induce $\boldsymbol{\varphi}$ via $\eta : {}^L H \to {}^L G$. Then this set is non-empty (because H contains anisotropic maximal tori), and it contains a canonical element $\boldsymbol{\varphi_H}$, called *well-positioned*, which is uniquely characterized by the following property: For one (and hence any) almost canonical representative φ_H of $\boldsymbol{\varphi_H}$, the composition $\eta \circ \varphi_H$ is an almost canonical representative of $\boldsymbol{\varphi}$.

We now choose an arbitrary equivalence class $\boldsymbol{\varphi}$ of discrete Langlands parameters for G^*, and obtain $\boldsymbol{\varphi_H}$ from $\boldsymbol{\varphi}$ as above. Choose an almost canonical representative φ_H of $\boldsymbol{\varphi_H}$, and let $\varphi := \eta \circ \varphi_H$. Thus φ is an almost canonical representative of $\boldsymbol{\varphi}$. By construction, the Borel pair in $\widehat{G}$ (resp. $\widehat{H}$) determined by φ (resp. φ_H) as on p. 182 of [**Kot90**] is $(\mathcal{T}, \mathcal{B})$ (resp. $(\mathcal{T}_{\widehat{H}}, \mathcal{B}_{\widehat{H}})$).

Let π_0 be the unique generic member (with respect to the unique equivalence class of Whittaker data) of the L-packet $\Pi_{\boldsymbol{\varphi}}$; see [**Kos78**] and [**Vog78**]. As proved by Shelstad in [**She08**] (see [**She10a**, Thm. 3.6]), we have

$$\underline{\Delta}^{\mathrm{spec}}_{\mathrm{Wh}}(\boldsymbol{\varphi_H}, \pi_0) = 1. \tag{6.2.5.1}$$

Here $\underline{\Delta}^{\mathrm{spec}}_{\mathrm{Wh}}(\cdot,\cdot)$ are the (absolute) spectral transfer factors between H and G^*, under the Whittaker normalization. (In fact, (6.2.5.1) holds for all discrete $\boldsymbol{\varphi_H}$ inducing $\boldsymbol{\varphi}$, not just the well-positioned one; cf. [**Kal16**, §5.6]. We will not need this.) By [**She10b**, Lem. 12.3], the transfer factors $\underline{\Delta}^{\mathrm{spec}}_{\mathrm{Wh}}(\cdot,\cdot)$ are *compatible* with the Whittaker-normalized transfer factors $\underline{\Delta}_{\mathrm{Wh}}(\cdot,\cdot)$, in the sense that the endoscopic character relations defined by the former are satisfied when the test functions satisfy orbital integral relations with respect to the latter.

We now fix $\underline{\mathcal{D}}$ and $\mathcal{D}_H$ as in §6.1.9, and assume that $\underline{\mathcal{D}} \in \mathrm{ED}(\underline{V})^o_{\mathrm{Wh}}$. We keep the notation in §6.1.9. By Lemma 6.1.12, the map $j_{\mathcal{D}_H,\underline{\mathcal{D}}}$ constructed in §6.1.9 is an admissible isomorphism. We note that $(j_{\mathcal{D}_H,\underline{\mathcal{D}}}, B_{\underline{\mathcal{D}}}, B_{\mathcal{D}_H})$ is *aligned* with $\boldsymbol{\varphi_H}$ in the sense of [**Kot90**, p. 184], which follows from our assumption that $\boldsymbol{\varphi_H}$ is well-positioned. In the following, we abbreviate $j_{\mathcal{D}_H,\underline{\mathcal{D}}}$ as $\underline{j}$, and abbreviate $B_{\underline{\mathcal{D}}}$ as $\underline{B}$.

In [**Kot90**, §7], a normalization

$$\Delta_{\underline{j},\underline{B}}(\cdot,\cdot)$$

of the transfer factors between H and G^* is defined. Write $\Delta^{\mathrm{spec}}_{\underline{j},\underline{B}}(\cdot,\cdot)$ for the spectral transfer factors normalized compatibly with $\Delta_{\underline{j},\underline{B}}(\cdot,\cdot)$. Then since $(\underline{j},\underline{B})$ is aligned with $\boldsymbol{\varphi_H}$, we have (see [**Kot90**, p. 185])

$$\Delta^{\mathrm{spec}}_{\underline{j},\underline{B}}(\boldsymbol{\varphi_H}, \pi(\varphi, \omega^{-1}\underline{B})) = \langle a_\omega, s\rangle, \tag{6.2.5.2}$$

for all $\omega \in \Omega_{\mathbb{C}}(G^*, T_{\underline{\mathcal{D}}})$. Here a_ω is defined in [**Kot90**, §5], and we shall not need the definition of $\langle a_\omega, s\rangle$ except the fact that

$$\langle a_1, s\rangle = 1.$$

Now by Vogan's classification theorem for generic representations [**Vog78**, Thm. 6.2] and by Lemma 6.2.4, we know that $\pi_0 = \pi(\varphi, \underline{B})$. Hence by setting $\omega = 1$ in (6.2.5.2) we obtain

$$\Delta^{\mathrm{spec}}_{\underline{j},\underline{B}}(\boldsymbol{\varphi_H}, \pi_0) = 1. \tag{6.2.5.3}$$

Comparing (6.2.5.1) and (6.2.5.3), we see that

$$\underline{\Delta}^{\mathrm{spec}}_{\mathrm{Wh}} = \Delta^{\mathrm{spec}}_{\underline{j},\underline{B}}. \tag{6.2.5.4}$$

Now as we recalled above, $\underline{\Delta}_{\mathrm{Wh}}^{\mathrm{spec}}$ is compatible with $\underline{\Delta}_{\mathrm{Wh}}$. Hence it follows from (6.2.5.4) that

$$\Delta_{\underline{j},\underline{B}} = \underline{\Delta}_{\mathrm{Wh}}.$$

We record this in the following lemma.

Lemma 6.2.6. — *Let* $\underline{\mathcal{D}} \in \mathrm{ED}(\underline{V})^o_{\mathrm{Wh}}$ *and* $\mathcal{D}_H \in \mathrm{ED}(V^+)^o \times \mathrm{ED}(V^-)^o$*. Let* $\underline{j} = j_{\mathcal{D}_H,\underline{\mathcal{D}}}$ *and let* $\underline{B} = B_{\underline{\mathcal{D}}}$*. Then* $\underline{\Delta}_{\mathrm{Wh}} = \Delta_{\underline{j},\underline{B}}$*.* □

Transfer factors between H and G

6.2.7. — Recall from §5.1 that we have fixed an isomorphism $\phi_V : V \otimes \mathbb{C} \xrightarrow{\sim} \underline{V} \otimes \mathbb{C}$ between quadratic spaces over $\mathbb{C}$, and used ϕ_V to define the inner twisting $\psi_V : G_{\mathbb{C}} \xrightarrow{\sim} G^*_{\mathbb{C}}$ and the cocycle $u_V : \Gamma_\infty \to G^*(\mathbb{C})$, satisfying (5.1.2.1). As we have explained in Remark 5.1.4, these extra data allow us to derive from $\underline{\Delta}_{\mathrm{Wh}}(\cdot,\cdot)$ a normalization of the transfer factors between H and G, which we denote by $\Delta_{\mathrm{Wh}}(\cdot,\cdot)$.

We now recall the characterization of Δ_{Wh} in terms of $\underline{\Delta}_{\mathrm{Wh}}$ following [**Kal11**, §2.2]. Let T_H, T, and $\underline{T}$ be anisotropic maximal tori in H, G, and G^*, respectively. (Recall that H, G, and G^* all contain anisotropic maximal tori.) Assume that u_V takes values in $\underline{T}(\mathbb{C})$. (We shall see in §6.2.15 below that this can indeed be arranged.) Let $j : T_{H,\mathbb{C}} \to T_{\mathbb{C}}$ and $\underline{j} : T_{H,\mathbb{C}} \to \underline{T}_{\mathbb{C}}$ be arbitrary admissible isomorphisms; see §5.6. Note that T_H, T, and $\underline{T}$ are all isomorphic to $\mathrm{U}(1)^m$, and so j and $\underline{j}$ are necessarily defined over $\mathbb{R}$. Let $\gamma^H \in T_H(\mathbb{R})$, and let

$$\gamma := j(\gamma^H), \quad \underline{\gamma} := \underline{j}(\gamma^H).$$

Assume that γ and $\underline{\gamma}$ are strongly regular. Then Δ_{Wh} is characterized by the following formula:

$$\Delta_{\mathrm{Wh}}(\gamma^H, \gamma) = \underline{\Delta}_{\mathrm{Wh}}(\gamma^H, \underline{\gamma}) \langle \mathrm{inv}(\gamma, \underline{\gamma}), s_{\gamma^H,\underline{\gamma}} \rangle^{-1}, \tag{6.2.7.1}$$

where $\mathrm{inv}(\gamma, \underline{\gamma})$ and $s_{\gamma^H,\underline{\gamma}}$ are defined as follows.

– Define $\mathrm{inv}(\gamma, \underline{\gamma})$ to be the image of the cocycle $(\rho \mapsto u_V(\rho))$ under the Tate–Nakayama isomorphism $\mathbf{H}^1(\mathbb{R}, \underline{T}) \xrightarrow{\sim} \widehat{\mathbf{H}}^{-1}(\Gamma_\infty, X_*(\underline{T}))$. In our case, since $\underline{T} \cong \mathrm{U}(1)^m$, the norm map on $X_*(\underline{T})$ is zero, and so $\widehat{\mathbf{H}}^{-1}(\Gamma_\infty, X_*(\underline{T}))$ is simply $X_*(\underline{T})_{\Gamma_\infty}$.

– Define $s_{\gamma^H,\underline{\gamma}}$ to be the image of $s \in Z(\widehat{H})$ (which is part of the endoscopic datum) under the composite map

$$Z(\widehat{H}) \hookrightarrow \widehat{T_H} \xrightarrow{\widehat{\underline{j}}} \widehat{\underline{T}}.$$

Here the first map is the common restriction to $Z(\widehat{H})$ of any isomorphism $\mathcal{T}_{\widehat{H}} \xrightarrow{\sim} \widehat{T_H}$ of the form $\mathfrak{d}^{-1}_{B_H,\mathcal{B}_{\widehat{H}}}$ for any Borel subgroup B_H of $H_{\mathbb{C}}$ containing $T_{H,\mathbb{C}}$; see §5.6. We know that $s_{\gamma^H,\underline{\gamma}}$ is invariant under Γ_∞ since, in our case, it is of order at most 2 and the non-trivial element of Γ_∞ acts on $\widehat{\underline{T}}$ by inversion. Thus $s_{\gamma^H,\underline{\gamma}}$ can be paired with $\mathrm{inv}(\gamma,\underline{\gamma})$.

Definition 6.2.8. — We call $\Delta_{\mathrm{Wh}}(\cdot,\cdot)$ as in (6.2.7.1) the *Whittaker-normalized transfer factors between H and G.*

Definition 6.2.9. — Let $\mathrm{ED}(\underline{V})^o_{\mathrm{Wh},\phi_V}$ be the set of tuples $(\underline{V}_j,\lambda_j)_{1\le j\le m}$, where $(\underline{V}_j)_j \in \mathrm{ED}(\underline{V})^o_{\mathrm{Wh}}$ (see Definition 6.2.2), and $\lambda_1,\cdots,\lambda_m \in \{1,\sqrt{-1}\}$, satisfying the following conditions.

(1) For each $1\le j\le m$, we have $\phi_V^{-1}(\underline{V}_j)\subset V\otimes\lambda_j^{-1}$.

(2) There exists $j_0\in\mathbb{Z}$ such that for each $1\le j\le m$, we have $\lambda_j=\sqrt{-1}$ if and only if $\underline{V}_j$ is negative definite and $j\le j_0$.

(3) If d is odd, then the restriction of $\phi_V^{-1}:\underline{V}\otimes\mathbb{C}\to V\otimes\mathbb{C}$ to the orthogonal complement of $\bigoplus_{j=1}^m \underline{V}_j$ in $\underline{V}$ is defined over $\mathbb{R}$.

Remark 6.2.10. — The set $\mathrm{ED}(\underline{V})^o_{\mathrm{Wh},\phi_V}$ is non-empty. This follows from the condition in Definition 5.1.1, the fact that V and $\underline{V}$ have the same discriminant, and Remark 6.2.3.

6.2.11. — Let $(\underline{V}_j,\lambda_j)_j\in\mathrm{ED}(\underline{V})^o_{\mathrm{Wh},\phi_V}$ as in Definition 6.2.9. We construct an element $(V_j)_j\in\mathrm{ED}(V)^o$ as follows. For each j, let $\{f_j,f_j'\}$ be a basis of $\underline{V}_j$ inducing the given orientation on $\underline{V}_j$. Then the vectors $\lambda_j\phi^{-1}(f_j),\lambda_j\phi^{-1}(f_j')\in V\otimes\mathbb{C}$ lie in $V\otimes 1$. We identify $V\otimes 1$ with V, and let V_j be the oriented plane spanned by $\{\lambda_j\phi^{-1}(f_j),\lambda_j\phi^{-1}(f_j')\}$. Then $(V_j)_j$ is an element of $\mathrm{ED}(V)$. By Lemma 6.1.8, we have $(V_j)_j\in\mathrm{ED}(V)^o$. The construction $(\underline{V}_j,\lambda_j)_j\mapsto(V_j)_j$ gives a map

$$\mathrm{ED}(\underline{V})^o_{\mathrm{Wh},\phi_V}\longrightarrow\mathrm{ED}(V)^o. \tag{6.2.11.1}$$

Definition 6.2.12. — We define a subset $\mathrm{ED}(V)^o_{\mathrm{nice}}$ of $\mathrm{ED}(V)^o$ as follows. When d is odd, we let $\mathrm{ED}(V)^o_{\mathrm{nice}}$ consist of those $(V_j)_j\in\mathrm{ED}(V)^o=\mathrm{ED}(V)$ for which there exists $j_0\in\mathbb{Z}$ such that

$$\{j\mid 1\le j\le m, V_j \text{ is negative definite}\} = \left\{j\mid 1\le j\le m, j>j_0, \text{ and } (-1)^j=\mathrm{sgn}(\delta)\right\}.$$

When d is even (but not divisible by 4), we let $\mathrm{ED}(V)^o_{\mathrm{nice}}$ consist of those $(V_j)_j\in\mathrm{ED}(V)^o$ for which there exists $j_0\in\mathbb{Z}$ such that

$$\{j\mid 1\le j\le m, V_j \text{ is negative definite}\} = \left\{j\mid 1\le j\le m, j>j_0, \text{ and } (-1)^j=1\right\}.$$

Example 6.2.13. — Let $\mathcal{D}=(V_j)_j$ be an arbitrary element of $\mathrm{ED}(V)^o$. Recall that V has signature (p,q). If d is odd and $q=2$, then $\mathcal{D}$ is in $\mathrm{ED}(V)^o_{\mathrm{nice}}$ if and only if V_m is negative definite. If d is odd and $q\le 1$, then $\mathcal{D}$ is automatically in $\mathrm{ED}(V)^o_{\mathrm{nice}}$. If d is even (but not divisible by 4) and $q=2$, then $\mathcal{D}$ is in $\mathrm{ED}(V)^o_{\mathrm{nice}}$ if and only if

V_{m-1} is negative definite. If d is even (but not divisible by 4) and $q = 0$, then $\mathcal{D}$ is automatically in $\mathrm{ED}(V)^o_{\mathrm{nice}}$.

Lemma 6.2.14. — *The image of the map (6.2.11.1) is contained in* $\mathrm{ED}(V)^o_{\mathrm{nice}}$.

Proof. — This is clear from the definitions. □

6.2.15. — Now let $(\underline{V}_j, \lambda_j)_j$ be an element of $\mathrm{ED}(\underline{V})^o_{\mathrm{Wh},\phi_V}$, with image $(V_j)_j \in \mathrm{ED}(V)^o_{\mathrm{nice}}$ under the map (6.2.11.1). Write $\underline{\mathcal{D}}$ for the element $(\underline{V}_j)_j \in \mathrm{ED}(\underline{V})^o_{\mathrm{Wh}}$, and write $\mathcal{D}$ for the element $(V_j)_j \in \mathrm{ED}(V)^o_{\mathrm{nice}}$. Write $\vec{\lambda}$ for the tuple $(\lambda_j)_j$. Let

$$f_{\underline{\mathcal{D}}} : \mathrm{U}(1)^m \longrightarrow T_{\underline{\mathcal{D}}}$$

be the parameterized anisotropic maximal torus in G^* associated to $\underline{\mathcal{D}}$, and let

$$f_{\mathcal{D}} : \mathrm{U}(1)^m \longrightarrow T_{\mathcal{D}}$$

be the parameterized anisotropic maximal torus in G associated to $\mathcal{D}$. Also, let $(T_{\underline{\mathcal{D}}}, B_{\underline{\mathcal{D}}})$ be the fundamental pair in G^* associated to $\underline{\mathcal{D}}$, and let $(T_{\mathcal{D}}, B_{\mathcal{D}})$ be the fundamental pair in G associated to $\mathcal{D}$. We abbreviate $(T_{\underline{\mathcal{D}}}, B_{\underline{\mathcal{D}}})$ as $(\underline{T}, \underline{B})$, and abbreviate $(T_{\mathcal{D}}, B_{\mathcal{D}})$ as (T, B).

Note that we have

$$f_{\underline{\mathcal{D}}} = \psi_V \circ f_{\mathcal{D}}, \tag{6.2.15.1}$$

which is clear from the definition of ψ_V in §5.1.2. In particular, the cocycle u_V takes values in $\underline{T}(\mathbb{C})$. More precisely, for $\rho = \tau$ the complex conjugation, $u_V(\tau)$ acts as -1 on $\underline{V}_j$ for those j such that $\lambda_j = \sqrt{-1}$, and acts as the identity on the orthogonal complement of these $\underline{V}_j$'s. It follows that $u_V(\tau) \in \underline{T}(\mathbb{R})$. Another consequence of the relation (6.2.15.1) is that ψ_V sends the Borel pair $(T_{\mathbb{C}}, B)$ in $G_{\mathbb{C}}$ to the Borel pair $(\underline{T}_{\mathbb{C}}, \underline{B})$ in $G^*_{\mathbb{C}}$.

Take any $\mathcal{D}_H \in \mathrm{ED}(V^+)^o \times \mathrm{ED}(V^-)^o$, and define

$$j_{\mathcal{D}_H, \underline{\mathcal{D}}} : T_{\mathcal{D}_H} \xrightarrow{\sim} T_{\underline{\mathcal{D}}}$$
$$j_{\mathcal{D}_H, \mathcal{D}} : T_{\mathcal{D}_H} \xrightarrow{\sim} T_{\mathcal{D}}$$

as in §6.1.9 (where $\underline{\mathcal{D}}$ and $\mathcal{D}$ are fixed in the last paragraph.) We abbreviate $j_{\mathcal{D}_H, \underline{\mathcal{D}}}$ as $\underline{j}$, and abbreviate $j_{\mathcal{D}_H, \mathcal{D}}$ as j. Let $(T_{\mathcal{D}_H}, B_{\mathcal{D}_H})$ be the fundamental pair in H associated to $\mathcal{D}_H$. We abbreviate $(T_{\mathcal{D}_H}, B_{\mathcal{D}_H})$ as (T_H, B_H). Take a test element $\gamma^H \in T_{\mathcal{D}_H}(\mathbb{R})$, and let

$$\gamma := j(\gamma^H), \qquad \underline{\gamma} := \underline{j}(\gamma^H).$$

Assume that γ and $\underline{\gamma}$ are strongly regular.

Lemma 6.2.16. — *Keep the setting of §6.2.15. Let* $\langle \mathrm{inv}(\gamma, \underline{\gamma}), s_{\gamma^H, \underline{\gamma}} \rangle$ *be the pairing defined in §6.2.7. Then we have*

$$\langle \mathrm{inv}(\gamma, \underline{\gamma}), s_{\gamma^H, \underline{\gamma}} \rangle = (-1)^{k(m^-, \vec{\lambda})},$$

where

$$k(m^-, \vec{\lambda}) := \#\left\{ j \mid 1 \le j \le m^-, \lambda_j = \sqrt{-1} \right\}.$$

Proof. — By [**Kal11**, Lem. 2.3.3], the element $\mathrm{inv}(\gamma, \underline{\gamma}) \in X_*(\underline{T})_{\Gamma_\infty}$ is equal to the image of any element $\mu \in X_*(\underline{T})$ such that $\mu(-1) = u_V(\tau)$, where τ is the complex conjugation. We identify $X_*(\underline{T})$ with $\mathbb{Z}^m$ via $f_{\underline{\mathcal{D}}} : \mathrm{U}(1)^m \xrightarrow{\sim} \underline{T}$, and let $\{\epsilon_1^\vee, \cdots, \epsilon_m^\vee\}$ be the natural basis. By the description of $u_V(\tau)$ in §6.2.15, we can take μ to be

$$\mu = \sum_{1 \le j \le m, \lambda_j = \sqrt{-1}} \epsilon_j^\vee.$$

On the other hand, if we identify $\widehat{\underline{T}}$ with $(\mathbb{C}^\times)^m$ under $\widehat{f_{\underline{\mathcal{D}}}}$, then the element $s_{\gamma^H, \underline{\gamma}} \in \widehat{\underline{T}}$ is given by

$$(\underbrace{-1, \cdots, -1}_{m^-}, \underbrace{1, \cdots, 1}_{m^+}) \in (\mathbb{C}^\times)^m.$$

(Remember that Convention 6.1.10 is in force in the definition of $j_{\mathcal{D}_H, \underline{\mathcal{D}}}$ in §6.1.9.) The lemma follows by evaluating μ at the above element. □

Lemma 6.2.17. — *Keep the setting of §6.2.15. We have*

$$\Delta_{\underline{j}, \underline{B}}(\gamma^H, \underline{\gamma}) = (-1)^{q(G) + q(G^*)} \Delta_{j,B}(\gamma^H, \gamma).$$

Here $\Delta_{\underline{j}, \underline{B}}$ (resp. $\Delta_{j,B}$) is the normalization of the transfer factors between H and G^* (between H and G), associated to $(\underline{j}, \underline{B})$ (resp. (j, B)), as defined in [**Kot90**, §7]*. The numbers $q(G)$ and $q(G^*)$ are as in Definition 1.1.4.*

Proof. — By the formula for $\Delta_{j,B}$ on p. 184 of [**Kot90**], we have

$$\Delta_{\underline{j}, \underline{B}}(\gamma^H, \underline{\gamma}) = (-1)^{q(G^*) + q(H)} \chi_{G^*, H}(\underline{\gamma}) \Delta_{\underline{B}}(\underline{\gamma}^{-1}) \Delta_{B_H}((\gamma^H)^{-1})^{-1}$$
$$\Delta_{j,B}(\gamma^H, \gamma) = (-1)^{q(G) + q(H)} \chi_{G,H}(\gamma) \Delta_B(\gamma^{-1}) \Delta_{B_H}((\gamma^H)^{-1})^{-1}.$$

Here $\Delta_{\underline{B}}$, Δ_B, and Δ_{B_H} are as in Definition 1.1.3, and we do not explain the definitions of $\chi_{G^*,H}$ and $\chi_{G,H}$. Since ψ sends the Borel pair $(T_{\mathbb{C}}, B)$ to $(\underline{T}_{\mathbb{C}}, \underline{B})$, we know that

$$\Delta_{\underline{B}}(\underline{\gamma}^{-1}) = \Delta_B(\gamma^{-1}).$$

It remains to show that

$$\chi_{G^*,H}(\underline{\gamma}) = \chi_{G,H}(\gamma).$$

Unraveling the definitions of these terms on p. 184 of [**Kot90**], we are reduced to checking that the following diagram commutes up to $\widehat{G}$-conjugation:

$$\begin{array}{ccc} {}^L T & \xrightarrow{\eta_B} & {}^L G \\ \downarrow & & \| \\ {}^L \underline{T} & \xrightarrow{\eta_{\underline{B}}} & {}^L G \end{array}$$

where the left vertical arrow is induced by $\psi_V|_T : T \xrightarrow{\sim} \underline{T}$ (defined over $\mathbb{R}$). This is true by the characterizations (a) (b) on p. 183 of [**Kot90**], in view of the fact that $\psi_V(B) = \underline{B}$. □

Corollary 6.2.18. — *Keep the setting of §6.2.15, and keep the notation in Lemmas 6.2.16 and 6.2.17. We have*

$$\Delta_{j,B} = (-1)^{q(G)+q(G^*)+k(m^-,\vec{\lambda})}\Delta_{\mathrm{Wh}}.$$

Proof. — Comparing (6.2.7.1) with Lemmas 6.2.16 and 6.2.17, we have

$$\frac{\underline{\Delta}_{\mathrm{Wh}}}{\Delta_{\underline{j},\underline{B}}} = (-1)^{q(G)+q(G^*)+k(m^-,\vec{\lambda})} \cdot \frac{\Delta_{\mathrm{Wh}}}{\Delta_{j,B}}.$$

By Lemma 6.2.6 we have $\underline{\Delta}_{\mathrm{Wh}} = \Delta_{\underline{j},\underline{B}}$. The corollary follows. □

Recall that V has signature (p, q), with $d = p + q$ not divisible by 4.

Lemma 6.2.19. — *We have*

$$(-1)^{q(G)+q(G^*)} = \begin{cases} (-1)^{\lceil \frac{m-p}{2} \rceil}, & \text{if } d \text{ is odd,} \\ 1, & \text{if } d \text{ is even.} \end{cases}$$

Proof. — For any signature (a, b), we have $q(\mathrm{SO}(a, b)) = ab/2$. In the odd case, V has signature $(p, q) = (p, 2m + 1 - p)$, and $\underline{V}$ has signature $(m + 1, m)$ or $(m, m + 1)$. Hence

$$\begin{aligned} q(G^*) - q(G) &\equiv \frac{(m+1)m}{2} - \frac{p(2m+1-p)}{2} \\ &= \frac{(m-p)(m+1-p)}{2} \equiv \lceil \frac{m-p}{2} \rceil \mod 2. \end{aligned}$$

In the even case, our assumption that G and G^* contain anisotropic maximal tori implies that the signatures of V and $\underline{V}$ are pairs of even numbers. Hence $q(G)$ and $q(G^*)$ are both even. □

Proposition 6.2.20. — *Keep the running assumption that V has signature (p, q), with $p > q$ and $d = p + q$ not divisible by 4. Let $\mathcal{D}$ be an arbitrary element of* $\mathrm{ED}(V)^o_{\mathrm{nice}}$ *(see Definition 6.2.12), and let $\mathcal{D}_H \in \mathrm{ED}(V^+)^o \times \mathrm{ED}(V^-)^o$. Define $j_{\mathcal{D}_H,\mathcal{D}}$ and $(T_{\mathcal{D}_H}, B_{\mathcal{D}_H})$ as in §6.1.9. We abbreviate $(j_{\mathcal{D}_H,\mathcal{D}}, B_{\mathcal{D}_H})$ as (j, B). Let $\Delta_{j,B}$ be the normalization of the transfer factors between H and G associated to (j, B), as defined in* [**Kot90**, §7].

(1) *Assume that d is odd. In this case, either assume that q is even and $q/2 \leq \lceil m^+/2 \rceil$, or assume that q is odd and $(q - 1)/2 \leq \lfloor m^+/2 \rfloor$. Then*

$$\Delta_{j,B} = \begin{cases} (-1)^{\lceil \frac{m}{2} \rceil + \lceil \frac{m^+}{2} \rceil + \lceil \frac{m-p}{2} \rceil}\Delta_{\mathrm{Wh}}, & \text{if } q \text{ is even,} \\ (-1)^{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{m^+}{2} \rfloor + \lceil \frac{m-p}{2} \rceil}\Delta_{\mathrm{Wh}}, & \text{if } q \text{ is odd.} \end{cases}$$

In particular, we have

$$\Delta_{j,B} = \begin{cases} (-1)^{\lceil \frac{m^+}{2} \rceil} \Delta_{\mathrm{Wh}}, & \text{when } q = 0 \text{ and } m^+ \text{ is arbitrary,} \\ (-1)^{\lfloor \frac{m^+}{2} \rfloor} \Delta_{\mathrm{Wh}}, & \text{when } q = 1 \text{ and } m^+ \text{ is arbitrary,} \\ (-1)^{1+\lceil \frac{m^+}{2} \rceil} \Delta_{\mathrm{Wh}}, & \text{when } q = 2 \text{ and } m^+ > 0. \end{cases}$$

(2) *Assume that d is even. Thus q is even since G contains anisotropic maximal tori. We have*

$$\Delta_{j,B} = \begin{cases} (-1)^{\lfloor \frac{m^-}{2} \rfloor} \Delta_{\mathrm{Wh}}, & \text{if } q/2 \leq \lfloor m^+/2 \rfloor, \\ (-1)^{\frac{m+1}{2}} \Delta_{\mathrm{Wh}}, & \text{if } m^+ = 1 \text{ and } q = 2. \end{cases}$$

In particular, we have

$$\Delta_{j,B} = \begin{cases} (-1)^{\lfloor \frac{m^-}{2} \rfloor} \Delta_{\mathrm{Wh}}, & \text{when } q = 0 \text{ and } m^+ \text{ is arbitrary,} \\ (-1)^{\lfloor \frac{m^-}{2} \rfloor} \Delta_{\mathrm{Wh}}, & \text{when } q = 2 \text{ and } m^+ \geq 2, \\ (-1)^{1+\lfloor \frac{m^-}{2} \rfloor} \Delta_{\mathrm{Wh}}, & \text{when } q = 2 \text{ and } m^+ = 1. \end{cases}$$

Proof. — First note that under the natural action of $G(\mathbb{R})$ on $\mathrm{ED}(V)^o$, the subset $\mathrm{ED}(V)^o_{\mathrm{nice}}$ of $\mathrm{ED}(V)^o$ is a single orbit. Thus $\Delta_{j,B}$ is in fact independent of the choice of $\mathcal{D} \in \mathrm{ED}(V)^o_{\mathrm{nice}}$. Hence we may assume that $\mathcal{D}$ is the same as the element introduced in §6.2.15. In view of Corollary 6.2.18 and Lemma 6.2.19, to prove the proposition it suffices to compute the sign $(-1)^{k(m^-,\vec{\lambda})}$ in each case. We recall that

$$k(m^-, \vec{\lambda}) := \# \left\{ j \mid 1 \leq j \leq m^-, \lambda_j = \sqrt{-1} \right\}.$$

(1) Let N be the number of negative definite planes among the m^+ planes

$$\underline{V}_{m^-+1}, \underline{V}_{m^-+2}, \cdots, \underline{V}_m.$$

By Definition 6.2.2, $\underline{V}_m$ is negative definite if and only if V has positive determinant, which happens if and only if q is even. Hence we have $N = \lceil m^+/2 \rceil$ when q is even, and $N = \lfloor m^+/2 \rfloor$ when q is odd. Thus our assumption on q can be rewritten as $\lfloor q/2 \rfloor \leq N$.

If there exists $1 \leq j_1 \leq m^-$ such that $\underline{V}_{j_1}$ is negative definite and $\lambda_{j_1} = 1$, then the integer j_0 in condition (2) in Definition 6.2.9 would be strictly less than j_1, from which it easily follows that the number of negative definite planes among $V_1, \cdots, V_m$ is at least $N+1$. Thus $q \geq 2(N+1)$, a contradiction. Hence such j_1 does not exist. Then by condition (2) in Definition 6.2.9, we have

$$k(m^-, \vec{\lambda}) = \# \left\{ j \mid 1 \leq j \leq m^-, V_j \text{ is negative definite} \right\}.$$

When q is even, we have

$$k(m^-, \vec{\lambda}) = \begin{cases} \lceil m^-/2 \rceil, & \text{if } m \text{ is odd} \\ \lfloor m^-/2 \rfloor, & \text{if } m \text{ is even} \end{cases} \equiv \lceil m/2 \rceil + \lceil m^+/2 \rceil \mod 2.$$

When q is odd, we have

$$k(m^-,\vec{\lambda}) = \begin{cases} \lfloor m^-/2 \rfloor, & \text{if } m \text{ is odd} \\ \lceil m^-/2 \rceil, & \text{if } m \text{ is even} \end{cases} \equiv \lfloor m/2 \rfloor + \lfloor m^+/2 \rfloor \mod 2.$$

We conclude the proof by combining the above computation of $k(m^-,\vec{\lambda})$ with Corollary 6.2.18 and Lemma 6.2.19.

(2) Since $d = 2m$ is not divisible by 4, we know that m is odd, and by Definition 6.2.2 we know that $\underline{V}_m$ is positive definite. Hence among the m^+ planes

$$\underline{V}_{m^-+1}, \underline{V}_{m^-+2}, \cdots, \underline{V}_m,$$

the number of negative definite planes is $\lfloor m^+/2 \rfloor$. When $q/2 \leq \lfloor m^+/2 \rfloor$, by the same argument as in part (1) we have

$$k(m^-,\vec{\lambda}) = \#\left\{ j \mid 1 \leq j \leq m^-, V_j \text{ is negative definite} \right\},$$

and this is equal to $\lfloor m^-/2 \rfloor$. When $m^+ = 1$ and $q = 2$, we easily see that

$$k(m^-,\vec{\lambda}) = \frac{m-1}{2} - 1.$$

In both cases we conclude the proof by combining the computation of $k(m^-,\vec{\lambda})$ with Corollary 6.2.18 and Lemma 6.2.19. $\square$

6.3. Transfer factors, when d is divisible by 4

6.3.1. — We keep the same setting as in §6.2.1, except that now we assume that d is divisible by 4. We keep the assumption that G and G^* contain anisotropic maximal tori, which forces the signature of V to be a pair of even numbers. In particular, δ is trivial, and so $\underline{V}$ and G^* are split. We would like to establish analogues of the results in §6.2 in the current case. The new feature is that there are now two different equivalence classes of Whittaker data for G^*. As in §6.2.1, we fix $(H, {}^L H, s, \eta)$, with H containing anisotropic maximal tori.

In the following we assume that V is of signature (p,q) with $p > q$, and that $d = p + q$ is divisible by 4.

Transfer factors between H and G^*

Definition 6.3.2. — We define two subsets $\mathrm{ED}(\underline{V})^o_{\mathrm{Wh\text{-}I}}$ and $\mathrm{ED}(\underline{V})^o_{\mathrm{Wh\text{-}II}}$ of $\mathrm{ED}(\underline{V})^o$ (see §6.1.7) as follows. Let $\mathrm{ED}(\underline{V})^o_{\mathrm{Wh\text{-}I}}$ consist of those $(\underline{V}_j)_j \in \mathrm{ED}(\underline{V})^o$ such that $\underline{V}_j$ is $(-1)^{j+1}$-definite for each j. Let $\mathrm{ED}(\underline{V})^o_{\mathrm{Wh\text{-}II}}$ consist of those $(\underline{V}_j)_j \in \mathrm{ED}(\underline{V})^o$ such that $\underline{V}_j$ is $(-1)^j$-definite for each j.

6.3.3. — Let (T_1, B_1) (resp. (T_2, B_2)) be the fundamental pair associated to an element of $\mathrm{ED}(\underline{V})^o_{\mathrm{Wh\text{-}I}}$ (resp. an element of $\mathrm{ED}(\underline{V})^o_{\mathrm{Wh\text{-}II}}$). Then (T_1, B_2) and (T_2, B_2)

both satisfy the condition that every simple root is non-compact, which can be proved in the same way as Lemma 6.2.4. As in [**Taï17**, §4.2.1], the two pairs (T_1, B_1) and (T_2, B_2) correspond to two different equivalence classes of Whittaker data $\mathfrak{w}_{\mathrm{I}}$ and $\mathfrak{w}_{\mathrm{II}}$ of G^* respectively, characterized by the condition that in any L-packet of discrete series representations of $G^*(\mathbb{R})$, the element corresponding to (T_1, B_2) (resp. (T_2, B_2)) is generic with respect to $\mathfrak{w}_{\mathrm{I}}$ (resp. $\mathfrak{w}_{\mathrm{II}}$). Then $\mathfrak{w}_{\mathrm{I}}$ and $\mathfrak{w}_{\mathrm{II}}$ exhaust the equivalence classes of Whittaker data. We call $\mathfrak{w}_{\mathrm{I}}$ the equivalence class of *type-I Whittaker data*, and call $\mathfrak{w}_{\mathrm{II}}$ the equivalence class of *type-II Whittaker data*. See *loc. cit.* for more details.

Definition 6.3.4. — We denote by $\underline{\Delta}_{\mathrm{Wh}}(\cdot,\cdot)$ the Whittaker-normalized transfer factors between H and G^* with respect to $\mathfrak{w}_{\mathrm{I}}$, called the *type-I Whittaker normalization.* Denote by $\tilde{\underline{\Delta}}_{\mathrm{Wh}}(\cdot,\cdot)$ the analogous objects with respect to $\mathfrak{w}_{\mathrm{II}}$.

Lemma 6.3.5. — *Let* $\underline{\mathcal{D}} \in \mathrm{ED}(\underline{V})^o_{\mathrm{Wh\text{-}I}}$, *and let* $\mathcal{D}_H \in \mathrm{ED}(V^+)^o \times \mathrm{ED}(V^-)^o$. *Let* $\underline{j}$, (T_H, B_H), *and* $(\underline{T}, \underline{B})$ *be the objects associated to* $\underline{\mathcal{D}}$ *and* $\mathcal{D}_H$ *as in §6.2.5. We have*

$$\underline{\Delta}_{\mathrm{Wh}} = \Delta_{\underline{j},\underline{B}}, \tag{6.3.5.1}$$

$$\tilde{\underline{\Delta}}_{\mathrm{Wh}} = (-1)^{m^-}\Delta_{\underline{j},\underline{B}}. \tag{6.3.5.2}$$

In particular,

$$\tilde{\underline{\Delta}}_{\mathrm{Wh}} = (-1)^{m^-}\underline{\Delta}_{\mathrm{Wh}}.$$

Proof. — The proof of (6.3.5.1) is the same as the argument in §6.2.5 leading to Lemma 6.2.6. For (6.3.5.2), by the same argument we are reduced to checking that

$$\langle a_\omega, s\rangle = (-1)^{m^-}, \tag{6.3.5.3}$$

where $\omega \in \Omega_{\mathbb{C}}(G^*, \underline{T})$ is an element such that $(\underline{T}, \omega\underline{B})$ is the fundamental pair associated to an element of $\mathrm{ED}(\underline{V})^o_{\mathrm{Wh\text{-}II}}$. (Such ω is unique up to right multiplication by $\Omega_{\mathbb{R}}(G^*, \underline{T})$.) We can take

$$\omega = (12)(34)\cdots(m-1,m) \in \mathfrak{S}_m \subset \Omega_{\mathbb{C}}(G^*, \underline{T}),$$

and then the class $a_\omega \in \mathbf{H}^1(\mathbb{R}, \underline{T})$ (defined in [**Kot90**, §5]) is represented by the cocycle sending the complex conjugation to $-1 \in \underline{T}(\mathbb{R})$. This implies (6.3.5.3). □

Transfer factors between H and G.

Definition 6.3.6. — As in §6.2.7, having fixed ψ_V and u_V, and having fixed the Whittaker datum $\mathfrak{w}_{\mathrm{I}}$, we obtain a normalization of the transfer factors between H and G, called the *type-I Whittaker normalization.* We denote this normalization by Δ_{Wh}.

Remark 6.3.7. — Analogously we also have the type-II Whittaker normalization between H and G. By (6.3.5.2), it is equal to $(-1)^{m^-}\Delta_{\mathrm{Wh}}$.

Definition 6.3.8. — We let $\mathrm{ED}(V)^o_{\mathrm{nice}}$ be the subset of $\mathrm{ED}(V)^o$ consisting of those $(V_j)_j$ for which there exists $j_0 \in \mathbb{Z}$ such that

$$\{j \mid 1 \le j \le m, V_j \text{ is negative definite}\} = \left\{j \mid 1 \le j \le m, j > j_0, \text{ and } (-1)^j = 1\right\}.$$

Recall our running assumption that V has signature (p, q), with $p > q$ and $d = p+q$ divisible by 4. Recall that p and q are even since G contains anisotropic maximal tori.

Proposition 6.3.9. — *Let $\mathcal{D} \in \mathrm{ED}(V)^o_{\mathrm{nice}}$ and $\mathcal{D}_H \in \mathrm{ED}(V^+)^o \times \mathrm{ED}(V^-)^o$. Define $j_{\mathcal{D}_H,\mathcal{D}}$ and $(T_{\mathcal{D}_H}, B_{\mathcal{D}_H})$ as in §6.1.9. We abbreviate $(j_{\mathcal{D}_H,\mathcal{D}}, B_{\mathcal{D}_H})$ as (j, B). Let $\Delta_{j,B}$ be the normalization of the transfer factors between H and G associated to (j, B), as defined in* [**Kot90**, §7]*. When $q/2 \le \lceil m^+/2 \rceil$, we have*

$$\Delta_{j,B} = (-1)^{\lfloor \frac{m^-}{2} \rfloor} \Delta_{\mathrm{Wh}}.$$

In particular, we have

$$\Delta_{j,B} = \begin{cases} (-1)^{\lfloor \frac{m^-}{2} \rfloor} \Delta_{\mathrm{Wh}}, & \text{when } q = 0 \text{ and } m^+ \text{ is arbitrary,} \\ (-1)^{\lfloor \frac{m^-}{2} \rfloor} \Delta_{\mathrm{Wh}}, & \text{when } q = 2 \text{ and } m^+ \ge 1. \end{cases}$$

Proof. — The proof is the same as Proposition 6.2.20. Note that the bound $q/2 \le \lfloor m^+/2 \rfloor$ in Proposition 6.2.20 (2) is replaced by $q/2 \le \lceil m^+/2 \rceil$ here. This is because in the current case, for any $(\underline{V}_j)_j \in \mathrm{ED}(\underline{V})^o_{\mathrm{Wh\text{-}I}}$, $\underline{V}_m$ is always negative definite. □

Comparison with Waldspurger's explicit formula

6.3.10. — We fix the additive character $\psi : \mathbb{R} \to \mathbb{C}^\times, x \mapsto e^{2\pi i x}$ in all the discussion below. Given any Borel subgroup B_0 of G^* defined over $\mathbb{R}$, by the general construction in [**KS99**, §5.3] we have a canonical map (depending only on ψ)

(6.3.10.1)

$$\{\mathbb{R}\text{- splittings of } G^* \text{ relative to } B_0\} \longrightarrow \left\{\text{generic characters } N_{B_0}(\mathbb{R}) \to \mathbb{C}^\times\right\},$$

where the left hand side is the set of $\mathbb{R}$-splittings of G^* of the form $(T_0, B_0, \{X_\alpha\})$. In our particular situation, since G^* is split, $\mathbb{R}$-splittings of G^* are the same as splittings.

We denote by $\mathcal{S}plit(G^*)$ the set of $G^*(\mathbb{R})$-conjugacy classes of ($\mathbb{R}$-) splittings of G^*, and denote by $\mathcal{W}hitt(G^*)$ the set of equivalence classes (i.e. $G^*(\mathbb{R})$-conjugacy classes) of Whittaker data for G^*. The map (6.3.10.1) induces a canonical bijection (depending only on ψ):

$$\mathscr{W}^{G^*} : \mathcal{S}plit(G^*) \xrightarrow{\sim} \mathcal{W}hitt(G^*).$$

Here both sides are torsors under the abelian group $G^{*,\mathrm{ad}}(\mathbb{R})/G^*(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$.

The two elements of $\mathcal{W}hitt(G^*)$ are of course $\mathfrak{w}_{\mathrm{I}}$ and $\mathfrak{w}_{\mathrm{II}}$; see §6.3.3. On the other hand, there is an independent way to label the two elements of $\mathcal{S}plit(G^*)$. Recall that in [**Wal10**, §1.6], Waldspurger associates an element $\eta \in \mathbb{R}^\times/\mathbb{R}^{\times,2} \cong \{\pm 1\}$ to the quintuple $(G^*, \mathbf{spl}, \underline{V}, \underline{q}, \rho_{\mathrm{std}})$, where $\mathbf{spl}$ is an arbitrary element of $\mathcal{S}plit(G^*)$ and ρ_{std} is the standard representation $G^* \to \mathrm{GL}(\underline{V})$. This gives rise to a map

$$\eta_{\underline{V}} : \mathcal{S}plit(G^*) \longrightarrow \{\pm 1\} \qquad \mathbf{spl} \longmapsto \eta(G^*, \mathbf{spl}, \underline{V}, \underline{q}, \rho_{\mathrm{std}}). \tag{6.3.10.2}$$

This map is easily seen to be surjective, and hence bijective. Thus we can use it to label the two elements of $\mathcal{S}plit(G^*)$.

The following result will be used in the proof of Proposition 8.9.5 below, and it may be of independent interest in representation theory.

Theorem 6.3.11. — *Let* $\mathbf{spl}_{\mathrm{I}} = \eta_{\underline{V}}^{-1}(-1) \in \mathcal{S}plit(G^*)$. *Then* $\mathscr{W}^{G^*}(\mathbf{spl}_{\mathrm{I}}) = \mathfrak{w}_{\mathrm{I}}$.

Proof. — Write $\mathfrak{w}'$ for $\mathscr{W}^{G^*}(\mathbf{spl}_{\mathrm{I}})$. Consider an elliptic endoscopic datum

$$\mathfrak{e}_{d^+,\delta^+,d^-,\delta^-} = (H, {}^L H, s, \eta)$$

such that H contains anisotropic maximal tori. As in §6.1.1 we have $\delta^{\pm} = (-1)^{d^{\pm}/2}$. Let $m^{\pm} := d^{\pm}/2$. Let $\underline{\Delta}_{\mathrm{Wh}}$ and $\tilde{\underline{\Delta}}_{\mathrm{Wh}}$ be the transfer factors between H and G^* as in Definition 6.3.4. By Lemma 6.3.5 we have

$$\underline{\Delta}_{\mathrm{Wh}} = (-1)^{m^-} \tilde{\underline{\Delta}}_{\mathrm{Wh}}.$$

Hence it suffices to show that $\underline{\Delta}_{\mathrm{Wh}}$ is equal to the Whittaker normalization $\underline{\Delta}_{\mathfrak{w}'}$ defined by the Whittaker datum $\mathfrak{w}'$, for one single choice of (d^+, d^-) with m^- odd. In the following we show that

$$\underline{\Delta}_{\mathrm{Wh}} = \underline{\Delta}_{\mathfrak{w}'} \tag{6.3.11.1}$$

without assuming that m^- is odd.

Let $\underline{\mathcal{D}} = (\underline{V}_j)_j$ and $\mathcal{D}_H$ be as in Lemma 6.3.5, and keep the other notations in that lemma. As usual, we use the isomorphism $f_{\underline{\mathcal{D}}} : \mathrm{U}(1)^m \xrightarrow{\sim} \underline{T}$ associated to $\underline{\mathcal{D}}$ to identify $X^*(\underline{T})$ with $\mathbb{Z}^m$. By Lemma 6.3.5, we have

$$\underline{\Delta}_{\mathrm{Wh}} = \Delta_{\underline{j},\underline{B}}. \tag{6.3.11.2}$$

We now recall the explicit formula for $\Delta_{\underline{j},\underline{B}}$ given in [**Kot90**, §7], cf. also [**Mor11**, §3.2].(1) Let Λ be the set of $\underline{B}$-positive roots for $(G^*_{\mathbb{C}}, \underline{T}_{\mathbb{C}})$ which do not come from H via $\underline{j}$. Namely,

$$\Lambda = \left\{ \epsilon_i + \epsilon_k, \epsilon_i - \epsilon_k \mid 1 \le i \le m^-,\ m^- + 1 \le k \le m \right\}.$$

Fix a strongly regular element $\underline{\gamma} \in \underline{T}(\mathbb{R})$, and let $\gamma^H := \underline{j}^{-1}(\underline{\gamma}) \in T_H(\mathbb{R})$. Then

$$\Delta_{\underline{j},\underline{B}}(\gamma^H, \underline{\gamma}) = (-1)^{q(G^*)+q(H)} \chi_{\underline{B}}(\underline{\gamma}) \prod_{\alpha \in \Lambda} (1 - \alpha(\underline{\gamma})) = \chi_{\underline{B}}(\underline{\gamma}) \prod_{\alpha \in \Lambda} (1 - \alpha(\underline{\gamma})), \tag{6.3.11.3}$$

(1) Note the following typo in [**Mor11**, §3.2]: The term $(1 - \alpha(\gamma^{-1}))$ there should be $(1 - \alpha(\gamma))$.

where $\chi_{\underline{B}}$ is a quasi-character on $\underline{T}(\mathbb{R})$ whose definition is recalled in [**Mor11**, Def. 3.2.1]. In [**Mor11**, Ex. 3.2.4] Morel proves, in a special case, the following formula:

$$\chi_{\underline{B}} = (\rho_{B_H} \circ \underline{j}^{-1})\rho_{\underline{B}}^{-1}, \tag{6.3.11.4}$$

where $\rho_{\underline{B}}$ and ρ_{B_H} are defined to be the half sums of the $\underline{B}$-positive roots and the B_H-positive roots respectively, and they are actual (as opposed to square roots of) quasi-characters in the special case considered in *loc. cit.* In our case, $\rho_{\underline{B}}$ and ρ_{B_H} are again actual quasi-characters. We explain why (6.3.11.4) still holds in our case. In fact, in the proof of (6.3.11.4) in *loc. cit.*, the only special property being used is that the cocycle $a \in Z^1(W_{\mathbb{R}}, \widehat{\underline{T}})$ used to define $\chi_{\underline{B}}$ could be arranged so that it sends the element $\tau \in W_{\mathbb{R}}$ (see the beginning of [**Mor11**, §3.1]) to $1 \in \widehat{T}$. In our case, this condition is not even needed. This is because $\underline{T} \cong \mathrm{U}(1)^m$, and so the image of a in $\mathbf{H}^1(W_{\mathbb{R}}, \widehat{\underline{T}})$, which determines $\chi_{\underline{T}}$ via the local Langlands correspondence for $\underline{T}$, only depends on $a|_{W_{\mathbb{C}}} : W_{\mathbb{C}} \to \widehat{\underline{T}}$. Hence Morel's proof of (6.3.11.4) remains valid in our case.

By (6.3.11.4), we have

$$\chi_{\underline{B}} = -m^+\epsilon_1 - m^+\epsilon_2 - \cdots - m^+\epsilon_{m^-}. \tag{6.3.11.5}$$

Having identified both $\underline{T}$ and T_H with $\mathrm{U}(1)^m$ (via $f_{\underline{\mathcal{D}}}$ and $f_{\mathcal{D}_H}$ respectively), we write

$$\gamma^H = \underline{\gamma} = (y_1, y_2, \cdots, y_m)$$

with each $y_i \in \mathrm{U}(1)(\mathbb{R}) \subset \mathbb{C}^\times$. In conclusion, by (6.3.11.2), (6.3.11.3), and (6.3.11.5), we have

(6.3.11.6)

$$\underline{\Delta}_{\mathrm{Wh}}(\gamma^H, \underline{\gamma}) = \prod_{\substack{1 \le i \le m^- \\ m^-+1 \le k \le m}} y_i^{-1}(1 - y_i y_k^{-1})(1 - y_i y_k) = \prod_{\substack{1 \le i \le m^- \\ m^-+1 \le k \le m}} 2(\Re y_i - \Re y_k).$$

We now compute $\Delta_{\mathfrak{w}'}$. Let Δ_0 be the Langlands–Shelstad normalization associated to the splitting $\mathbf{spl}_{\mathrm{I}}$. In [**Wal10**] Waldspurger gives an explicit formula for Δ_0 excluding the factor Δ_{IV}. Let us denote the value of Waldspurger's formula by Δ_{Wal}, so that $\Delta_0 = \Delta_{\mathrm{Wal}}\Delta_{IV}$. Thus we have (see [**KS99**, §5.3], [**KS12**, §5.5])

$$\Delta_{\mathfrak{w}'} = \epsilon_L(U, \psi)\Delta_0 = \epsilon_L(U, \psi)\Delta_{\mathrm{Wal}}\Delta_{IV}, \tag{6.3.11.7}$$

where U is the virtual Γ_∞-representation $X^*(T_0) \otimes \mathbb{C} - X^*(T_{H,0}) \otimes \mathbb{C}$, with T_0 a maximal split torus in G^* and $T_{H,0}$ a maximal split torus in H, and $\epsilon_L(\cdot, \psi)$ is the local epsilon factor (according to the "Langlands normalization"; see [**KS99**, §5.3]) defined using the additive character $\psi : \mathbb{R} \to \mathbb{C}^\times, x \mapsto e^{2\pi i x}$ and the usual Lebesgue measure on $\mathbb{R}$ (which is self-dual with respect to ψ). Since G^* is split, T_0 is necessarily split, so $X^*(T_0)$ is a direct sum of trivial representations of Γ_∞. As for $X^*(T_{H,0})$, it is a direct sum of trivial representations when m^- is even, and a direct sum of trivial representations and two copies of $X^*(\mathrm{U}(1))$ when m^- is odd. Therefore, by [**Tat79**, (3.2.4), (3.4.1)] we have

$$\epsilon_L(U,\psi) = (-1)^{m^-}. \tag{6.3.11.8}$$

By definition we have

$$\Delta_{IV}(\gamma^H, \underline{\gamma}) = \prod_{\alpha\in\Lambda} \left|\alpha(\underline{\gamma})\right|^{-1/2} \left|1-\alpha(\underline{\gamma})\right| = \prod_{\substack{1\le i\le m^- \\ m^-+1\le k\le m}} 2\left|\Re y_i - \Re y_k\right|. \tag{6.3.11.9}$$

Waldspurger's explicit formula reads (see [**Wal10**, §1.10])

$$\Delta_{\mathrm{Wal}}(\gamma^H, \underline{\gamma}) = \prod_{i=1}^{m^-} \operatorname{sgn}\left(\eta_{\underline{V}}(\mathbf{spl}_{\mathrm{I}}) c_i (1+\Re y_i) \prod_{\substack{1\le k\le m \\ k\ne i}} (\Re y_i - \Re y_k)\right), \tag{6.3.11.10}$$

where $c_i \in \{\pm 1\}$ is such that $\underline{V}_i$ is c_i-definite. Recall that $(\underline{V}_i)_i \in \mathrm{ED}(\underline{V})^o_{\mathrm{Wh\text{-}I}}$, which implies $c_i = (-1)^{i+1}$. Note that $1+\Re y_i > 0$, and we have

$$\prod_{\substack{1\le i,k\le m^- \\ i\ne k}} \operatorname{sgn}(\Re y_i - \Re y_k) = (-1)^{m^-(m^--1)/2} = (-1)^{\lfloor m^-/2\rfloor},$$

$$\prod_{i=1}^{m^-} \operatorname{sgn} c_i = \prod_{i=1}^{m^-} (-1)^{i+1} = (-1)^{\lfloor m^-/2\rfloor}.$$

Therefore (6.3.11.10) can be rewritten as follows (remember that $\eta_{\underline{V}}(\mathbf{spl}_{\mathrm{I}}) = -1$)

$$\Delta_{\mathrm{Wal}}(\gamma^H, \gamma) = (-1)^{m^-} \prod_{\substack{1\le i\le m^- \\ m^-+1\le k\le m}} \operatorname{sgn}(\Re y_i - \Re y_k). \tag{6.3.11.11}$$

Combining (6.3.11.6) (6.3.11.7), (6.3.11.8), (6.3.11.9), and (6.3.11.11), we obtain (6.3.11.1), as desired. □

CHAPTER 7

TRANSFER MAPS DEFINED BY THE SATAKE ISOMORPHISM

In this chapter, we fix an odd prime p.

7.1. Recall of the Satake isomorphism

We recall the Satake isomorphism, following [**Car79, Bor79, HR10, ST16**]. Let F be a finite extension of $\mathbb{Q}_p$. Let q be the residue cardinality of F and let ϖ_F be a uniformizer of F. In this section we let G be an arbitrary unramified reductive group over F.

7.1.1. — Let K be the hyperspecial subgroup of $G(F)$ determined by a hyperspecial point v_0 in the building of G. Let S be a maximal split torus in G whose apartment contains v_0, and let T be the centralizer of S in G. Let Ω (resp. $\Omega(F)$) be the absolute (resp. relative) Weyl group of G defined using T (resp. S). In other words,

$$\Omega := \mathrm{Nor}_G(T)/T,$$
$$\Omega(F) := \mathrm{Nor}_G(S)/T.$$

There is a natural Γ_F-action on Ω, and $\Omega^{\Gamma_F} = \Omega(F)$. See [**Bor79**, §6.1] for more details.

We equip $G(F)$ with the Haar measure giving volume 1 to K. Let $\mathcal{H}(G(F) /\!\!/ K)$ be the Hecke algebra of $\mathbb{C}$-valued compactly supported locally constant K-bi-invariant distributions on $G(F)$. Using the fixed Haar measure, we identify $\mathcal{H}(G(F) /\!\!/ K)$ with the set of $\mathbb{C}$-valued compactly supported locally constant K-bi-invariant functions on $G(F)$. In the same way we define $\mathcal{H}(T(F) /\!\!/ T(F) \cap K)$, and we simply write it as $\mathcal{H}(T(F)/T(F) \cap K)$ since $T(F)$ is abelian. For any choice of a Borel subgroup B of

G containing T, the *Satake isomorphism* is the following $\mathbb{C}$-algebra isomorphism:
(7.1.1.1)

$$\mathcal{S}^G_{K,S} : \mathcal{H}(G(F) /\!\!/ K) \xrightarrow{\sim} \mathcal{H}(T(F)/T(F) \cap K)^{\Omega(F)}$$
$$f \longmapsto f_T,\ f_T(t) = \delta_{B(F)}(t)^{-1/2} \int_{N_B(F)} f(nt)dn,\ \forall t \in T(F),$$

where N_B is the unipotent radical of B, and we normalize the Haar measure dn on $N_B(F)$ such that $N_B(F) \cap K$ has volume 1. It is known that $\mathcal{S}^G_{K,S}$ depends only on K and S, not on B (see for instance [**ST16**, §6.1]).

7.1.2. — We explain how to make both sides of the Satake isomorphism more canonical, that is, independent of the choices of K and S. First note that we have canonical isomorphisms

$$\mathcal{H}(T(F)/T(F) \cap K) \cong \mathcal{H}(S(F)/S(F) \cap K) \cong \mathbb{C}[X_*(S)];$$

see [**Bor79**, §9.5] and cf. [**Car79**, §7.2]. Moreover, if S' is another maximal split torus in G, then there is a canonical isomorphism

$$\mathbb{C}[X_*(S)]^{\Omega(F)} \xrightarrow{\sim} \mathbb{C}[X_*(S')]^{\Omega'(F)}$$

induced by conjugation by any $g \in G(F)$ such that $gSg^{-1} = S'$. (Here $\Omega'(F)$ denotes the analogue of $\Omega(F)$ with S replaced by S'.) Let

$$\mathscr{A}_G := \varprojlim_S \mathbb{C}[X_*(S)]^{\Omega(F)},$$

where the projective limit is over all maximal split tori S in G, and the transition maps are the above-mentioned canonical isomorphisms. For our fixed v_0 and K, the Satake isomorphisms (7.1.1.1) for various choices of S whose apartments contain v_0 induce the same isomorphism

$$\mathcal{S}^G_K : \mathcal{H}(G(F) /\!\!/ K) \xrightarrow{\sim} \mathscr{A}_G. \tag{7.1.2.1}$$

This is because any such S extends to a maximal split torus in the reductive model of G over $\mathcal{O}_F$ corresponding to v_0, and hence any two such choices of S must be conjugate by an element of K; cf. [**SGA70**, XXVI, Prop. 6.16].

If K and K_1 are two different hyperspecial subgroups of $G(F)$, we have a canonical isomorphism

$$(\mathcal{S}^G_{K_1})^{-1} \circ \mathcal{S}^G_K : \mathcal{H}(G(F) /\!\!/ K) \xrightarrow{\sim} \mathcal{H}(G(F) /\!\!/ K_1),$$

where $\mathcal{S}^G_K$ and $\mathcal{S}^G_{K_1}$ are as in (7.1.2.1). In fact, this isomorphism can be described more concretely as follows. Recall that all hyperspecial subgroups of $G(F)$ are conjugate under $G^{\mathrm{ad}}(F)$. For any $g \in G^{\mathrm{ad}}(F)$ such that $\mathrm{Int}(g)(K_1) = K$, we have an isomorphism $\mathcal{H}(G(F) /\!\!/ K) \xrightarrow{\sim} \mathcal{H}(G(F) /\!\!/ K_1)$ sending each f to $f \circ \mathrm{Int}(g)$. We claim that this isomorphism is equal to $(\mathcal{S}^G_{K_1})^{-1} \circ \mathcal{S}^G_K$, and is in particular independent of the choice of g. To verify this, choose S with respect to K as in §7.1.1, and let

$S_1 := \mathrm{Int}(g^{-1})(S)$. Then S_1 is a maximal split torus in G whose apartment contains a hyperspecial point defining K_1. Let T (resp. T_1) be the centralizer of S (resp. S_1). By the functoriality of the definition (7.1.1.1), we only need to check that the map

$$\mathcal{H}(T(F)/T(F) \cap K)^{\Omega(F)} \longrightarrow \mathcal{H}(T_1(F)/T_1(F) \cap K_1)^{\Omega(F)}$$
$$f \longmapsto f \circ \mathrm{Int}(g)$$

is compatible with the canonical isomorphisms

$$\mathcal{H}(T(F)/T(F) \cap K)^{\Omega(F)} \cong \mathscr{A}_G \cong \mathcal{H}(T_1(F)/T_1(F) \cap K_1)^{\Omega(F)}.$$

For this, it suffices to check that the isomorphism $\mathbb{C}[X_*(S)]^{\Omega(F)} \xrightarrow{\sim} \mathbb{C}[X_*(S_1)]^{\Omega(F)}$ induced by $\mathrm{Int}(g) : S \xrightarrow{\sim} S_1$ is the same as that induced by $\mathrm{Int}(g_0) : S \xrightarrow{\sim} S_1$ for any $g_0 \in G(F)$ with $\mathrm{Int}(g_0)(S) = S_1$. We can further reduce to the case where $S = S_1$, and then it suffices to check that $\gamma = \mathrm{Int}(g)|_S \in \mathrm{Aut}(S)$ comes from $\Omega(F)$. This is true because γ lies in Ω and it stabilizes S. The claim is proved. We let

$$\mathcal{H}^{\mathrm{ur}}(G) := \varprojlim_K \mathcal{H}(G(F) /\!\!/ K),$$

where the projective limit is over all hyperspecial subgroups K and the transition maps are the canonical isomorphisms.

In conclusion, the Satake isomorphism can be viewed as a canonical $\mathbb{C}$-algebra isomorphism

$$\mathcal{S}^G : \mathcal{H}^{\mathrm{ur}}(G) \xrightarrow{\sim} \mathscr{A}_G, \tag{7.1.2.2}$$

where both sides are canonically associated to G, not depending on any extra choices.

7.1.3. — As in [**Bor79**, §6], the $\mathbb{C}$-algebra $\mathscr{A}_G$ has an alternative interpretation in terms of the L-group of G. To explain this, fix a finite unramified extension F'/F splitting G, and let σ_F be the arithmetic Frobenius generator of $\mathrm{Gal}(F'/F)$. Since F' splits G, we may form the L-group of G using $\mathrm{Gal}(F'/F)$. We use the symbol ${}^L G^{\mathrm{ur}}$ to denote this version of the L-group, i.e.,

$${}^L G^{\mathrm{ur}} := \widehat{G} \rtimes \mathrm{Gal}(F'/F) = \widehat{G} \rtimes \langle \sigma_F \rangle.$$

Inside the $\mathbb{C}$-algebra of $\mathbb{C}$-valued functions on the set of semi-simple $\widehat{G}$-conjugacy classes in $\widehat{G} \rtimes \sigma_F$, we let

$$\mathbb{C}[\mathrm{ch}({}^L G^{\mathrm{ur}})]$$

be the sub-algebra generated by the restrictions of characters of finite-dimensional representations of ${}^L G^{\mathrm{ur}}$. Then there is a canonical isomorphism

$$\mathscr{A}_G \cong \mathbb{C}[\mathrm{ch}({}^L G^{\mathrm{ur}})] \tag{7.1.3.1}$$

characterized as follows. Let $f \in \mathscr{A}_G$. Fix a maximal split torus S in G, and let T be the centralizer of S. Then $\mathscr{A}_G \cong \mathbb{C}[X_*(S)]^{\Omega(F)} \subset \mathbb{C}[X_*(T)]$, so we can view f as a function on the $\mathbb{C}$-torus $\widehat{T}$. As usual (cf. §5.3.1), $\widehat{G}$ is equipped with a Borel pair $(\mathcal{T}, \mathcal{B})$ and an isomorphism $\mathrm{BRD}(G) \xrightarrow{\sim} \mathrm{BRD}(\mathcal{T}, \mathcal{B})^\vee$. In particular, if we choose a

Borel subgroup B of G containing T, then we get an isomorphism of $\mathbb{C}$-tori $\widehat{T} \xrightarrow{\sim} \mathcal{T}$. In this way we obtain from f a function $f_{\mathcal{T}} : \mathcal{T} \to \mathbb{C}$. The construction $f \mapsto f_{\mathcal{T}}$ is independent of the choices of S and B. The image of f under (7.1.3.1) is characterized by the condition that its value at the $\widehat{G}$-conjugacy class of $t \rtimes \sigma_F$ is equal to $f_{\mathcal{T}}(t)$, for all $t \in \mathcal{T}$.

In the sequel, we shall often make the identification (7.1.3.1) without explicitly mentioning it. Thus we can evaluate an element of $\mathscr{A}_G$ at a semi-simple $\widehat{G}$-conjugacy class in $\widehat{G} \rtimes \sigma_F$ to get a complex number.

In view of (7.1.3.1), we can also view the Satake isomorphism as a canonical isomorphism

$$\mathcal{S}^G : \mathcal{H}^{\mathrm{ur}}(G) \xrightarrow{\sim} \mathbb{C}[\mathrm{ch}({}^L G^{\mathrm{ur}})]. \tag{7.1.3.2}$$

7.1.4. — Next we recall a result of Kottwitz. Let λ be a cocharacter of G defined over F. Assume that λ is *minuscule*, in the sense that the representation $\mathrm{Ad} \circ \lambda$ of $\mathbb{G}_m$ on $\mathrm{Lie}\, G_{\overline{F}}$ has no weights other than $\{-1, 0, 1\}$. Let K and S be as in §7.1.1, and assume that λ factors through S. Denote by $\Omega(F) \cdot \lambda$ the $\Omega(F)$-orbit of λ in $X_*(S)$. Let $f_{K,\lambda} \in \mathcal{H}(G(F) /\!\!/ K)$ be the characteristic function of $K\lambda(\varpi_F)K$ inside $G(F)$. By the Cartan decomposition, the dependence of $f_{K,\lambda}$ on λ is only through the set $\Omega(F) \cdot \lambda$.

Theorem 7.1.5 (**[Kot84a**, Lem. 1.1.3, §2]**).** — *We have*

$$\mathcal{S}^G_{K,S}(f_{K,\lambda}) = q^{\langle \rho, \lambda_{\mathrm{dom}} \rangle} \sum_{\lambda' \in \Omega(F)\cdot\lambda} [\lambda'] \ \in \mathbb{C}[X_*(S)]^{\Omega(F)},$$

where ρ is the half sum of a fixed set of positive (absolute) roots in $X^*(Z_G(S))$, and λ_{dom} is any element of $\Omega(F) \cdot \lambda$ which is dominant with respect to the same choice of positive roots. Moreover, the element of $\mathscr{A}_G$ corresponding to $\mathcal{S}^G_{K,S}(f_{K,\lambda}) \in \mathbb{C}[X_*(S)]^{\Omega(F)}$ depends only on the $G(F)$-conjugacy class of λ, not on K or S. □

Definition 7.1.6. — Let λ be a minuscule cocharacter of G defined over F. We write

$$f_\lambda \in \mathcal{H}^{\mathrm{ur}}(G)$$

for the element corresponding to $f_{K,\lambda} \in \mathcal{H}(G(F) /\!\!/ K)$, for some choice of K and S as in §7.1.1 such that λ factors through S. By Theorem 7.1.5, f_λ depends only on the $G(F)$-orbit of λ, not on any extra choices.

7.1.7. — We now discuss the compatibility between the Satake isomorphisms and the constant term maps. Let K, S, and T be as in §7.1.1. Let M be a Levi component of a parabolic subgroup P of G. Assume that $M \supset T$. Let N_P be the unipotent radical of P. Then $M(F) \cap K$ is a hyperspecial subgroup of $M(F)$. We define the *constant* *term map*

(7.1.7.1)

$$(\cdot)_M : \mathcal{H}(G(F) /\!\!/ K) \longrightarrow \mathcal{H}(M(F) /\!\!/ M(F) \cap K)$$

$$f \longmapsto f_M,\ f_M(m) = \delta_{P(F)}(m)^{-1/2} \int_{N_P(F)} f(nm)dn,\ m \in M(F),$$

where the Haar measure dn on $N_P(F)$ is normalized by the condition that $N_P(F) \cap K$ has volume 1.

Remark 7.1.8. — The constant term map can be defined more generally for C_c^∞ functions; see for instance [**GKM97**, §7.13] or [**ST16**, §6.1]. In [**ST16**] the map (7.1.7.1) is called the *partial Satake transform.* When $M = T$, the map (7.1.7.1) is the same as $\mathcal{S}^G_{K,S}$ in (7.1.1.1).

Lemma 7.1.9. — *In the setting of §7.1.7, let $\Omega_M(F)$ be the relative Weyl group of M defined using S. Then $\Omega_M(F)$ is a subgroup of $\Omega(F)$ when both groups are viewed as subgroups of* $\mathrm{GL}(X_*(S))$. *Moreover, we have a commutative diagram:*

$$\begin{array}{ccc} \mathcal{H}(G(F) /\!\!/ K) & \xrightarrow{\mathcal{S}^G_{K,S}} & \mathbb{C}[X_*(S)]^{\Omega(F)} \\ \big\downarrow{\scriptstyle (\cdot)_M} & & \big\downarrow \\ \mathcal{H}(M(F) /\!\!/ M(F) \cap K) & \xrightarrow{\mathcal{S}^M_{M(F)\cap K,S}} & \mathbb{C}[X_*(S)]^{\Omega_M(F)} \end{array}$$

where the right vertical arrow is the inclusion.

Proof. — This is well known. See for instance [**HR10**, §12.3] or [**ST16**, §2, §6]. □

Proposition 7.1.10. — *In the setting of §7.1.7, the constant term map (7.1.7.1) induces a canonical map*

(7.1.10.1) $$(\cdot)_M : \mathcal{H}^{\mathrm{ur}}(G) \longrightarrow \mathcal{H}^{\mathrm{ur}}(M)$$

which depends only on M, not on K, S, P.

Proof. — This follows from Lemma 7.1.9, and the fact that for all maximal split tori S in M, the inclusion maps $\mathbb{C}[X_*(S)]^{\Omega(F)} \to \mathbb{C}[X_*(S)]^{\Omega_M(F)}$ induce the same map $\mathscr{A}_G \to \mathscr{A}_M$. □

Remark 7.1.11. — There is a canonical $\widehat{G}$-conjugacy class of embeddings ${}^L M^{\mathrm{ur}} \hookrightarrow {}^L G^{\mathrm{ur}}$, and these embeddings induce via pull-back a common canonical map

(7.1.11.1) $$\mathbb{C}[\mathrm{ch}({}^L G^{\mathrm{ur}})] \longrightarrow \mathbb{C}[\mathrm{ch}({}^L M^{\mathrm{ur}})].$$

Under the canonical Satake isomorphism (7.1.3.2) and its analogue for M, the canonical constant term map (7.1.10.1) corresponds to (7.1.11.1); cf. [**ST16**, Rmk. 2.8]. From this description, one sees that (7.1.10.1) depends on the embedding $M \hookrightarrow G$ only up to $G(F)$-conjugacy.

7.2. The twisted transfer map

We recall the formalism of the twisted transfer map. We keep the notation and setting of §7.1. We still let G be an arbitrary unramified reductive group over F. Fix a positive integer a and let F_a be the degree a unramified extension of F.

7.2.1. — We first recall some facts concerning Weil restriction of scalars.

Let $R := \operatorname{Res}_{F_a/F} G$. Then $\widehat{R}$ together with the $\operatorname{Gal}(F^{\mathrm{ur}}/F)$-action on it can be identified with $\prod_{i=1}^{a} \widehat{G}$, on which the arithmetic Frobenius generator σ_F of $\operatorname{Gal}(F^{\mathrm{ur}}/F)$ acts by

$$\sigma_F(x_1, \cdots, x_a) = (\sigma_F(x_2), \cdots, \sigma_F(x_{a-1}), \sigma_F(x_1)).$$

We have a canonical isomorphism $\mathscr{A}_R \cong \mathscr{A}_{G_{F_a}}$, where $\mathscr{A}_{G_{F_a}}$ is formed with respect to G_{F_a} over the base field F_a instead of F. This isomorphism is characterized as follows. Let S' be a maximal F_a-split torus in G_{F_a}. Then $\operatorname{Res}_{F_a/F} S'$ is an F-rational torus in R, and its maximal F-split subtorus U is a maximal F-split torus in R. We have

$$(\operatorname{Res}_{F_a/F} S') \otimes_F F_a \cong \prod_{\iota \in \operatorname{Gal}(F_a/F)} S'.$$

Let $\pi : (\operatorname{Res}_{F_a/F} S') \otimes_F F_a \to S'$ be the projection to the factor corresponding to $\mathrm{id} \in \operatorname{Gal}(F_a/F)$. Composing the inclusion map $U \hookrightarrow \operatorname{Res}_{F_a/F} S'$ (or more precisely, its base change to F_a) with π, we obtain a map $U_{F_a} \to S'$, which is in fact an F_a-isomorphism. The resulting isomorphism $X_*(U) \xrightarrow{\sim} X_*(S')$ then induces the canonical isomorphism $\mathscr{A}_R \cong \mathscr{A}_{G_{F_a}}$.

Under the isomorphism $\mathscr{A}_R \cong \mathscr{A}_{G_{F_a}}$, suppose an element $f' \in \mathscr{A}_R$ corresponds to $f \in \mathscr{A}_{G_{F_a}}$. We would like to have a formula, in terms of f, for the evaluation of f' at an element

$$(g_1, \cdots, g_a) \rtimes \sigma_F \in {}^L R^{\mathrm{ur}} = (\prod_{i=1}^{a} \widehat{G}) \rtimes \langle \sigma_F \rangle,$$

where $g_1, \cdots, g_a$ are arbitrary semi-simple elements of $\widehat{G}$. (Here $\langle \sigma_F \rangle$ is understood as either the unramified Weil group W_F^{ur} or a sufficiently large finite quotient of it. In all cases σ_F is a generator.) Working through the definitions, we obtain the desired formula as follows:

$$f'((g_1, \cdots, g_a) \rtimes \sigma_F) = f(g_1 \sigma(g_2) \cdots \sigma^{a-1}(g_a) \rtimes \sigma_F^a). \tag{7.2.1.1}$$

Here, $g_1 \sigma(g_2) \cdots \sigma^{a-1}(g_a) \rtimes \sigma_F^a$ is an element of ${}^L(G_{F_a})^{\mathrm{ur}} = \widehat{G} \rtimes \langle \sigma_F^a \rangle$, the unramified Langlands dual group of G_{F_a} formed with respect to the base field F_a (so the Galois part is generated by σ_F^a), and hence we can evaluate f at this element.

7.2.2. — Consider an endoscopic datum $(H, \mathcal{H}, s, \eta)$ for G. For simplicity, assume that $\mathcal{H} = {}^L H$ and $s \in \eta(Z(\widehat{H})^{\Gamma_F})$; these assumptions will be met in our applications.

We assume that $(H, {}^L H, s, \eta)$ is *unramified*, meaning that the following two conditions are satisfied:

(1) The group H is unramified over F. In particular, the action of Γ_F on $\widehat{H}$ factors through $\mathrm{Gal}(F^{\mathrm{ur}}/F)$.

(2) The map $\eta : {}^L H \to {}^L G$ is induced by an L-embedding ${}^L H^{\mathrm{ur}} \to {}^L G^{\mathrm{ur}}$. Here ${}^L H^{\mathrm{ur}}$ and ${}^L G^{\mathrm{ur}}$ denote the L-groups formed with Γ', where Γ' is either the unramified Weil group W_F^{ur} or a sufficiently large finite quotient of it. In all cases we denote by σ_F the arithmetic Frobenius generator of Γ'.

Let $R = \mathrm{Res}_{F_a/F}\, G$. Define a homomorphism

$$\tilde{\eta} : {}^L H^{\mathrm{ur}} = \widehat{H} \rtimes \langle \sigma_F \rangle \longrightarrow {}^L R^{\mathrm{ur}} = \left(\prod_{i=1}^{a} \widehat{G} \right) \rtimes \langle \sigma_F \rangle,$$

by

$$\widehat{H} \ni x \longmapsto (\eta(x), \cdots, \eta(x)) \rtimes 1,$$

and

$$1 \rtimes \sigma_F \longmapsto (s^{-1}\eta(\sigma_F)\sigma_F^{-1}, \eta(\sigma_F)\sigma_F^{-1}, \cdots, \eta(\sigma_F)\sigma_F^{-1}) \rtimes \sigma_F. \tag{7.2.2.1}$$

Let

$$\tilde{\eta}^* : \mathbb{C}[\mathrm{ch}({}^L R^{\mathrm{ur}})] \longrightarrow \mathbb{C}[\mathrm{ch}({}^L H^{\mathrm{ur}})]$$

be the map induced by the pull-back along $\tilde{\eta}$. As we have explained in §7.1.3, the source and target of $\tilde{\eta}^*$ are canonically identified with $\mathscr{A}_R$ and $\mathscr{A}_H$ respectively. Also, as in §7.2.1 we have $\mathscr{A}_R \cong \mathscr{A}_{G_{F_a}}$. We can thus view $\tilde{\eta}^*$ as a map

$$\tilde{\eta}^* : \mathscr{A}_{G_{F_a}} \longrightarrow \mathscr{A}_H.$$

We call this map the *twisted transfer map*. If we identify the two sides with $\mathcal{H}^{\mathrm{ur}}(G_{F_a})$ and $\mathcal{H}^{\mathrm{ur}}(H)$ respectively using the canonical Satake isomorphisms, we obtain a map $\mathcal{H}^{\mathrm{ur}}(G_{F_a}) \to \mathcal{H}^{\mathrm{ur}}(H)$ which is also called the twisted transfer map.

Lemma 7.2.3. — *Let $f \in \mathscr{A}_{G_{F_a}}$, and let x be a semi-simple element of $\widehat{H}$. Write $\eta(x \rtimes \sigma_F)^a = z \rtimes \sigma_F^a$, with $z \in \widehat{G}$. Then the evaluation of $\tilde{\eta}^*(f) \in \mathscr{A}_H$ at $x \rtimes \sigma_F \in {}^L H^{\mathrm{ur}}$ is equal to*

$$f(s^{-1} z \rtimes \sigma_F^a).$$

Here we have $s^{-1}z \in \widehat{G}$, and $s^{-1}z \rtimes \sigma_F^a$ is an element of ${}^L(G_{F_a})^{\mathrm{ur}} = \widehat{G} \rtimes \langle \sigma_F^a \rangle$, so we can evaluate f at $s^{-1}z \rtimes \sigma_F^a$.

Proof. — Write y for $\eta(x \rtimes \sigma_F)\sigma_F^{-1} \in \widehat{G}$. Let $f' \in \mathscr{A}_R$ be the element corresponding to f under $\mathscr{A}_R \cong \mathscr{A}_{G_{F_a}}$. We compute

$$\begin{aligned} \tilde{\eta}^*(f)(x \rtimes \sigma_F) &= f'(\tilde{\eta}(x \rtimes \sigma_F)) = f'((s^{-1}y, y, \cdots, y) \rtimes \sigma_F) \\ &= f(s^{-1} y \sigma(y) \cdots \sigma^{a-1}(y) \rtimes \sigma_F^a) \\ &= f(s^{-1} z \rtimes \sigma_F^a). \end{aligned}$$

Here the third equality follows from (7.2.1.1). □

Remark 7.2.4. — In the above definition of $\tilde{\eta}$ we have taken advantage of the simplifying assumptions $\mathcal{H} = {}^L H$ and $s \in \eta(Z(\widehat{H})^{\Gamma_F})$. For the definition in more general situations, see [**Kot90**, §7] or [**KSZ**, §7.4]. Under our simplifying assumptions, the formula (7.2.2.1) can also be replaced by

$$1 \rtimes \sigma_F \longmapsto (t_1\eta(\sigma_F)\sigma_F^{-1}, t_2\eta(\sigma_F)\sigma_F^{-1}, \cdots, t_a\eta(\sigma_F)\sigma_F^{-1}) \rtimes \sigma_F$$

for any choices of $t_1, \cdots, t_a \in \eta(Z(\widehat{H})^{\Gamma_F})$ such that $t_1t_2\cdots t_a = s^{-1}$. In fact, such a replacement does not change the conclusion of Lemma 7.2.3. We have chosen $t_1 = s^{-1}$ and $t_2 = \cdots = t_a = 1$ for definiteness.

7.2.5. — As a special case of the twisted transfer map, consider the trivial endoscopic datum $(G, {}^L G, 1, \mathrm{id})$ for G, which makes sense since G is quasi-split. Then we obtain the so-called *base change map*

$$\mathscr{A}_{G_{F_a}} \longrightarrow \mathscr{A}_G,$$

also viewed as a map

$$\mathcal{H}^{\mathrm{ur}}(G_{F_a}) \to \mathcal{H}^{\mathrm{ur}}(G).$$

7.3. Explicit description of the twisted transfer map

We now make the construction in §7.2 explicit for unramified special orthogonal groups.

7.3.1. — We first make explicit the group $\mathscr{A}_G$ and the evaluation of its elements at semi-simple $\widehat{G}$-conjugacy classes in $\widehat{G} \rtimes \sigma_F$.

We now keep the setting and notation of §5, specialized to the case where F is a finite extension of $\mathbb{Q}_p$. In particular, G denotes $\mathrm{SO}(V)$ where V is a quadratic space over F of dimension d and discriminant δ. As always we write m for $\lfloor d/2 \rfloor$. Assume that G is unramified over F. By Proposition 1.2.8, if d is odd, or if d is even and δ is trivial, our assumption implies that G is split. If d is even and δ is non-trivial, our assumption implies that δ has a representative in $\mathcal{O}_F^{\times}/\mathcal{O}_F^{\times,2}$, and that G is split over $F(\alpha)$; here recall that $\alpha \in \overline{F}$ is a fixed square root of a fixed lift of δ in $F^{\times}$.

To simplify notation, for each positive integer n we define

$$\mathscr{A}_{\mathsf{B}}[X_1, \cdots, X_n] := \mathbb{C}[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]^{\{\pm 1\}^n \rtimes \mathfrak{S}_n},$$
$$\mathscr{A}_{\mathsf{D}}[X_1, \cdots, X_n] := \mathbb{C}[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]^{(\{\pm 1\}^n)' \rtimes \mathfrak{S}_n}.$$

Here the group $\{\pm 1\}^n \rtimes \mathfrak{S}_n$ acts on $\mathbb{C}[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]$ as follows. The non-trivial element of the i-th copy of $\{\pm 1\}$ acts by swapping X_i and X_i^{-1}, and $\mathfrak{S}_n$ acts by permuting the n variables $X_1, \cdots, X_n$ (and simultaneously permuting $X_1^{-1}, \cdots, X_n^{-1}$). As usual, $(\{\pm 1\}^n)'$ is the kernel of the multiplication map $\{\pm 1\}^n \to \{\pm 1\}$. When $n = 1$, by definition we have $\mathscr{A}_{\mathsf{D}}[X_1] = \mathbb{C}[X_1^{\pm 1}]$.

First assume that d is odd. Then G is split. Fix a Borel pair (T, B) in G. We then get an isomorphism $\mathrm{BRD}(T, B) \xrightarrow{\sim} \mathrm{BRD}(\mathcal{T}, \mathcal{B})^\vee$ from the L-group datum fixed in §5.3. The right hand side is canonically identified with $\mathrm{BRD}(\mathsf{B}_m)$. Thus we get an isomorphism $X_*(T) \xrightarrow{\sim} \mathbb{Z}^m$, and an isomorphism

$$\mathscr{A}_G \cong \mathbb{C}[X_*(T)]^\Omega \xrightarrow{\sim} \mathbb{C}[\mathbb{Z}^m]^{\{\pm 1\}^m \rtimes \mathfrak{S}_m} \cong \mathscr{A}_\mathsf{B}[X_1, \cdots, X_m],$$

which is independent of the choice of (T, B). If an element of $\mathscr{A}_G$ corresponds to $F(X_1, \cdots, X_m) \in \mathscr{A}_\mathsf{B}[X_1, \cdots, X_m]$, then the evaluation of this element at $\mathrm{symdiag}(t_1, \cdots, t_m) \rtimes \sigma_F \in \mathcal{T} \rtimes \sigma_F$ (see §5.2.1 for the notation) is given by $F(t_1, \cdots, t_m) \in \mathbb{C}$.

If d is even and δ is trivial, then G is still split, and similarly as in the odd case we have a canonical identification

$$\mathscr{A}_G \cong \mathscr{A}_\mathsf{D}[X_1, \cdots, X_m].$$

(This is true for $m = 1$ as well.) As in the odd case, the evaluation of an element of $\mathscr{A}_G$ corresponding to $F(X_1, \cdots, X_m) \in \mathscr{A}_\mathsf{D}[X_1, \cdots, X_m]$ at $\mathrm{symdiag}(t_1, \cdots, t_m) \rtimes \sigma_F \in \mathcal{T} \rtimes \sigma_F$ is given by $F(t_1, \cdots, t_m)$.

Now consider the case where d is even and δ is non-trivial. Let S be a maximal split torus in G, let T be the centralizer of S, and let B be a Borel subgroup of G containing T. We then get an isomorphism $\mathrm{BRD}(T, B) \xrightarrow{\sim} \mathrm{BRD}(\mathcal{T}, \mathcal{B})^\vee$ from the L-group datum fixed in §5.3. The right hand side is canonically identified with $\mathrm{BRD}(\mathsf{D}_m)$. We thus get an isomorphism $X_*(T) \xrightarrow{\sim} \mathbb{Z}^m$. Under this isomorphism, $X_*(S) = X_*(T)^{\Gamma_F}$ corresponds to the subgroup $\mathbb{Z}^{m-1} \times \{0\} = \{(x_1, \cdots, x_{m-1}, 0) \mid x_i \in \mathbb{Z}\}$ of $\mathbb{Z}^m$, and the $\Omega(F)$-action on $X_*(T)$ corresponds to the natural action of $(\{\pm 1\}^m)' \rtimes \mathfrak{S}_{m-1}$ on $\mathbb{Z}^m$, that is, the non-trivial element of the i-th copy of $\{\pm 1\}$ acts by multiplication by -1 on the i-th coordinate, and $\mathfrak{S}_{m-1}$ acts by permuting the first $m-1$ coordinates. We have natural identifications

$$\mathbb{C}[\mathbb{Z}^{m-1} \times \{0\}]^{(\{\pm 1\}^m)' \rtimes \mathfrak{S}_{m-1}} \cong \mathbb{C}[\mathbb{Z}^{m-1}]^{\{\pm 1\}^{m-1} \rtimes \mathfrak{S}_{m-1}} \cong \mathscr{A}_\mathsf{B}[X_1, \cdots, X_{m-1}].$$

Hence we obtain an identification

$$\mathscr{A}_G \cong \mathscr{A}_\mathsf{B}[X_1, \cdots, X_{m-1}].$$

As in the previous cases, this identification is independent of the choices of S and B. If an element of $\mathscr{A}_G$ corresponds to $F(X_1, \cdots, X_{m-1}) \in \mathscr{A}_\mathsf{B}[X_1, \cdots, X_{m-1}]$, then the evaluation of this element at $\mathrm{symdiag}(t_1, \cdots, t_m) \rtimes \sigma_F \in \mathcal{T} \rtimes \sigma_F$ is given by $F(t_1, \cdots, t_{m-1})$.

7.3.2. — Let G be as in §7.3.1. In §5.4, we constructed representatives $\mathfrak{e}_{d^+,\delta^+,d^-,\delta^-}$ of the isomorphism classes of elliptic endoscopic data for G, where $(d^+, \delta^+, d^-, \delta^-)$ belongs to a set $\mathscr{P}_V$ as in Definition 5.4.2. In order to ensure ellipticity, in the definition of $\mathscr{P}_V$ we have the condition that if d is even and at least 4 then neither of (d^+, δ^+) and (d^-, δ^-) is equal to $(2, 1)$. We now take a quadruple $(d^+, \delta^+, d^-, \delta^-)$ satisfying all the conditions in the definition of $\mathscr{P}_V$ except the condition just mentioned. The construction in §5.4 still applies to $(d^+, \delta^+, d^-, \delta^-)$ and yields an endoscopic datum

$$\mathfrak{e}_{d^+,\delta^+,d^-,\delta^-} = (H, {}^L H, s, \eta)$$

for G, which may no longer be elliptic. In fact, the non-elliptic endoscopic data for G arising in this way account for all the non-elliptic endoscopic data (up to isomorphism) that can possibly appear as the localization of global elliptic endoscopic data, in the case where G is the localization of a special orthogonal group over a number field.

Throughout we assume that $d^+ \neq 0$. We now assume that $F = \mathbb{Q}_p$, and write σ for σ_F. We keep assuming that G is unramified. As in §7.2.2, we assume that the endoscopic datum $\mathfrak{e}_{d^+,\delta^+,d^-,\delta^-}$ is unramified. In the odd case the last assumption is automatic, and in the even case it implies that δ^+ and δ^- both have (unique) representatives in $\mathbb{Z}_p^\times/\mathbb{Z}_p^{\times,2}$, in view of Proposition 1.2.8. It is easy to check that the converse is also true. Note that $\mathbb{Z}_p^\times/\mathbb{Z}_p^{\times,2} \cong \mathbb{Z}/2\mathbb{Z}$ as p is odd. Hence each of $\delta, \delta^+, \delta^-$ can take only two values: the trivial or the non-trivial element of $\mathbb{Z}_p^\times/\mathbb{Z}_p^{\times,2}$.

Fix a positive integer a. We still write F_a for the degree a unramified extension of $F = \mathbb{Q}_p$. We now make explicit the twisted transfer map $\tilde{\eta}^* : \mathscr{A}_{G_{F_a}} \to \mathscr{A}_H$ defined in §7.2.2. As always we write m for $\lfloor d/2 \rfloor$, and write $m^\pm$ for $\lfloor d^\pm/2 \rfloor$.

7.3.2.1. *The odd case.* — In this case, $\mathscr{A}_{G_{F_a}}$ is identified with $\mathscr{A}_{\mathsf{B}}[X_1, \cdots, X_m]$, and $\mathscr{A}_H = \mathscr{A}_{H^+} \otimes_{\mathbb{C}} \mathscr{A}_{H^-}$ is identified with

$$\mathscr{A}_{\mathsf{B}}[Z_1, \cdots, Z_{m^+}] \otimes_{\mathbb{C}} \mathscr{A}_{\mathsf{B}}[Y_1, \cdots, Y_{m^-}],$$

which we identify with a $\mathbb{C}$-subalgebra of $\mathbb{C}[Z_1^{\pm 1}, \cdots, Z_{m^+}^{\pm 1}, Y_1^{\pm 1}, \cdots, Y_{m^-}^{\pm 1}]$. Consider an element

$$t_{\widehat{H}} = (\mathrm{symdiag}(t_1, \cdots, t_{m^+}), \mathrm{symdiag}(u_1, \cdots, u_{m^-}))$$

of the maximal torus $\mathcal{T}_{\widehat{H}} = \mathcal{T}_{V^+} \times \mathcal{T}_{V^-}$ in $\widehat{H}$. We have

$$\eta(t_{\widehat{H}} \rtimes \sigma) = \mathrm{symdiag}(u_1, \cdots, u_{m^-}, t_1, \cdots, t_{m^+}) \rtimes \sigma \in \mathcal{T} \rtimes \sigma.$$

Since σ acts trivially on $\mathcal{T}$, we have

$$\eta(t_{\widehat{H}} \rtimes \sigma)^a = \mathrm{symdiag}(u_1^a, \cdots, u_{m^-}^a, t_1^a, \cdots, t_{m^+}^a) \rtimes \sigma^a,$$
$$s^{-1}\eta(t_{\widehat{H}} \rtimes \sigma)^a = \mathrm{symdiag}(-u_1^a, \cdots, -u_{m^-}^a, t_1^a, \cdots, t_{m^+}^a) \rtimes \sigma^a.$$

Suppose $f \in \mathscr{A}_{G_{F_a}}$ corresponds to $F(X_1, \cdots, X_m) \in \mathscr{A}_{\mathsf{B}}[X_1, \cdots, X_m]$. By Lemma 7.2.3, the evaluation of $\tilde{\eta}^*(f)$ at $t_{\widehat{H}} \rtimes \sigma$ is equal to

$$F(-u_1^a, \cdots, -u_{m^-}^a, t_1^a, \cdots, t_{m^+}^a).$$

Thus the map $\tilde{\eta}^*$ is explicitly given by

$$\mathscr{A}_{\mathsf{B}}[X_1, \cdots, X_m] \longrightarrow \mathscr{A}_{\mathsf{B}}[Z_1, \cdots, Z_{m^+}] \otimes_{\mathbb{C}} \mathscr{A}_{\mathsf{B}}[Y_1, \cdots, Y_{m^-}]$$
$$F(X_1, \cdots, X_m) \longmapsto F(-Y_1^a, \cdots, -Y_{m^-}^a, Z_1^a, \cdots, Z_{m^+}^a).$$

7.3.2.2. *The even case, with trivial* δ^+ *and trivial* δ^-. — In this case, δ is also trivial since $\delta = \delta^+\delta^-$. We have

$$\mathscr{A}_{G_{F_a}} \cong \mathscr{A}_{\mathsf{D}}[X_1, \cdots, X_m],$$

and

$$\mathscr{A}_H = \mathscr{A}_{H^+} \otimes_{\mathbb{C}} \mathscr{A}_{H^-} \cong \mathscr{A}_{\mathsf{D}}[Z_1, \cdots, Z_{m^+}] \otimes_{\mathbb{C}} \mathscr{A}_{\mathsf{D}}[Y_1, \cdots, Y_{m^-}].$$

By similar computation as in §7.3.2.1, we find that $\tilde{\eta}^*$ is explicitly given by

$$\begin{aligned} \mathscr{A}_{\mathsf{D}}[X_1, \cdots, X_m] &\longrightarrow \mathscr{A}_{\mathsf{D}}[Z_1, \cdots, Z_{m^+}] \otimes_{\mathbb{C}} \mathscr{A}_{\mathsf{D}}[Y_1, \cdots, Y_{m^-}] \\ F(X_1, \cdots, X_m) &\longmapsto F(-Y_1^a, \cdots, -Y_{m^-}^a, Z_1^a, \cdots, Z_{m^+}^a). \end{aligned}$$

7.3.2.3. *The even case, with non-trivial* δ^+ *and trivial* δ^-. — In this case, δ is non-trivial in $\mathbb{Z}_p^\times/\mathbb{Z}_p^{\times,2}$. It is a square in $F_a^\times$ if and only if a is even. Thus we have

$$\mathscr{A}_{G_{F_a}} \cong \begin{cases} \mathscr{A}_{\mathsf{D}}[X_1, \cdots, X_m], & \text{if } a \text{ is even}, \\ \mathscr{A}_{\mathsf{B}}[X_1, \cdots, X_{m-1}], & \text{if } a \text{ is odd}, \end{cases}$$

and

$$\mathscr{A}_H = \mathscr{A}_{H^+} \otimes_{\mathbb{C}} \mathscr{A}_{H^-} \cong \mathscr{A}_{\mathsf{B}}[Z_1, \cdots, Z_{m^+-1}] \otimes_{\mathbb{C}} \mathscr{A}_{\mathsf{D}}[Y_1, \cdots, Y_{m^-}].$$

Consider an element

$$t_{\widehat{H}} = (\mathrm{symdiag}(t_1, \cdots, t_{m^+}), \mathrm{symdiag}(u_1, \cdots, u_{m^-})) \in \mathcal{T}_{\widehat{H}} = \mathcal{T}_{V^+} \times \mathcal{T}_{V^-}.$$

Since δ^- is trivial, σ belongs to the first case in (5.4.3.2). Hence

$$\eta(t_{\widehat{H}} \rtimes \sigma) = \mathrm{symdiag}(u_1, \cdots, u_{m^-}, t_1, \cdots, t_{m^+}) \rtimes \sigma \in \mathcal{T} \rtimes \sigma.$$

Now the action of σ on $\mathcal{T}$ sends $\mathrm{symdiag}(x_1, \cdots, x_m)$ to $\mathrm{symdiag}(x_1, \cdots, x_{m-1}, x_m^{-1})$; cf. §5.3.2. We introduce the notation

$$\nu_a := \frac{(-1)^{a+1} + 1}{2}. \tag{7.3.2.1}$$

Hence

$$\begin{aligned} \eta(t_{\widehat{H}} \rtimes \sigma)^a &= \mathrm{symdiag}(u_1^a, \cdots, u_{m^-}^a, t_1^a, \cdots, t_{m^+-1}^a, t_{m^+}^{\nu_a}) \rtimes \sigma^a, \\ s^{-1}\eta(t_{\widehat{H}} \rtimes \sigma)^a &= \mathrm{symdiag}(-u_1^a, \cdots, -u_{m^-}^a, t_1^a, \cdots, t_{m^+-1}^a, t_{m^+}^{\nu_a}) \rtimes \sigma^a, \end{aligned}$$

Suppose a is even. — Suppose $f \in \mathscr{A}_{G_{F_a}}$ corresponds to $F(X_1, \cdots, X_m) \in \mathscr{A}_{\mathsf{D}}[X_1, \cdots, X_m]$. By Lemma 7.2.3, the evaluation of $\tilde{\eta}^*(f)$ at $t_{\widehat{H}} \rtimes \sigma$ is equal to

$$F(-u_1^a, \cdots, -u_{m^-}^a, t_1^a, \cdots, t_{m^+-1}^a, 1).$$

Thus the map $\tilde{\eta}^*$ is explicitly given by

$$\begin{aligned} \mathscr{A}_{\mathsf{D}}[X_1, \cdots, X_m] &\longrightarrow \mathscr{A}_{\mathsf{B}}[Z_1, \cdots, Z_{m^+-1}] \otimes_{\mathbb{C}} \mathscr{A}_{\mathsf{D}}[Y_1, \cdots, Y_{m^-}] \\ F(X_1, \cdots, X_m) &\longmapsto F(-Y_1^a, \cdots, -Y_{m^-}^a, Z_1^a, \cdots, Z_{m^+-1}^a, 1). \end{aligned}$$

Suppose a is odd. — Suppose $f \in \mathscr{A}_{G_{F_a}}$ corresponds to $F(X_1, \cdots, X_{m-1}) \in \mathscr{A}_{\mathsf{B}}[X_1, \cdots, X_{m-1}]$. By Lemma 7.2.3, the evaluation of $\tilde{\eta}^*(f)$ at $t_{\widehat{H}} \rtimes \sigma$ is equal to

$$F(-u_1^a, \cdots, -u_{m^-}^a, t_1^a, \cdots, t_{m^+-1}^a).$$

Thus the map $\tilde{\eta}^*$ is explicitly given by

$$\begin{aligned} \mathscr{A}_{\mathsf{B}}[X_1, \cdots, X_{m-1}] &\longrightarrow \mathscr{A}_{\mathsf{B}}[Z_1, \cdots, Z_{m^+-1}] \otimes_{\mathbb{C}} \mathscr{A}_{\mathsf{D}}[Y_1, \cdots, Y_{m^-}] \\ F(X_1, \cdots, X_{m-1}) &\longmapsto F(-Y_1^a, \cdots, -Y_{m^-}^a, Z_1^a, \cdots, Z_{m^+-1}^a). \end{aligned}$$

7.3.2.4. *The even case, with trivial δ^+ and non-trivial δ^-.* — In this case, δ is non-trivial. We have

$$\mathscr{A}_{G_{F_a}} \cong \begin{cases} \mathscr{A}_{\mathsf{D}}[X_1, \cdots, X_m], & \text{if } a \text{ is even}, \\ \mathscr{A}_{\mathsf{B}}[X_1, \cdots, X_{m-1}], & \text{if } a \text{ is odd}, \end{cases}$$

and

$$\mathscr{A}_H = \mathscr{A}_{H^+} \otimes_{\mathbb{C}} \mathscr{A}_{H^-} \cong \mathscr{A}_{\mathsf{D}}[Z_1, \cdots, Z_{m^+}] \otimes_{\mathbb{C}} \mathscr{A}_{\mathsf{B}}[Y_1, \cdots, Y_{m^--1}].$$

Consider an element

$$t_{\widehat{H}} = (\mathrm{symdiag}(t_1, \cdots, t_{m^+}), \mathrm{symdiag}(u_1, \cdots, u_{m^-})) \in \mathcal{T}_{\widehat{H}} = \mathcal{T}_{V^+} \times \mathcal{T}_{V^-}.$$

Since δ^- is non-trivial, we are in the second case in (5.4.3.2). Hence

$$\eta(t_{\widehat{H}} \rtimes \sigma) = \mathrm{symdiag}(u_1, \cdots, u_{m^-}, t_1, \cdots, t_{m^+}) \cdot S \rtimes \sigma \in {}^L G^{\mathrm{ur}},$$

where S is the permutation matrix switching $\hat{e}_{m^-}$ and $\hat{e}_{d-m^-+1}$, and switching $\hat{e}_m$ and $\hat{e}_{m+1}$. The conjugation action of $S \rtimes \sigma$ on $\mathcal{T}$ is given by

$$\mathrm{symdiag}(x_1, \cdots, x_m) \longmapsto \mathrm{symdiag}(x_1, \cdots, x_{m^--1}, x_{m^-}^{-1}, x_{m^-+1}, \cdots, x_m).$$

Moreover, $(S \rtimes \sigma)^a = S^a \rtimes \sigma^a$, and S is of order 2. Therefore, with the notation (7.3.2.1), we have

$$s^{-1}\eta(t_{\widehat{H}} \rtimes \sigma)^a = \mathrm{symdiag}(-u_1^a, \cdots, -u_{m^--1}^a, -u_{m^-}^{\nu_a}, t_1^a, \cdots, t_{m^+}^a) \cdot S^{\nu_a} \rtimes \sigma^a.$$

If a is even, the above element lies in $\mathcal{T} \rtimes \sigma^a$. If a is odd, the above element is conjugate by some $g \in \widehat{G}$ to the element

$$\mathrm{symdiag}(-u_1^a, \cdots, -u_{m^--1}^a, t_{m^+}^a, t_1^a, \cdots, t_{m^+-1}^a, -u_{m^-}) \rtimes \sigma^a \in \mathcal{T} \rtimes \sigma^a.$$

For instance, one can take g to be the permutation matrix in $\widehat{G}$ switching $\hat{e}_{m^-}$ and $\hat{e}_m$ and switching $\hat{e}_{d-m^-+1}$ and $(-1)^{m^-+m}\hat{e}_{m+1}$. Indeed, we have $g^{-1} = g$, $(S \rtimes \sigma^a)g = g \rtimes \sigma^a$, and

$$\begin{aligned} g \cdot \mathrm{symdiag}(-u_1^a, \cdots, -u_{m^--1}^a, -u_{m^-}, t_1^a, \cdots, t_{m^+}^a) \cdot g \\ = \mathrm{symdiag}(-u_1^a, \cdots, -u_{m^--1}^a, t_{m^+}^a, t_1^a, \cdots, t_{m^+-1}^a, -u_{m^-}). \end{aligned}$$

Suppose a is even. — Suppose $f \in \mathscr{A}_{G_{F_a}}$ corresponds to $F(X_1, \cdots, X_m) \in \mathscr{A}_{\mathsf{D}}[X_1, \cdots, X_m]$. By Lemma 7.2.3, the evaluation of $\tilde{\eta}^*(f)$ at $t_{\widehat{H}} \rtimes \sigma$ is equal to

$$F(-u_1^a, \cdots, -u_{m^--1}^a, -1, t_1^a, \cdots, t_{m^+}^a).$$

Thus the map $\tilde{\eta}^*$ is explicitly given by

$$\begin{aligned}\mathscr{A}_{\mathsf{D}}[X_1, \cdots, X_m] &\longrightarrow \mathscr{A}_{\mathsf{D}}[Z_1, \cdots, Z_{m^+}] \otimes_{\mathbb{C}} \mathscr{A}_{\mathsf{B}}[Y_1, \cdots, Y_{m^--1}]\\ F(X_1, \cdots, X_m) &\longmapsto F(-Y_1^a, \cdots, -Y_{m^--1}^a, -1, Z_1^a, \cdots, Z_{m^+}^a).\end{aligned}$$

Suppose a is odd. — Suppose $f \in \mathscr{A}_{G_{F_a}}$ corresponds to $F(X_1, \cdots, X_{m-1}) \in \mathscr{A}_{\mathsf{B}}[X_1, \cdots, X_{m-1}]$. By Lemma 7.2.3, the evaluation of $\tilde{\eta}^*(f)$ at $t_{\widehat{H}} \rtimes \sigma$ is equal to

$$F(-u_1^a, \cdots - u_{m^--1}^a, t_{m^+}^a, t_1^a, \cdots t_{m^+-1}^a).$$

Thus the map $\tilde{\eta}^*$ is explicitly given by

$$\begin{aligned}\mathscr{A}_{\mathsf{B}}[X_1, \cdots, X_{m-1}] &\longrightarrow \mathscr{A}_{\mathsf{D}}[Z_1, \cdots, Z_{m^+}] \otimes_{\mathbb{C}} \mathscr{A}_{\mathsf{B}}[Y_1, \cdots, Y_{m^--1}]\\ F(X_1, \cdots, X_{m-1}) &\longmapsto F(-Y_1^a, \cdots, -Y_{m^--1}^a, Z_{m^+}^a, Z_1^a, \cdots, Z_{m^+-1}^a).\end{aligned}$$

7.3.2.5. *The even case, with non-trivial δ^+ and non-trivial δ^-.* — In this case, δ is trivial. We have

$$\mathscr{A}_{G_{F_a}} \cong \mathscr{A}_{\mathsf{D}}[X_1, \cdots, X_m]$$

and

$$\mathscr{A}_H = \mathscr{A}_{H^+} \otimes_{\mathbb{C}} \mathscr{A}_{H^-} \cong \mathscr{A}_{\mathsf{B}}[Z_1, \cdots, Z_{m^+-1}] \otimes_{\mathbb{C}} \mathscr{A}_{\mathsf{B}}[Y_1, \cdots, Y_{m^--1}].$$

Consider an element

$$t_{\widehat{H}} = (\mathrm{symdiag}(t_1, \cdots, t_{m^+}), \mathrm{symdiag}(u_1, \cdots, u_{m^-})) \in \mathcal{T}_{\widehat{H}} = \mathcal{T}_{V^+} \times \mathcal{T}_{V^-}.$$

Since δ^- is non-trivial, we are in the second case in (5.4.3.2). Hence

$$\eta(t_{\widehat{H}} \rtimes \sigma) = \mathrm{symdiag}(u_1, \cdots, u_{m^-}, t_1, \cdots, t_{m^+}) \cdot S \rtimes \sigma \in {}^L G^{\mathrm{ur}},$$

where S is the permutation matrix switching $\hat{e}_{m^-}$ and $\hat{e}_{d-m^-+1}$, and switching $\hat{e}_m$ and $\hat{e}_{m+1}$. Since δ is trivial, the action of σ on $\widehat{G}$ is trivial. We know that $S^2 = 1$, and the conjugation action of S on $\mathcal{T}$ is given by

$$\mathrm{symdiag}(x_1, \cdots, x_m) \longmapsto \mathrm{symdiag}(x_1, \cdots, x_{m^--1}, x_{m^-}^{-1}, x_{m^-+1}, \cdots, x_{m-1}, x_m^{-1}).$$

Hence with the notation (7.3.2.1) we have

$$s^{-1}\eta(t_{\widehat{H}} \rtimes \sigma)^a = \mathrm{symdiag}(-u_1^a, \cdots, -u_{m^--1}^a, -u_{m^-}^{\nu_a}, t_1^a, \cdots, t_{m^+-1}^a, t_{m^+}^{\nu_a}) \cdot S^{\nu_a} \rtimes \sigma^a.$$

If a is even, the above element lies in $\mathcal{T} \rtimes \sigma^a$. If a is odd, we claim that the above element is $\widehat{G}$-conjugate to

$$\mathrm{symdiag}(-u_1^a, \cdots, -u_{m^--1}^a, -1, t_1^a, \cdots, t_{m^+-1}^a, 1) \rtimes \sigma^a \in \mathcal{T} \rtimes \sigma^a.$$

To show the claim, it suffices to show that $\mathrm{symdiag}(x_1, \cdots, x_{m^-}, y_1, \cdots, y_{m^+}) \cdot S$ is $\widehat{G}$-conjugate to $\mathrm{symdiag}(x_1, \cdots, x_{m^--1}, -1, y_1, \cdots, y_{m^+-1}, 1)$ for arbitrary $x_i, y_i \in \mathbb{C}^\times$. Let $\mathcal{J}$ be the special orthogonal group of the 4-dimensional quadratic space $\mathrm{span}\{\hat{e}_{m^-}, \hat{e}_m, \hat{e}_{m+1}, \hat{e}_{d-m^-+1}\}$ over $\mathbb{C}$. We write elements of $\mathcal{J}$ as 4×4 matrices using the given basis. We identify $\mathcal{J}$ as a subgroup of $\widehat{G}$, by letting elements of $\mathcal{J}$

act trivially on $\hat{e}_i$ for all $i \notin \{m^-, m, m+1, d-m^-+1\}$. Then $S \in \mathcal{J}$, and the 4×4 matrix of S is

$$\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}.$$

Let U and K be elements of $\mathcal{J}$ whose 4×4 matrices are $\mathrm{symdiag}(x_{m^-}, y_{m^+})$ and $\mathrm{symdiag}(-1, 1)$ respectively. Now since US is semi-simple (as can be easily seen in GL_4), it must be conjugate in $\mathcal{J}$ to some element of the diagonal maximal torus $\{\mathrm{symdiag}(a,b) \mid a, b \in \mathbb{C}^\times\}$ in $\mathcal{J}$, which must be either K or $-K$ by considering the characteristic polynomial. But K and $-K$ are actually conjugate in $\mathcal{J}$. Hence US is conjugate to K in $\mathcal{J}$. Now inside $\widehat{G}$ we have

$$\begin{aligned} &\mathrm{symdiag}(x_1, \cdots, x_{m^-}, y_1, \cdots, y_{m^+})S \\ &\qquad = \mathrm{symdiag}(x_1, \cdots, x_{m^- -1}, 1, y_1, \cdots, y_{m^+ -1}, 1)US, \end{aligned}$$

and $\mathrm{symdiag}(x_1, \cdots, x_{m^- -1}, 1, y_1, \cdots, y_{m^+ -1}, 1)$ commutes with $\mathcal{J}$. Hence the above element is $\widehat{G}$-conjugate to

$$\begin{aligned} &\mathrm{symdiag}(x_1, \cdots, x_{m^- -1}, 1, y_1, \cdots, y_{m^+ -1}, 1)K \\ &\qquad = \mathrm{symdiag}(x_1, \cdots, x_{m^- -1}, -1, y_1, \cdots, y_{m^+ -1}, 1), \end{aligned}$$

as desired. Our claim follows.

Now suppose $f \in \mathscr{A}_{G_{F_a}}$ corresponds to $F(X_1, \cdots, X_m) \in \mathscr{A}_{\mathsf{D}}[X_1, \cdots, X_m]$. By Lemma 7.2.3 and the above claim, the evaluation of $\tilde{\eta}^*(f)$ at $t_{\widehat{H}} \rtimes \sigma$ is equal to

$$F(-u_1^a, \cdots, -u_{m^- -1}^a, -1, t_1^a, \cdots, t_{m^+ -1}^a, 1)$$

for both parities of a. Thus the map $\tilde{\eta}^*$ is explicitly given by

$$\begin{aligned} \mathscr{A}_{\mathsf{D}}[X_1, \cdots, X_m] &\longrightarrow \mathscr{A}_{\mathsf{B}}[Z_1, \cdots, Z_{m^+ -1}] \otimes_{\mathbb{C}} \mathscr{A}_{\mathsf{B}}[Y_1, \cdots, Y_{m^- -1}] \\ F(X_1, \cdots, X_m) &\longmapsto F(-Y_1^a, \cdots, -Y_{m^- -1}^a, -1, Z_1^a, \cdots, Z_{m^+ -1}^a, 1). \end{aligned}$$

7.3.3. — In the following, we collect the explicit description of $\tilde{\eta}^*$ in all the cases obtained in §7.3.2.

7.3.3.1. *The odd case.* —

$$\begin{aligned} \mathscr{A}_{\mathsf{B}}[X_1, \cdots, X_m] &\longrightarrow \mathscr{A}_{\mathsf{B}}[Z_1, \cdots, Z_{m^+}] \otimes_{\mathbb{C}} \mathscr{A}_{\mathsf{B}}[Y_1, \cdots, Y_{m^-}] \\ F(X_1, \cdots, X_m) &\longmapsto F(-Y_1^a, \cdots, -Y_{m^-}^a, Z_1^a, \cdots, Z_{m^+}^a). \end{aligned}$$

7.3.3.2. *The even case, with trivial δ^+ and trivial δ^-.* —

$$\begin{aligned} \mathscr{A}_{\mathsf{D}}[X_1, \cdots, X_m] &\longrightarrow \mathscr{A}_{\mathsf{D}}[Z_1, \cdots, Z_{m^+}] \otimes_{\mathbb{C}} \mathscr{A}_{\mathsf{D}}[Y_1, \cdots, Y_{m^-}] \\ F(X_1, \cdots, X_m) &\longmapsto F(-Y_1^a, \cdots, -Y_{m^-}^a, Z_1^a, \cdots, Z_{m^+}^a). \end{aligned}$$

7.3.3.3. *The even case, with non-trivial δ^+ and trivial δ^-.* —

Suppose a is even. —

$$\begin{aligned}\mathscr{A}_{\mathsf{D}}[X_1,\cdots,X_m] &\longrightarrow \mathscr{A}_{\mathsf{B}}[Z_1,\cdots,Z_{m^+-1}]\otimes_{\mathbb{C}}\mathscr{A}_{\mathsf{D}}[Y_1,\cdots,Y_{m^-}]\\ F(X_1,\cdots,X_m) &\longmapsto F(-Y_1^a,\cdots,-Y_{m^-}^a,Z_1^a,\cdots,Z_{m^+-1}^a,1).\end{aligned}$$

Suppose a is odd. —

$$\begin{aligned}\mathscr{A}_{\mathsf{B}}[X_1,\cdots,X_{m-1}] &\longrightarrow \mathscr{A}_{\mathsf{B}}[Z_1,\cdots,Z_{m^+-1}]\otimes_{\mathbb{C}}\mathscr{A}_{\mathsf{D}}[Y_1,\cdots,Y_{m^-}]\\ F(X_1,\cdots,X_{m-1}) &\longmapsto F(-Y_1^a,\cdots,-Y_{m^-}^a,Z_1^a,\cdots,Z_{m^+-1}^a).\end{aligned}$$

7.3.3.4. *The even case, with trivial δ^+ and non-trivial δ^-.* —

Suppose a is even. —

$$\begin{aligned}\mathscr{A}_{\mathsf{D}}[X_1,\cdots,X_m] &\longrightarrow \mathscr{A}_{\mathsf{D}}[Z_1,\cdots,Z_{m^+}]\otimes_{\mathbb{C}}\mathscr{A}_{\mathsf{B}}[Y_1,\cdots,Y_{m^--1}]\\ F(X_1,\cdots,X_m) &\longmapsto F(-Y_1^a,\cdots,-Y_{m^--1}^a,-1,Z_1^a,\cdots,Z_{m^+}^a).\end{aligned}$$

Suppose a is odd. —

$$\begin{aligned}\mathscr{A}_{\mathsf{B}}[X_1,\cdots,X_{m-1}] &\longrightarrow \mathscr{A}_{\mathsf{D}}[Z_1,\cdots,Z_{m^+}]\otimes_{\mathbb{C}}\mathscr{A}_{\mathsf{B}}[Y_1,\cdots,Y_{m^--1}]\\ F(X_1,\cdots,X_{m-1}) &\longmapsto F(-Y_1^a,\cdots,-Y_{m^--1}^a,Z_{m^+}^a,Z_1^a,\cdots,Z_{m^+-1}^a).\end{aligned}$$

7.3.3.5. *The even case, with non-trivial δ^+ and non-trivial δ^-.* —

$$\begin{aligned}\mathscr{A}_{\mathsf{D}}[X_1,\cdots,X_m] &\longrightarrow \mathscr{A}_{\mathsf{B}}[Z_1,\cdots,Z_{m^+-1}]\otimes_{\mathbb{C}}\mathscr{A}_{\mathsf{B}}[Y_1,\cdots,Y_{m^--1}]\\ F(X_1,\cdots,X_m) &\longmapsto F(-Y_1^a,\cdots,-Y_{m^--1}^a,-1,Z_1^a,\cdots,Z_{m^+-1}^a,1).\end{aligned}$$

7.4. Computation of twisted transfers

7.4.1. — We keep the setting of §7.3.1, assume that $F=\mathbb{Q}_p$, and import the constructions and notations in §§5.5.2–5.5.3. In particular, we fix W,r,t, and a hyperbolic basis $\mathbb{B}_{W^\perp}$ of $W^\perp$, and from these data we obtain a Levi subgroup $M\subset G$ (defined over $\mathbb{Q}_p$). Since G is by assumption unramified over $\mathbb{Q}_p$, so is M.

Let $\mathfrak{p}=(d^+,\delta^+,d^-,\delta^-)$ be a quadruple satisfying all the conditions in the definition of the set $\mathscr{P}_W$, except that even when $\dim W$ is even and at least 4 we still allow $(d^+,\delta^+)=(2,1)$ or $(d^-,\delta^-)=(2,1)$ (or both); cf. the discussion at the beginning of §7.3.2. Let A be a subset of $[r]$ and B be a subset of $[t]$. Although $(A,B,d^+,\delta^+,d^-,\delta^-)$ is more general than an element of $\mathscr{P}_{r,t}\times'\mathscr{P}_W$ as in Definition 5.5.4, the construction in §5.5.6 still applies to it and yields an endoscopic G-datum for M :

$$\mathfrak{e}_{A,B,\mathfrak{p}}=(M',{}^LM',s_M,\eta_M),$$

which may no longer be bi-elliptic. Also, we obtain an endoscopic datum for M :

$$\mathfrak{e}_{\mathfrak{p}}(M)=\mathfrak{e}_{d^+,\delta^+,d^-,\delta^-}(M)=(M',{}^LM',s'_M,\eta_M)$$

and an endoscopic datum for G:

$$\mathfrak{e}_{d^++2|A|+4|B|,\delta^+,d^-+2|A^c|+4|B^c|,\delta^-}=(H,{}^LH,s,\eta),$$

both of which are possibly non-elliptic (due to the possible appearance of $(2,1)$ in the subscripts). Note that the last two endoscopic data are unramified if and only if both δ^+ and δ^- have even p-adic valuations. (In the odd case this is automatic.) In the following we assume that this is the case.

Fix a positive integer a. As in §7.2, we have the twisted transfer map induced by the unramified endoscopic datum $(H, {}^L H, \eta, s)$ for G:

$$b : \mathcal{H}^{\mathrm{ur}}(G_{\mathbb{Q}_{p^a}}) \longrightarrow \mathcal{H}^{\mathrm{ur}}(H).$$

Let μ be the cocharacter of G such that the $\mathbb{G}_m$-action on V via μ has weight 1 on $f_1 \in \mathbb{B}_{W^\perp}$, weight -1 on $f_{2(r+2t)} \in \mathbb{B}_{W^\perp}$, and weight zero on the orthogonal complement of these two vectors. Thus μ is given by

$$\begin{aligned} \mathbb{G}_m \longrightarrow \mathbb{G}_m^r \times \mathrm{GL}_2^t &\xrightarrow{(5.5.2.1)} \mathrm{SO}(W^\perp) \longrightarrow G \\ z \longmapsto (z, 1, \cdots, 1, I_2, &\cdots, I_2) \end{aligned}$$

if $r > 0$, and is given by

$$\begin{aligned} \mathbb{G}_m \longrightarrow \mathrm{GL}_2^t &\xrightarrow{(5.5.2.1)} \mathrm{SO}(W^\perp) \longrightarrow G \\ z \longmapsto (\mathrm{diag}(z,1), I_2, &\cdots, I_2) \end{aligned}$$

if $r = 0$. Let

$$f_{-\mu} \in \mathcal{H}^{\mathrm{ur}}(G_{\mathbb{Q}_{p^a}})$$

be as in Definition 7.1.6, with $F = \mathbb{Q}_{p^a}$ and $\lambda = -\mu$. Define

$$f^H := b(f_{-\mu}) \in \mathcal{H}^{\mathrm{ur}}(H).$$

The construction in §5.5.9 still applies to the current slightly more general situation (with the possibly non-elliptic data). Hence M' is identified with a Levi subgroup of H (up to $H(F)$-conjugation). We have the canonical constant term map (see Proposition 7.1.10):

$$(\cdot)_{M'} : \mathcal{H}^{\mathrm{ur}}(H) \longrightarrow \mathcal{H}^{\mathrm{ur}}(M').$$

In the following we describe $(f^H)_{M'}$.

Recall from §5.5.2 that $M = M^{\mathrm{GL}} \times M^{\mathrm{SO}}$, where M^{GL} is identified with $\mathbb{G}_m^r \times \mathrm{GL}_2^t$ via (5.5.2.1), and $M^{\mathrm{SO}} = \mathrm{SO}(W)$. The maximal split torus in M^{GL} given by the product of $\mathbb{G}_m^r$ with the diagonal tori in the copies of GL_2 is naturally identified with $\mathbb{G}_m^{r+2t}$. Correspondingly, the algebra $\mathscr{A}_{M^{\mathrm{GL}}}$ is naturally identified with

$$\mathbb{C}[\xi_1^{\pm 1}] \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{C}[\xi_r^{\pm 1}] \otimes_{\mathbb{C}} \mathbb{C}[\zeta_1^{\pm 1}, \zeta_2^{\pm 1}]^{\mathfrak{S}_2} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{C}[\zeta_{2t-1}^{\pm 1}, \zeta_{2t}^{\pm 1}]^{\mathfrak{S}_2}.$$

(Here $\mathfrak{S}_2$ acts on each $\mathbb{C}[\zeta_j^{\pm}, \zeta_{j+1}^{\pm}]$ by swapping ζ_j and ζ_{j+1}.) In the sequel we shall view elements of the above algebra, such as $\zeta_1 + \zeta_2$, as an element of $\mathscr{A}_{M^{\mathrm{GL}}}$ or $\mathcal{H}^{\mathrm{ur}}(M^{\mathrm{GL}})$. We have $M' = M^{\mathrm{GL}} \times M^{\prime,\mathrm{SO}}$ (see §5.5.6), and correspondingly we have

$$\mathcal{H}^{\mathrm{ur}}(M') = \mathcal{H}^{\mathrm{ur}}(M^{\mathrm{GL}}) \otimes_{\mathbb{C}} \mathcal{H}^{\mathrm{ur}}(M^{\prime,\mathrm{SO}}).$$

We retain the notation $\nabla_i(\cdot)$ as in Definition 5.5.5.

Proposition 7.4.2. — *The element* $p^{a(2-d)/2}(f^H)_{M'} \in \mathcal{H}^{\mathrm{ur}}(M')$ *is of the form*

$$k(A,B)\otimes 1 + 1 \otimes h,$$

with $k(A,B) \in \mathcal{H}^{\mathrm{ur}}(M^{\mathrm{GL}})$ *and* $h \in \mathcal{H}^{\mathrm{ur}}(M^{\prime,\mathrm{SO}})$*. The element* h *depends only on the parameter* $\mathfrak{p} = (d^+, \delta^+, d^-, \delta^-)$*, not on* (A,B)*. The element* $k(A,B)$ *is given by*

$$k(A,B) = \sum_{i=1}^{r} \nabla_i(A)(\xi_i^a + \xi_i^{-a}) + \sum_{j=1}^{t} \nabla_j(B)(\zeta_{2j-1}^a + \zeta_{2j-1}^{-a} + \zeta_{2j}^a + \zeta_{2j}^{-a}).$$

Proof. — Write F_a for $\mathbb{Q}_{p^a}$. Fix a maximal F_a-split torus S in G_{F_a}. In this proof we omit notations for the Satake isomorphisms. We use Theorem 7.1.5 to compute (the Satake transform of) $f_{-\mu}$. We have $\langle \rho, (-\mu)_{\mathrm{dom}}\rangle = (d-2)/2$, and so

$$p^{a(2-d)/2} f_{-\mu} = \sum_{\lambda \in \Omega(F)\cdot(-\mu)} [\lambda] \in \mathbb{C}[X_*(S)]^{\Omega(F)} \cong \mathscr{A}_{G_{F_a}}$$

by that theorem. Let $m = \lfloor d/2 \rfloor$ be the absolute rank of G. As in §7.3.1, $\mathscr{A}_{G_{F_a}}$ is identified with one of the three algebras

$$\mathscr{A}_{\mathsf{B}}[X_1,\cdots,X_m], \quad \mathscr{A}_{\mathsf{D}}[X_1,\cdots,X_m], \quad \mathscr{A}_{\mathsf{B}}[X_1,\cdots,X_{m-1}].$$

Correspondingly, we have

$$p^{a(2-d)/2} f_{-\mu} = \begin{cases} X_1 + X_1^{-1} + \cdots + X_m + X_m^{-1} \in \mathscr{A}_{\mathsf{B}}[X_1,\cdots,X_m], \\ X_1 + X_1^{-1} + \cdots + X_m + X_m^{-1} \in \mathscr{A}_{\mathsf{D}}[X_1,\cdots,X_m], \\ X_1 + X_1^{-1} + \cdots + X_{m-1} + X_{m-1}^{-1} \in \mathscr{A}_{\mathsf{B}}[X_1,\cdots,X_{m-1}]. \end{cases}$$

For any positive integer l, we introduce short-hand notations

$$\mathscr{A}_{\mathsf{B}}[\mathcal{Y}_l] := \mathscr{A}_{\mathsf{B}}[Y_1,\cdots,Y_l], \qquad \mathscr{A}_{\mathsf{D}}[\mathcal{Y}_l] := \mathscr{A}_{\mathsf{D}}[Y_1,\cdots,Y_l],$$
$$\mathscr{A}_{\mathsf{B}}[\mathcal{Z}_l] := \mathscr{A}_{\mathsf{B}}[Z_1,\cdots,Z_l], \qquad \mathscr{A}_{\mathsf{D}}[\mathcal{Z}_l] := \mathscr{A}_{\mathsf{D}}[Z_1,\cdots,Z_l],$$

and

$$\mathcal{Y}_l^a := \sum_{i=1}^{l} Y_i^a + Y_i^{-a}, \qquad \mathcal{Z}_l^a := \sum_{i=1}^{l} Z_i^a + Z_i^{-a}.$$

We then compute, according to §7.3.3, that (the Satake transform of) $p^{a(2-d)/2} f^H$ in $\mathscr{A}_H$ is given by:

(7.4.2.1)

$$\begin{cases} -\mathcal{Y}_{m^-}^a + \mathcal{Z}_{m^+}^a \in \mathscr{A}_{\mathsf{B}}[\mathcal{Z}_{m^+}] \otimes \mathscr{A}_{\mathsf{B}}[\mathcal{Y}_{m^-}], & d \text{ odd} \\ -\mathcal{Y}_{m^-}^a + \mathcal{Z}_{m^+}^a \in \mathscr{A}_{\mathsf{D}}[\mathcal{Z}_{m^+}] \otimes \mathscr{A}_{\mathsf{D}}[\mathcal{Y}_{m^-}], & d \text{ even}, \delta^+ = \delta^- = 1, \\ -\mathcal{Y}_{m^-}^a + \mathcal{Z}_{m^+-1}^a + 1 + (-1)^a \in \mathscr{A}_{\mathsf{B}}[\mathcal{Z}_{m^+-1}] \otimes \mathscr{A}_{\mathsf{D}}[\mathcal{Y}_{m^-}], & d \text{ even}, \delta^+ \neq 1, \delta^- = 1, \\ -\mathcal{Y}_{m^--1}^a + \mathcal{Z}_{m^+}^a - 1 - (-1)^a \in \mathscr{A}_{\mathsf{D}}[\mathcal{Z}_{m^+}] \otimes \mathscr{A}_{\mathsf{B}}[\mathcal{Y}_{m^--1}], & d \text{ even}, \delta^+ = 1, \delta^- \neq 1, \\ -\mathcal{Y}_{m^--1}^a + \mathcal{Z}_{m^+-1}^a \in \mathscr{A}_{\mathsf{B}}[\mathcal{Z}_{m^+-1}] \otimes \mathscr{A}_{\mathsf{B}}[\mathcal{Y}_{m^--1}], & d \text{ even}, \delta^+ \neq 1, \delta^- \neq 1. \end{cases}$$

Recall that $M'^{,\mathrm{SO}} = \mathrm{SO}(W^+) \times \mathrm{SO}(W^-)$. Write $n^\pm$ for the absolute rank of $\mathrm{SO}(W^\pm)$. Similar to $\mathscr{A}_{H^+}$, we identify $\mathscr{A}_{\mathrm{SO}(W^+)}$ with one of

$$\mathscr{A}_{\mathsf{B}}[\mathcal{Z}_{n^+}], \quad \mathscr{A}_{\mathsf{D}}[\mathcal{Z}_{n^+}], \quad \mathscr{A}_{\mathsf{B}}[\mathcal{Z}_{n^+-1}],$$

in the odd case, in the even case with $\delta^+ = 1$, and in the even case with $\delta^+ \neq 1$ respectively. Similarly, we identify $\mathscr{A}_{\mathrm{SO}(W^-)}$ with one of

$$\mathscr{A}_{\mathsf{B}}[\mathcal{Y}_{n^-}], \quad \mathscr{A}_{\mathsf{D}}[\mathcal{Y}_{n^-}], \quad \mathscr{A}_{\mathsf{B}}[\mathcal{Y}_{n^--1}],$$

in the odd case, in the even case with $\delta^- = 1$, and in the even case with $\delta^- \neq 1$ respectively. The constant term map $\mathscr{A}_H \to \mathscr{A}_{M'}$ is of the form

$$\begin{cases} \mathscr{A}_{\mathsf{B}}[\mathcal{Z}_{m^+}] \otimes \mathscr{A}_{\mathsf{B}}[\mathcal{Y}_{m^-}] \to \mathscr{A}_{M^{\mathrm{GL}}} \otimes \mathscr{A}_{\mathsf{B}}[\mathcal{Z}_{n^+}] \otimes \mathscr{A}_{\mathsf{B}}[\mathcal{Y}_{n^-}], \\ \mathscr{A}_{\mathsf{D}}[\mathcal{Z}_{m^+}] \otimes \mathscr{A}_{\mathsf{D}}[\mathcal{Y}_{m^-}] \to \mathscr{A}_{M^{\mathrm{GL}}} \otimes \mathscr{A}_{\mathsf{D}}[\mathcal{Z}_{n^+}] \otimes \mathscr{A}_{\mathsf{D}}[\mathcal{Y}_{n^-}], \\ \mathscr{A}_{\mathsf{B}}[\mathcal{Z}_{m^+-1}] \otimes \mathscr{A}_{\mathsf{D}}[\mathcal{Y}_{m^-}] \to \mathscr{A}_{M^{\mathrm{GL}}} \otimes \mathscr{A}_{\mathsf{B}}[\mathcal{Z}_{n^+-1}] \otimes \mathscr{A}_{\mathsf{D}}[\mathcal{Y}_{n^-}], \\ \mathscr{A}_{\mathsf{D}}[\mathcal{Z}_{m^+}] \otimes \mathscr{A}_{\mathsf{B}}[\mathcal{Y}_{m^--1}] \to \mathscr{A}_{M^{\mathrm{GL}}} \otimes \mathscr{A}_{\mathsf{D}}[\mathcal{Z}_{n^+}] \otimes \mathscr{A}_{\mathsf{B}}[\mathcal{Y}_{n^--1}], \\ \mathscr{A}_{\mathsf{B}}[\mathcal{Z}_{m^+-1}] \otimes \mathscr{A}_{\mathsf{B}}[\mathcal{Y}_{m^--1}] \to \mathscr{A}_{M^{\mathrm{GL}}} \otimes \mathscr{A}_{\mathsf{B}}[\mathcal{Z}_{n^+-1}] \otimes \mathscr{A}_{\mathsf{B}}[\mathcal{Y}_{n^--1}], \end{cases}$$

where the division into the five cases is the same as in (7.4.2.1). In each case, using Lemma 7.1.9, we see that the map is determined by the following rule: Write

$$A = \{i_1, \cdots, i_u\}, \qquad A^c = \left\{\tilde{i}_1, \cdots, \tilde{i}_{r-u}\right\},$$
$$B = \{j_1, \cdots, j_v\}, \qquad B^c = \left\{\tilde{j}_1, \cdots, \tilde{j}_{t-v}\right\}.$$

We send $Z_1, \cdots, Z_u$ to $\xi_{i_1}, \cdots, \xi_{i_u}$, send $Y_1, \cdots, Y_{r-u}$ to $\xi_{\tilde{i}_1}, \cdots, \xi_{\tilde{i}_{r-u}}$, send $Z_{u+1}, \cdots, Z_{u+2v}$ to

$$\zeta_{2j_1-1}, \zeta_{2j_1}, \zeta_{2j_2-1}, \zeta_{2j_2}, \cdots, \zeta_{2j_v-1}, \zeta_{2j_v},$$

send $Y_{r-u+1}, \cdots, Y_{r-u+2t-2v}$ to

$$\zeta_{2\tilde{j}_1-1}, \zeta_{2\tilde{j}_1}, \zeta_{2\tilde{j}_2-1}, \zeta_{2\tilde{j}_2}, \cdots, \zeta_{2\tilde{j}_{t-v}-1}, \zeta_{2\tilde{j}_{t-v}},$$

send the remaining Z_i's to $Z_1, Z_2, \cdots$, and send the remaining Y_i's to $Y_1, Y_2, \cdots$. From this description of the constant term map and the previous computation (7.4.2.1) of $p^{a(2-d)/2} f^H$, we see that $p^{a(2-d)/2}(f^H)_{M'} \in \mathcal{H}^{\mathrm{ur}}(M')$ is of the form

$$k(A, B) \otimes 1 + 1 \otimes h,$$

where $k(A, B)$ is given as in the statement of the proposition, and

$$h \in \mathscr{A}_{\mathrm{SO}(W^+)} \otimes \mathscr{A}_{\mathrm{SO}(W^-)}$$

is given by

$$\begin{cases} -\mathcal{Y}_{n^-}^a + \mathcal{Z}_{n^+}^a \in \mathscr{A}_{\mathsf{B}}[\mathcal{Z}_{n^+}] \otimes \mathscr{A}_{\mathsf{B}}[\mathcal{Y}_{n^-}], \\ -\mathcal{Y}_{n^-}^a + \mathcal{Z}_{n^+}^a \in \mathscr{A}_{\mathsf{D}}[\mathcal{Z}_{n^+}] \otimes \mathscr{A}_{\mathsf{D}}[\mathcal{Y}_{n^-}], \\ -\mathcal{Y}_{n^-}^a + \mathcal{Z}_{n^+-1}^a + 1 + (-1)^a \in \mathscr{A}_{\mathsf{B}}[\mathcal{Z}_{n^+-1}] \otimes \mathscr{A}_{\mathsf{D}}[\mathcal{Y}_{n^-}], \\ -\mathcal{Y}_{n^--1}^a + \mathcal{Z}_{n^+}^a - 1 - (-1)^a \in \mathscr{A}_{\mathsf{D}}[\mathcal{Z}_{n^+}] \otimes \mathscr{A}_{\mathsf{B}}[\mathcal{Y}_{n^--1}], \\ -\mathcal{Y}_{n^--1}^a + \mathcal{Z}_{n^+-1}^a \in \mathscr{A}_{\mathsf{B}}[\mathcal{Z}_{n^+-1}] \otimes \mathscr{A}_{\mathsf{B}}[\mathcal{Y}_{n^--1}], \end{cases}$$

in the five cases as before. Clearly h depends only on $\mathfrak{p}$, not on (A, B). □

CHAPTER 8

STABILIZATION

8.1. Standard definitions and facts on Langlands–Shelstad transfer

8.1.1. — For any field F of characteristic zero and homomorphism $I \to J$ of algebraic groups over F, we write

$$\mathfrak{D}(I, J; F) := \ker(\mathbf{H}^1(F, I) \to \mathbf{H}^1(F, J)).$$

Now let F be a non-archimedean local field of characteristic zero, and G a reductive group over F. We recall the definition of κ-orbital integrals in the fashion of [**Lab99**, §2.7]. Let $\gamma \in G(F)$ be a semi-simple element, and write I_γ for $(G_\gamma)^0$. Recall from [**Lab99**, §2.3] that there is a natural surjection from $\mathfrak{D}(I_\gamma, G; F)$ to the set of conjugacy classes in the stable conjugacy class of γ, which is a bijection if $I_\gamma = G_\gamma$. We have a short exact sequence of pointed sets

$$1 \longrightarrow I_\gamma(F)\backslash G(F) \longrightarrow \mathbf{H}^0(F, I_\gamma\backslash G) \longrightarrow \mathfrak{D}(I_\gamma, G; F) \longrightarrow 1, \tag{8.1.1.1}$$

and a natural map (see [**Lab99**, §1.8])

$$\mathbf{H}^0(F, I_\gamma\backslash G) \longrightarrow \mathbf{H}^0_{\mathrm{ab}}(F, I_\gamma\backslash G),$$

where $\mathbf{H}^0_{\mathrm{ab}}(F, I_\gamma\backslash G)$ is a locally compact topological abelian group. Denote by $\mathfrak{K}(I_\gamma, G; F)$ the Pontryagin dual group of $\mathbf{H}^0_{\mathrm{ab}}(F, I_\gamma\backslash G)$. [(1)]

Choose Haar measures on $I_\gamma(F)$ and on $G(F)$, and equip $\mathfrak{D}(I_\gamma, G; F)$ with the counting measure. Then the short exact sequence (8.1.1.1) defines a measure dx on $\mathbf{H}^0(F, I_\gamma\backslash G)$; see [**Lab99**, §2.7]. For $f \in C_c^\infty(G(F))$ and $\kappa \in \mathfrak{K}(I_\gamma, G; F)$, define the *$\kappa$-orbital integral*

$$O^\kappa_\gamma(f) := \int_{x \in \mathbf{H}^0(F, I_\gamma\backslash G)} e(I_{x^{-1}\gamma x})\kappa(x) f(x^{-1}\gamma x) dx,$$

---

(1) Under the assumption that F is non-archimedean, $\mathfrak{K}(I_\gamma, G; F)$ is isomorphic to the group $\mathfrak{K}(I_\gamma/F)$ defined in [**Kot86**, §4.6].

where $e(I_{x^{-1}\gamma x})$ is the Kottwitz sign of $I_{x^{-1}\gamma x}$ (see [**Lab99**, Def. 1.7.1]). Also define the *stable orbital integral*

$$SO_\gamma(f) := O^1_\gamma(f).$$

Remark 8.1.2. — We give a more concrete description of $O^\kappa_\gamma(f)$. For each $[x] \in \mathfrak{D}(I_\gamma, G; F)$, fix an element $x \in G(\overline{F})$ mapping to $[x]$ under the composite map

$$G(\overline{F}) \longrightarrow \mathbf{H}^0(F, I_\gamma \backslash G) \longrightarrow \mathfrak{D}(I_\gamma, G; F).$$

Then $\gamma_x := x^{-1}\gamma x$ is in $G(F)$ and $\mathrm{Int}(x)$ induces an inner twisting $I_\gamma \to I_{\gamma_x}$. In particular, the Haar measure on $I_\gamma(F)$ transfers to a Haar measure on $I_{\gamma_x}(F)$. Using this and the fixed Haar measure on $G(F)$, we define the orbital integral

$$O_{\gamma_x}(f) := \int_{x \in I_{\gamma_x}(F)\backslash G(F)} f(x^{-1}\gamma x).$$

Then we have

$$O^\kappa_\gamma(f) = \sum_{[x]\in\mathfrak{D}(I_\gamma, G; F)} e(I_{\gamma_x})\kappa(x) O_{\gamma_x}(f).$$

8.1.3. — Fix an inner twisting $\psi : G \to G^*$ with G^* quasi-split, and fix an L-group datum for G, as in [**LS87**]. Let $(H, \mathcal{H}, s, \eta)$ be an endoscopic datum for G. For simplicity, we assume that $\mathcal{H} = {}^L H$ (cf. the discussion in §5.4.1). The notion of when a semi-simple element $\gamma_H \in H(F)$ (not necessarily G-regular) is an *image* of a semi-simple element $\gamma \in G(F)$ is defined in [**LS90**, §1.2].

Under the additional assumption that G^{der} is simply connected, Langlands–Shelstad [**LS90**, §2.4] have defined transfer factors for (G, H)-regular elements. Thus after fixing a normalization we have a number

$$\Delta(\gamma_H, \gamma) \in \mathbb{C},$$

for each semi-simple (G, H)-regular $\gamma_H \in H(F)$ and each semi-simple $\gamma \in G(F)$. Moreover, $\Delta(\gamma_H, \gamma)$ depends on γ_H (resp. γ) only via its stable conjugacy class (resp. conjugacy class) over F, and we have $\Delta(\gamma_H, \gamma) = 0$ unless γ_H is an image of γ.

Since we have assumed that $\mathcal{H} = {}^L H$, we can in fact define $\Delta(\gamma_H, \gamma)$ for (G, H)-regular γ_H without assuming that G^{der} is simply connected. In the more restrictive G-regular case, this is done in [**LS87**]; below we explain the (G, H)-regular case. For this, consider a z-extension $1 \to Z \to G_1 \to G \to 1$. This determines a central extension $1 \to Z \to H_1 \to H \to 1$ as in [**LS87**, §4.4]. As explained in *loc. cit.*, we have a homomorphism $\eta_1 : {}^L H_1 \to {}^L G_1$ such that $(H_1, {}^L H_1, s, \eta_1)$ is an endoscopic datum for G_1. The restriction of η_1 to $\widehat{H_1}$ is canonical, but η_1 itself is canonical only up to twisting by a cocycle in the center of $\widehat{H_1}$. In our current situation (with

$\mathcal{H} = {}^L H$), we can take η_1 such that the diagram

$$\begin{array}{ccc} {}^L H & \xrightarrow{\eta} & {}^L G \\ \downarrow & & \downarrow \\ {}^L H_1 & \xrightarrow{\eta_1} & {}^L G_1 \end{array} \tag{8.1.3.1}$$

commutes, where the vertical arrows are the natural ones associated to $H_1 \to H$ and $G_1 \to G$. This pins down η_1 canonically. We then define $\Delta(\gamma_H, \gamma)$ to be zero if γ_H is not an image of γ, and otherwise to be $\Delta(\gamma_{H_1}, \gamma_1)$ where $\gamma_{H_1} \in H_1(F)$ (resp. $\gamma_1 \in G_1(F)$) is a lift of γ_H (resp. γ) such that γ_{H_1} is an image of γ_1, and $\Delta(\gamma_{H_1}, \gamma_1)$ is defined with respect to the endoscopic datum $(H_1, {}^L H_1, s, \eta_1)$ for G_1 as in [**LS90**, §2.4]. In the latter case, the pair (γ_{H_1}, γ_1) always exists, and is unique up to simultaneous translation by $Z(F)$. To show that this definition of $\Delta(\gamma_H, \gamma)$ is independent of the lifts, it suffices to check that $\Delta(z\gamma_{H_1}, z\gamma_1) = \Delta(\gamma_{H_1}, \gamma_1)$ for all $z \in Z(F)$. For this it suffices to treat the case where γ_{H_1} is strongly G_1-regular. Then the desired statement is proved on p. 254 of [**LS87**] (with $\lambda = 1$). One can also check that the above definition is independent of the choice of the z-extension G_1. For this, using the standard fact (see [**Kot82**, Lem. 1.1]) that any two z-extensions of G can be dominated by a third z-extension, one is reduced to checking that when G^{der} is simply connected, for strongly G-regular $\gamma_H \in H(F)$, the definition of $\Delta(\gamma_H, \gamma)$ as above (i.e., $\Delta(\gamma_H, \gamma) := \Delta(\gamma_{H_1}, \gamma_1)$ with a given z-extension G_1 and with η_1 pinned down as above) agrees with the original definition of $\Delta(\gamma_H, \gamma)$ in [**LS87**]. This is a routine exercise which involves checking suitable functorial properties of all the terms $\Delta_I, \cdots, \Delta_{IV}$ in *loc. cit.*.

The Langlands–Shelstad Transfer Conjecture and the Fundamental Lemma are now unconditional theorems thanks to the work of Ngô [**Ngô10**], Waldspurger [**Wal97, Wal06**], Cluckers–Loeser [**CL10**], and Hales [**Hal95**]. We recall these statements in the following theorem(2), taking into account the extension to (G, H)-regular elements in [**LS90**, §2.4].

Theorem 8.1.4. — *Let G be a reductive group over a non-archimedean local field F of characteristic zero. Let $(H, {}^L H, s, \eta)$ be an endoscopic datum for G.*

(1) *(Langlands–Shelstad Transfer.) Fix a normalization of the transfer factors, and fix Haar measures on $G(F)$ and $H(F)$. For any $f \in C_c^\infty(G(F))$, there exists $f^H \in C_c^\infty(H(F))$, called the* Langlands–Shelstad transfer *of f, with the following*

(2) We state only the Fundamental Lemma for the unit element of the unramified Hecke algebra. The references [**Ngô10**], [**Wal06**], and [**CL10**] give this result for characteristic zero local fields with sufficiently large residue characteristic. In [**Hal95**] it is shown that the Fundamental Lemma for the unit for all sufficiently large residue characteristic is enough to imply the Fundamental Lemma (for the full unramified Hecke algebra) for characteristic zero local fields with arbitrary residue characteristic. See also [**LMW18**].

properties: For any semi-simple (G,H)-regular $\gamma_H \in H(F)$, we have

$$SO_{\gamma_H}(f^H) = \begin{cases} 0, & \gamma_H \text{ is not an image from } G, \\ \Delta(\gamma_H,\gamma)O^s_\gamma(f), & \gamma_H \text{ is an image of } \gamma \in G(F)_{\mathrm{ss}}. \end{cases} \tag{8.1.4.1}$$

In the second situation of (8.1.4.1) we have the following explanations.

– *The component s in $(H, {}^LH, s, \eta)$ defines an element of $\mathfrak{K}(I_\gamma, G; F)$ still denoted by s, and we use that to define O^s_γ.*

– *We define $SO_{\gamma_H}(f^H)$ and $O^s_\gamma(f)$ using the fixed Haar measures on $G(F)$ and $H(F)$ and compatible Haar measures on $G^0_\gamma(F)$ and $H^0_{\gamma_H}(F)$.*

(2) *(Fundamental Lemma.) Suppose G and $(H, {}^LH, s, \eta)$ are unramified (see §7.2.2). Normalize the Haar measures on $G(F)$ and $H(F)$ such that all hyperspecial subgroups have volume 1. Let K (resp. K_H) be an arbitrary hyperspecial subgroup of $G(F)$ (resp. $H(F)$). Then 1_{K_H} is a Langlands–Shelstad transfer of 1_K as in part (1), for the unramified normalization canonically associated to K of transfer factors defined in* [**Hal93**].

(3) *(Adelic Transfer.) Let G_0 be a reductive group over a number field F_0 and let $(H_0, {}^LH_0, s_0, \eta_0)$ be an endoscopic datum for G_0 over F_0. Suppose there is a finite set Σ of finite places of F_0 and a reductive model $\mathcal{G}$ of G_0 over $\mathcal{O}_{F_0}[1/\Sigma]$ such that for all finite places v of F_0 outside Σ the endoscopic datum $(H_0, {}^LH_0, s_0, \eta_0)$ localizes to an unramified endoscopic datum over $F_{0,v}$, and the transfer factors between $H_{F_{0,v}}$ and $G_{F_{0,v}}$ are normalized under the canonical unramified normalization associated to $\mathcal{G}(\mathcal{O}_{F_{0,v}})$. Let S be the union of Σ and the set of all archimedean places of F_0, and let $\mathbb{A}^S_{F_0}$ denote the adeles over F_0 away from S. For any $f \in C^\infty_c(G_0(\mathbb{A}^S_{F_0}))$, there exists $f^H \in C^\infty_c(H_0(\mathbb{A}^S_{F_0}))$ such that the $\mathbb{A}^S_{F_0}$-analogue of (8.1.4.1) holds. Here the notion of an adelic (G_0,H_0)-regular element is defined in* [**Kot90**, §7, pp. 178–179], *and all the orbital integrals are defined with respect to adelic Haar measures.*

Remark 8.1.5. — Part (1) of Theorem 8.1.4 appears to be stronger than the original form of the Langlands–Shelstad Conjecture in two ways. Firstly, the original conjecture is about transferring functions on G to functions on a central extension H_1 of H. More precisely, fix a z-extension $1 \to Z \to G_1 \to G \to 1$ and obtain H_1 as in §8.1.3. For a choice of $\eta_1 : {}^LH_1 \to {}^LG_1$ (recall that $\eta_1|_{\widehat{H_1}}$ is canonical), the conjecture concerns transferring functions in $C^\infty_c(G(F))$ to functions in $C^\infty_c(H_1(F), \lambda)$. Here λ is a character on $Z(F)$ determined by η_1, and $C^\infty_c(H_1(F), \lambda)$ denotes the set of functions in $C^\infty(H_1(F))$ that transform under $Z(F)$ by λ and whose supports are compact modulo $Z(F)$. Now under our assumption that $\mathcal{H} = {}^LH$, we may and shall pin down η_1 as in §8.1.3, and then $\lambda = 1$. In view of the definition of the transfer factors discussed in §8.1.3, we know that under the natural bijection $C^\infty_c(H_1(F), 1) \xrightarrow{\sim} C^\infty_c(H(F))$, a Langlands–Shelstad transfer of $f \in C^\infty_c(G(F))$ to $C^\infty_c(H_1(F), 1)$ in the original sense corresponds to a Langlands–Shelstad transfer of f to $C^\infty_c(H(F))$ in the sense of Theorem 8.1.4.

Secondly, in the original conjecture the identity (8.1.4.1) is only required to hold for all G-regular γ_H. In [**LS90**, §2.4], Langlands–Shelstad prove that this indeed implies (8.1.4.1) for all (G, H)-regular γ_H, under the assumption that G^{der} is simply connected. In view of the last paragraph, we know that this implication is still valid without assuming that G^{der} is simply connected (but always under the assumption that $\mathcal{H} = {}^L H$).

Similar remarks also apply to part (2) of Theorem 8.1.4.

8.2. Calculation of some invariants

In this section let G be the special orthogonal group of an arbitrary quadratic space of dimension $d \geq 2$ over $\mathbb{Q}$. Let $m := \lfloor d/2 \rfloor$.

Proposition 8.2.1. — *Assume that G is not the split* SO_2. *Then the Tamagawa number $\tau(G) = 2$.*

Proof. — By [**Kot84b**, (5.1.1)] and Weil's conjecture on Tamagawa numbers proved in [**Kot88**], we have

$$\tau(G) = \left|\pi_0(Z(\widehat{G})^{\Gamma_{\mathbb{Q}}})\right| / \left|\ker^1(\mathbb{Q}, Z(\widehat{G}))\right|. \tag{8.2.1.1}$$

First assume that $d \geq 3$. Then $\widehat{G}$ is a symplectic group of rank at least 1 or an even orthogonal group of rank at least 2, so $Z(\widehat{G}) \cong \mu_2$. In particular, $\ker^1(\mathbb{Q}, Z(\widehat{G})) = 0$ by Chebotarev's density theorem. On the other hand $\pi_0(Z(\widehat{G})^{\Gamma_{\mathbb{Q}}}) = Z(\widehat{G})$ has cardinality 2. Hence $\tau(G) = 2$.

Now assume that $d = 2$. Since G is not split, it is isomorphic to the norm-1 subtorus of $\mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_m$ for some quadratic extension $K/\mathbb{Q}$. We have $Z(\widehat{G}) = \widehat{G} = \mathbb{C}^\times$. The action of $\Gamma_{\mathbb{Q}}$ on $Z(\widehat{G})$ factors through $\mathrm{Gal}(K/\mathbb{Q})$, and the non-trivial element of $\mathrm{Gal}(K/\mathbb{Q})$ acts by $z \mapsto z^{-1}$. Hence $Z(\widehat{G})^{\Gamma_{\mathbb{Q}}} = \{\pm 1\}$. On the other hand, $\ker^1(\mathbb{Q}, Z(\widehat{G}))$ is the dual group of the finite abelian group $\ker^1(\mathbb{Q}, T)$ by [**Kot84b**, (3.4.5.1)], and the latter is trivial by the Hasse norm theorem (cf. [**PR94**, pp. 307–308]). Hence $\tau(G) = 2$. □

Definition 8.2.2. — Let H be reductive group over $\mathbb{R}$ assumed to contain elliptic maximal tori. Define

$$k(H) := \left|\mathrm{im}(\mathbf{H}^1(\mathbb{R}, T_e^{\mathrm{SC}}) \to \mathbf{H}^1(\mathbb{R}, T_e))\right|,$$

where T_e denotes an elliptic maximal torus in H and T_e^{SC} denotes the inverse image of T_e in H^{SC}. Since all elliptic maximal tori in H are conjugate under $H(\mathbb{R})$, $k(H)$ is well defined.

Proposition 8.2.3. — *Assume that $G_{\mathbb{R}}$ contains elliptic maximal tori. Then $k(G) = 2^{m-1}$.*

Proof. — If $d = 2$, then $G_{\mathbb{R}}$ is a torus, so obviously $k(G) = 1$. In this case $m = 1$, so the proposition is true. Now assume that $d \geq 3$. Let T_e be an elliptic maximal torus in $G_{\mathbb{R}}$, which is in fact anisotropic. As argued in the proofs of [**Mor10b**, Lem. 5.4.2] and [**Mor11**, Lem. 5.2.2], we have(3)

$$k(G) = \left|\pi_0(\widehat{T_e}^{\Gamma_\infty})\right| / \left|\pi_0(Z(\widehat{G})^{\Gamma_\infty})\right|.$$

We have $\pi_0(Z(\widehat{G})^{\Gamma_\infty}) \cong Z(\widehat{G}) \cong \mathbb{Z}/2\mathbb{Z}$, and since $T_e \cong \mathrm{U}(1)^m$ we have $\pi_0(\widehat{T_e}^{\Gamma_\infty}) \cong (\mathbb{Z}/2\mathbb{Z})^m$. Hence $k(G) = 2^{m-1}$. □

Recall that $\mathrm{GL}_{j,\mathbb{R}}$ contains elliptic maximal tori precisely when $j = 1, 2$.

Proposition 8.2.4. — *For any $j \geq 1$, $\tau(\mathrm{GL}_j) = 1$. For $j = 1, 2$, $k(\mathrm{GL}_{j,\mathbb{R}}) = 1$.*

Proof. — For $j \geq 1$, $Z(\widehat{\mathrm{GL}}_j) = \mathbb{C}^\times$, on which $\Gamma_{\mathbb{Q}}$ acts trivially. Hence

$$\pi_0(Z(\widehat{\mathrm{GL}}_j)^{\Gamma_{\mathbb{Q}}}) = \pi_0(\mathbb{C}^\times) = 1,$$

and

$$\ker^1(\mathbb{Q}, Z(\widehat{\mathrm{GL}}_j)) = 1$$

by Chebotarev's density theorem. Thus $\tau(\mathrm{GL}_j) = 1$ by (8.2.1.1). Since $\mathrm{GL}_{1,\mathbb{R}}$ is a torus, we have $k(\mathrm{GL}_{1,\mathbb{R}}) = 1$. Any elliptic maximal torus T_e in $\mathrm{GL}_{2,\mathbb{R}}$ is isomorphic to $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_m$, and $\mathbf{H}^1(\mathbb{R}, T_e)$ is trivial by Shapiro's lemma. Hence $k(\mathrm{GL}_{2,\mathbb{R}}) = 1$. □

Corollary 8.2.5. — *Let M be a Levi subgroup of G defined over $\mathbb{Q}$. Let M' be the group in a bi-elliptic endoscopic G-datum for M. Let H' be the induced endoscopic group for G. Assume that M is not a direct product of copies of GL_1 and GL_2 over $\mathbb{Q}$, and assume that all four $\mathbb{R}$-groups $G_{\mathbb{R}}, M_{\mathbb{R}}, M'_{\mathbb{R}}, H_{\mathbb{R}}$ contain elliptic maximal tori. Then we have*

$$\frac{\tau(G)}{\tau(H)}\frac{\tau(M')}{\tau(M)} = \frac{k(H)}{k(G)}\frac{k(M)}{k(M')}.$$

Proof. — We have $M \cong M^{\mathrm{GL}} \times M^{\mathrm{SO}}$, where M^{GL} is a product of copies of GL_1 and GL_2, and M^{SO} is a special orthogonal group which is not the split SO_2 over $\mathbb{Q}$. Then M' is either a direct product of M^{GL} with one special orthogonal group S_0 of the same parity and absolute rank as M^{SO}, or a direct product of M^{GL} with two special orthogonal groups S_1, S_2 of the same parity as M^{SO} whose absolute ranks add up to that of M^{SO}. In both cases, none of S_i is the split SO_2 over $\mathbb{Q}$ since M' is an elliptic endoscopic group for M.

In the first case, H is a special orthogonal group of the same parity and absolute rank as G. By Proposition 8.2.1 we have $\tau(G) = \tau(H)$ and $\tau(M) = \tau(M')$. By

(3) In *loc. cit.* it is stated that $\left|\mathrm{im}(\mathbf{H}^1(\mathbb{R}, T_e \cap G^{\mathrm{der}}) \to \mathbf{H}^1(\mathbb{R}, T_e))\right| = \left|\pi_0(\widehat{T_e}^{\Gamma_\infty})\right| / \left|\pi_0(Z(\widehat{G})^{\Gamma_\infty})\right|$, and in that context G^{der} is simply connected. For the correct generalization, one replaces the left hand side by $k(G)$.

Proposition 8.2.3 we have $k(G) = k(H)$ and $k(M) = k(M')$. The desired identity holds.

In the second case, H is a direct product of two special orthogonal groups H_1, H_2 whose absolute ranks m_1, m_2 add up to that of G, and neither of the two is the split SO_2 over $\mathbb{Q}$ since H is an elliptic endoscopic group for G. By Proposition 8.2.1 and the multiplicativity of $\tau(\cdot)$ with respect to direct products, we have $\tau(G) = 2$, $\tau(H) = \tau(H_1)\tau(H_2) = 4$, and

$$\tau(M') = \tau(S_1)\tau(S_2)\tau(M^{\mathrm{GL}}) = 4\tau(M^{\mathrm{GL}}) = 2\tau(M^{\mathrm{SO}})\tau(M^{\mathrm{GL}}) = 2\tau(M).$$

Hence the LHS of the desired identity is 1. By Proposition 8.2.3 and the multiplicativity of $k(\cdot)$ with respect to direct products, we have

$$k(G) = 2^{m-1} = 2 \cdot 2^{m_1-1}2^{m_2-1} = 2k(H_1)k(H_2) = 2k(H),$$

and similarly

$$k(M) = k(M^{\mathrm{GL}})k(M^{\mathrm{SO}}) = 2k(M^{\mathrm{GL}})k(S_1)k(S_2) = 2k(M').$$

Hence the RHS of the desired identity is also 1. □

8.3. The simplified geometric side of the stable trace formula

We recall the definition of the simplified geometric side of the stable trace formula, applicable to test functions which are stable cuspidal at infinity. This stems from Kottwitz's work in his unpublished notes. Our exposition follows [**Mor10b**, §5.4]. More discussion on the relationship between the simplified geometric side given here and the "usual" stable trace formula appearing in Arthur's work is given in §9.1 below.

Definition 8.3.1. — Let M be a reductive group over $\mathbb{R}$ containing elliptic maximal tori. Fix a Haar measure on $M(\mathbb{R})$. Let $\bar{M}$ be the inner form of M over $\mathbb{R}$ that is anisotropic modulo center (which exists by our assumption on M). Define

$$\bar{v}(M) := e(\bar{M}) \operatorname{vol}(\bar{M}(\mathbb{R})/A_M(\mathbb{R})^0),$$

where $e(\bar{M})$ is the Kottwitz sign of $\bar{M}$, $\bar{M}(\mathbb{R})$ is equipped with the Haar measure transferred from that on $M(\mathbb{R})$, and $A_M(\mathbb{R})^0$ is equipped with the canonical Haar measure obtained by choosing an $\mathbb{R}$-algebraic group isomorphism $\phi : A_M \xrightarrow{\sim} \mathbb{G}_m^n$ and pulling back the Lebesgue measure along the composite isomorphism

$$\log\phi : A_M(\mathbb{R})^0 \xrightarrow{\phi} (\mathbb{R}_{>0})^n \xrightarrow{(x_i)_i \mapsto (\log x_i)_i} \mathbb{R}^n.$$

(This measure on $A_M(\mathbb{R})^0$ is indeed canonical since a different choice of ϕ would replace $\log\phi$ by $g \circ \log\phi$ for some $g \in \mathrm{GL}_n(\mathbb{Z})$.)

Definition 8.3.2. — Let G be a reductive group over $\mathbb{R}$. Fix a quasi-character $\nu : A_G(\mathbb{R})^0 \to \mathbb{C}^\times$. Let M be a Levi subgroup of G such that M contains elliptic maximal tori (of M), and let $f \in C_c^\infty(G(\mathbb{R}), \nu^{-1})$ be a stable cuspidal function (see

[**Art89**, §4], [**Mor10b**, §5.4]). For $\gamma \in M(\mathbb{R})$ semi-simple elliptic, we define

$$S\Phi_M^G(\gamma, f) := (-1)^{\dim A_M} k(M)k(G)^{-1}\bar{v}(M_\gamma^0)^{-1} \sum_{\Pi} \Phi_M(\gamma^{-1}, \Theta_\Pi) \operatorname{Tr}(f \mid \Pi),$$

where Π runs through the discrete series L-packets belonging to ν, Θ_Π denotes the stable character associated to Π, and $\Phi_M(\cdot, \Theta_\Pi)$ is the normalized stable discrete series character as in §4.2.1. This definition depends on the choices of a Haar measure on $M_\gamma^0(\mathbb{R})$ (used to define $\bar{v}(M_\gamma^0)$) and a Haar measure on $G(\mathbb{R})$ (used to define $\operatorname{Tr}(f \mid \Pi)$).

Definition 8.3.3. — Let G be a reductive group over $\mathbb{Q}$. Assume that G is cuspidal in the sense of Definition 1.1.6. For $f = f^\infty f_\infty \in C_c^\infty(G(\mathbb{A}))$ with $f_\infty \in C_c^\infty(G(\mathbb{R}), \nu^{-1})$ stable cuspidal (where ν is a fixed quasi-character $A_G(\mathbb{R})^0 \to \mathbb{C}^\times$), and for $M \subset G$ a Levi subgroup that is cuspidal, define

$$ST_M^G(f) := \tau(M) \sum_{\gamma} \bar{\iota}^M(\gamma)^{-1} SO_\gamma(f_M^\infty) S\Phi_M^G(\gamma, f_\infty),$$

where γ runs through a set of representatives of the stable conjugacy classes of the $\mathbb{R}$-elliptic semi-simple elements of $M(\mathbb{Q})$, and

$$\bar{\iota}^M(\gamma) := \left|(M_\gamma/M_\gamma^0)(\mathbb{Q})\right|.$$

For $M \subset G$ a Levi subgroup that is not cuspidal, define

$$ST_M^G(f) := 0.$$

We define

$$ST^G(f) := \sum_M (n_M^G)^{-1} ST_M^G(f),$$

where M runs through the Levi subgroups of G up to $G(\mathbb{Q})$-conjugacy, and n_M^G is as in Definition 1.1.1.

Remark 8.3.4. — We explain how the Haar measures are normalized in the definitions of $ST_M^G(f)$ and $ST^G(f)$ so that the results are independent of the Haar measures. For each $SO_\gamma(f_M^\infty)$, we need Haar measures on $M_\gamma^0(\mathbb{A}_f)$ and $M(\mathbb{A}_f)$ to define the stable orbital integral $SO_\gamma(\cdot)$, and need Haar measures on $M(\mathbb{A}_f)$ and $G(\mathbb{A}_f)$ to define the constant term f_M^∞. We assume that the two measures on $M(\mathbb{A}_f)$ are the same. Then $SO_\gamma(f_M^\infty)$ depends only on the Haar measures on $M_\gamma^0(\mathbb{A}_f)$ and $G(\mathbb{A}_f)$. Now in the definition of $S\Phi_M^G(\gamma, f_\infty)$, we need Haar measures on $M_\gamma^0(\mathbb{R})$ and $G(\mathbb{R})$ (cf. Definition 8.3.2). We assume that the measures on $M_\gamma^0(\mathbb{A}_f)$ and $M_\gamma^0(\mathbb{R})$ multiply to the Tamagawa measure on $M_\gamma^0(\mathbb{A})$, and assume that the measures on $G(\mathbb{A}_f)$ and $G(\mathbb{R})$ multiply to the Tamagawa measure on $G(\mathbb{A})$. Then $ST_M^G(f)$ and $ST^G(f)$ are independent of the choices of Haar measures.

8.4. Test functions on endoscopic groups

8.4.1. — We now keep the notation and setting in §1.8.3 and Theorem 1.8.4. In particular, $G = \mathrm{SO}(V, q)$, where (V, q) is a quadratic space over $\mathbb{Q}$ of dimension $d \geq 5$, signature $(d-2, 2)$, and discriminant $\delta \in \mathbb{Q}^\times/\mathbb{Q}^{\times,2}$. Assume that the G-representation $\mathbb{V}$ fixed in §1.7.1 is absolutely irreducible. Fix a prime $p \notin \Sigma(\mathbf{O}(V), \mathbb{V}, \lambda, K, f^\infty)$, and fix an integer $a \geq a_0(\mathbf{O}(V), \mathbb{V}, \lambda, K, f^\infty, p)$. Let $f^{p,\infty}$ be as in §1.8.3.

Let $\mathfrak{e}_{d^+,\delta^+,d^-,\delta^-} = (H, {}^L H, s, \eta)$ be an elliptic endoscopic datum for $G = \mathrm{SO}(V)$, presented in the explicit form as in §5.4. In the following we will always assume that $d^+ \geq 2$, or equivalently, that in the decomposition $H = H^+ \times H^- = \mathrm{SO}(V^+) \times \mathrm{SO}(V^-)$ the factor H^+ is non-trivial. By §5.4.4, every isomorphism class in $\mathscr{E}(G)$ can be represented by such a datum.

We follow [**Kot90**, §7] to define a test function $f^H \in C_c^\infty(H(\mathbb{A}))$. By definition, $f^H = 0$ unless the following condition is satisfied:

(†) The $\mathbb{R}$-group $H_\mathbb{R}$ contains anisotropic maximal tori(4), and the $\mathbb{Q}_p$-group $H_{\mathbb{Q}_p}$ is unramified.

Note that for our explicit representative $(H, {}^L H, s, \eta)$, the group $H_{\mathbb{Q}_p}$ is unramified if and only if the localization of the endoscopic datum $(H, {}^L H, s, \eta)$ over $\mathbb{Q}_p$ is unramified. Also, if $H_\mathbb{R}$ contains anisotropic maximal tori, then H is cuspidal as a $\mathbb{Q}$-group, and neither of $H^\pm_\mathbb{R}$ is isomorphic to the split SO_2 over $\mathbb{R}$. It easily follows from the last condition that the localization of the (globally elliptic) endoscopic datum $(H, {}^L H, s, \eta)$ over $\mathbb{R}$ remains elliptic, as an endoscopic datum over $\mathbb{R}$. Conversely, if H is cuspidal, then since A_H is trivial by the (global) ellipticity of $(H, {}^L H, s, \eta)$, we know that $H_\mathbb{R}$ contains anisotropic maximal tori. In conclusion, (†) is equivalent to the following condition:

(‡) The $\mathbb{Q}$-group H is cuspidal, and the $\mathbb{Q}_p$-group $H_{\mathbb{Q}_p}$ is unramified.

Moreover, as we have seen, these conditions imply that the endoscopic datum $(H, {}^L H, s, \eta)$ is elliptic over $\mathbb{R}$ and unramified over $\mathbb{Q}_p$. In the following we assume that (†) and (‡) hold.

By definition f^H is of the form

$$f^H = f^H_\infty f^H_p f^{H,p,\infty}$$

with $f^H_\infty \in C_c^\infty(H(\mathbb{R}))$ stable cuspidal, and $f^H_p \in C_c^\infty(H(\mathbb{Q}_p)), f^{H,p,\infty} \in C_c^\infty(H(\mathbb{A}_f^p))$. (As Z_H^0 is anisotropic over $\mathbb{R}$ we do not need to specify central characters for the notion of stable cuspidal functions.)

We fix a Haar measure on $H(\mathbb{A}_f^p)$ arbitrarily, and fix the Haar measure on $H(\mathbb{Q}_p)$ such that hyperspecial subgroups have volume 1. Then there is a unique Haar measure

(4) In [**Kot90**, §7], the more general condition at the archimedean place is that elliptic maximal tori in $G_\mathbb{R}$ should "come from" $H_\mathbb{R}$. In our situation, since $G_\mathbb{R}$ contains anisotropic maximal tori, the condition simplifies to the one in the text.

on $H(\mathbb{R})$ such that the product measure on $H(\mathbb{A})$ is the Tamagawa measure. We fix this measure on $H(\mathbb{R})$ as well.

8.4.2. — The definition of f_∞^H will depend on the choice of an auxiliary datum $(j, B_{G,H})$, which we now specify. Here $j : T_H \xrightarrow{\sim} T_G$ is an admissible isomorphism between anisotropic maximal tori $T_H \subset H_\mathbb{R}$ and $T_G \subset G_\mathbb{R}$; see §5.6 for the notion of admissible isomorphisms over $\mathbb{C}$, and note that any $\mathbb{C}$-isomorphism $T_{H,\mathbb{C}} \xrightarrow{\sim} T_{G,\mathbb{C}}$ is automatically defined over $\mathbb{R}$ since both T_H and T_G are anisotropic over $\mathbb{R}$. The other part $B_{G,H}$ is a Borel subgroup of $G_\mathbb{C}$ containing $T_{G,\mathbb{C}}$; in other words, $(T_G, B_{G,H})$ is a fundamental pair in $G_\mathbb{R}$. Later we shall also use the choice of $(j, B_{G,H})$ to normalize the archimedean transfer factors between H and G. The dependence of f_∞^H on $(j, B_{G,H})$ is analogous to the dependence of a transfer of a function from G to H on the normalization of transfer factors. However this is only an analogy, as f_∞^H is not defined to be the transfer of a function on G.

We now fix $(j, B_{G,H})$ once and for all in the following way. We let the fundamental pair $(T_G, B_{G,H})$ arise, in the way described in §6.1.6, from an elliptic decomposition (Definition 6.1.2) $\mathcal{D}^H \in \mathrm{ED}(V_\mathbb{R})$. Moreover, in the even case we assume that $\mathcal{D}^H$ gives rise to the orientation o_V on $V_\mathbb{R}$ fixed in §6.1.7. In other words, $\mathcal{D}^H \in \mathrm{ED}(V_\mathbb{R})^o$ in the notation of §6.1.7. As the notation suggests, *we shall make possibly different choices of* $\mathcal{D}^H$ *for different* $(H, {}^LH, s, \eta)$; a uniform choice is sometimes not possible because of some further conditions to be imposed in the following paragraph. Once $\mathcal{D}^H$ has been chosen, we choose j as follows. Recall that H is of the form $H = H^+ \times H^- = \mathrm{SO}(V^+) \times \mathrm{SO}(V^-)$. To define $j : T_H \xrightarrow{\sim} T_G$, we choose an elliptic decomposition $\mathcal{D}_H = (\mathcal{D}_{H^+}, \mathcal{D}_{H^-})$ of $(V_\mathbb{R}^+, V_\mathbb{R}^-)$ which should induce the fixed orientations on $V^\pm$ in the even case; in other words $\mathcal{D}_H \in \mathrm{ED}(V_\mathbb{R}^+)^o \times \mathrm{ED}(V_\mathbb{R}^-)^o$ in the notation of §6.1.9. Then we define j to be $j_{\mathcal{D}_H, \mathcal{D}^H}$ in the notation of §6.1.9. By Lemma 6.1.13, this j is indeed an admissible isomorphism.

Now let us specify further conditions on $\mathcal{D}^H$. Since the signature of $V_\mathbb{R}$ is $(d-2, 2)$, we know that $\mathcal{D}^H$ involves exactly one negative definite plane as its member. In the odd case, we assume that $\mathcal{D}^H$ lies in $\mathrm{ED}(V_\mathbb{R})^o_{\mathrm{nice}}$ as in Definition 6.2.12. This means that the unique negative definite member of $\mathcal{D}^H$ is the last member; cf. Example 6.2.13. In the even case, unless $m = d/2$ is odd and $d^+ = 2$, we assume that $\mathcal{D}^H$ lies in $\mathrm{ED}(V_\mathbb{R})^o_{\mathrm{nice}}$ as in Definitions 6.2.12 and 6.3.8, meaning that the unique negative definite member is the last (resp. second last) member if m is even (resp. odd). If in the even case m is odd and $d^+ = 2$, we assume that the unique negative definite member of $\mathcal{D}^H$ is the last member. In this case, $\mathcal{D}^H$ is not in $\mathrm{ED}(V_\mathbb{R})^o_{\mathrm{nice}}$, but it differs from an element thereof by the transposition $(m-1, m) \in \mathfrak{S}_m$.

As long as d is not $\equiv 2 \mod 4$, we can clearly choose $\mathcal{D}^H$ satisfying all the above conditions independently of $(H, {}^LH, s, \eta)$. When $d \equiv 2 \mod 4$, we need to adjust the choice of $\mathcal{D}^H$ according to whether $d^+ = 2$ or not. For instance, for all $(H, {}^LH, s, \eta)$ with $d^+ \neq 2$ we may choose $\mathcal{D}^H$ to be some common $\mathcal{D}$, and then we may choose

$\mathcal{D}^H$ for $d^+ = 2$ to be $(m-1, m) \cdot \mathcal{D}$, i.e., $\mathcal{D}$ with the last two members swapped. In particular, we see that in all cases, we may and shall arrange that T_G is independent of $(H, {}^L H, s, \eta)$, which justifies our notation.

Since $\mathrm{SO}(V^+)$ is non-trivial, our assumptions on $\mathcal{D}^H$ imply that the factor $\mathrm{U}(1)$ of T_G corresponding to the unique negative definite member of $\mathcal{D}^H$ is sent under j^{-1} into $\mathrm{SO}(V^+) \subset H$.

8.4.3. — The fixed choice of $(j, B_{G,H})$ determines a Borel subgroup B_H of $H_{\mathbb{C}}$ containing $T_{H,\mathbb{C}}$, a subset Ω_* of $\Omega = \Omega_{\mathbb{C}}(G, T_G)$, and a bijection induced by multiplication

$$\Omega_H \times \Omega_* \longrightarrow \Omega$$

as follows. Here $\Omega_H := \Omega_{\mathbb{C}}(H, T_H)$ is viewed as a subgroup of Ω via

$$\Omega_H \hookrightarrow \mathrm{Aut}(T_{H,\mathbb{C}}) \xrightarrow[j]{\sim} \mathrm{Aut}(T_{G,\mathbb{C}}) \supset \Omega.$$

The Borel subgroup B_H is characterized by the condition that the B_H-positive roots on $T_{H,\mathbb{C}}$ are transported via j to $B_{G,H}$-positive roots on $T_{G,\mathbb{C}}$. (Note that (T_H, B_H) is nothing but the fundamental pair in $H_{\mathbb{R}}$ determined by $\mathcal{D}^H$ as in §6.1.6, where $\mathcal{D}^H$ is as in §8.4.2.) The subset $\Omega_* \subset \Omega$ consists of those $\omega \in \Omega$ such that the B_H-positive roots on $T_{H,\mathbb{C}}$ are transported via j to $\omega B_{G,H}$-positive roots on $T_{G,\mathbb{C}}$.

Let $\mathbb{V}^*$ be the contragredient representation of $\mathbb{V}$. Let $\varphi_{\mathbb{V}^*}$ be the discrete Langlands parameter of $G_{\mathbb{R}}$ corresponding to $\mathbb{V}^*$, i.e., the L-packet of $\varphi_{\mathbb{V}^*}$ consists of discrete series representations of $G(\mathbb{R})$ having the same infinitesimal character as the $G(\mathbb{C})$-representation $\mathbb{V}^* \otimes_{\mathbb{E}} \mathbb{C}$ (which is irreducible). Let $\Phi_H(\varphi_{\mathbb{V}^*})$ be the set of equivalence classes of discrete Langlands parameters of $H_{\mathbb{R}}$ that induce the equivalence class of $\varphi_{\mathbb{V}^*}$ via $\eta : {}^L H \to {}^L G$. As on [**Kot90**, p. 185], we have a bijection

$$\omega_*(\cdot) : \Phi_H(\varphi_{\mathbb{V}^*}) \xrightarrow{\sim} \Omega_*, \quad \varphi_H \longmapsto \omega_*(\varphi_H),$$

characterized by the condition that φ_H is aligned with $(\omega_*(\varphi_H)^{-1} \circ j, B_{G,H}, B_H)$ in the sense of [**Kot90**, p. 184].

For any $\varphi_H \in \Phi_H(\varphi_{\mathbb{V}^*})$, define

$$f_{\varphi_H} := d(H)^{-1} \sum_{\pi \in \Pi(\varphi_H)} f_\pi \in C_c^\infty(H(\mathbb{R})), \tag{8.4.3.1}$$

where the terms are explained in the following.

– The summation is over the discrete series representations π of $H(\mathbb{R})$ inside the L-packet $\Pi(\varphi_H)$ of φ_H.

– For each π, the function $f_\pi \in C_c^\infty(H(\mathbb{R}))$ is a pseudo-coefficient for π; see [**CD85**]. Note that this notion depends on the choice of a Haar measure on $H(\mathbb{R})$. We use the one fixed in §8.4.1.

– We define $d(H)$ to be the cardinality of $\Pi(\varphi_H)$. Note that this number is an invariant of $H_{\mathbb{R}}$, equal to the cardinality of the complex Weyl group divided by the cardinality of the real Weyl group of an elliptic (i.e., anisotropic) maximal torus.

The function f_{φ_H} is stable cuspidal. Using this, we build the function f^H_∞ in the following definition; cf. [**Kot90**, p. 186], [**Mor10b**, §6.2].

Definition 8.4.4. — We define

$$f^H_\infty := (-1)^{q(G_\mathbb{R})}\langle \mu_{T_G}, s\rangle_j \sum_{\varphi_H \in \Phi_H(\varphi_{\mathbb{V}^*})} \det(\omega_*(\varphi_H)) f_{\varphi_H} \in C^\infty_c(H(\mathbb{R})).$$

Here $\mu_{T_G} \in X_*(T_G)$ is the Hodge cocharacter of any h in the Shimura datum $\mathcal{X}$ that *factors through* T_G. The number $\langle \mu_{T_G}, s\rangle_j$ is defined to be the image of $(j^{-1} \circ \mu_{T_G}, s)$ under the canonical pairing

$$X_*(T_H) \times Z(\widehat{H}) \to \pi_1(H) \times Z(\widehat{H}) = X^*(Z(\widehat{H})) \times Z(\widehat{H}) \to \mathbb{C}^\times.$$

For each $\omega \in \Omega$, we write $\det(\omega)$ for the sign of ω.(5)

Remark 8.4.5. — By construction f^H_∞ is stable cuspidal.

Lemma 8.4.6. — *We have* $\langle \mu_{T_G}, s\rangle_j = 1$.

Proof. — Using the observation made at the end of §8.4.2, we compute that the image of $j^{-1} \circ \mu_{T_G} \in X_*(T_H)$ in $\pi_1(H) \cong \pi_1(H^+) \times \pi_1(H^-)$ has non-trivial projection in $\pi_1(H^+) \cong \mathbb{Z}/2\mathbb{Z}$ and trivial projection in $\pi_1(H^-)$. We conclude the proof by recalling that s has trivial component in $Z(\widehat{H^+})$. □

8.4.7. — We normalize the transfer factors between $(H, {}^L H, s, \eta)$ and G at various places as follows.

We use the canonical unramified normalization associated to K_p of the transfer factors at p (see [**Hal93**]), denoted by $(\Delta^G_H)_p$. Associated to the datum $(j, B_{G,H})$ fixed in §8.4.2, we have Kottwitz's normalization [**Kot90**, §7] for the transfer factors at ∞, which we denote by $\Delta_{j,B_{G,H}}$ (cf. §§6.2–6.3) and also by $(\Delta^G_H)_\infty$. We normalize the transfer factors away from p and ∞ such that at almost all unramified places we have the canonical unramified normalization (associated to the hyperspecial subgroup determined by some reductive model of G over $\mathbb{Z}[1/\Sigma]$ for some finite set Σ of primes) and such that the global product formula with $(\Delta^G_H)_p$ and $(\Delta^G_H)_\infty$ is satisfied (see [**LS87**, §6]). For each place $v \notin \{p, \infty\}$, we denote our normalization by $(\Delta^G_H)_v$.

We are now ready to give the definitions of the other two parts $f^{H,p,\infty}$ and f^H_p in f^H.

Definition 8.4.8. — Define $f^{H,p,\infty} \in C^\infty_c(H(\mathbb{A}^p_f))$ to be a Langlands–Shelstad transfer of $f^{p,\infty}$ as in Theorem 8.1.4 with respect to the Haar measure $dg^{p,\infty}$ on $G(\mathbb{A}^p_f)$ fixed in §1.8.3 and the Haar measure on $H(\mathbb{A}^p_f)$ fixed in §8.4.1. Here the transfer factors are normalized as in §8.4.7.

(5) This is indeed equal to the determinant of ω acting on the finite free $\mathbb{Z}$-module $X_*(T_G)$, which explains the notation. In §4.2.2 the sign function is denoted by $\epsilon(\cdot)$, but in the current chapter we prefer the notation $\det(\cdot)$.

Definition 8.4.9. — Let $\mu : \mathbb{G}_m \to G_{\mathbb{Q}_p}$ be a Hodge cocharacter of the Shimura datum $\mathbf{O}(V)$ defined over $\mathbb{Q}_p$ (see §1.5.1). Let $f_{-\mu}$ be the element of $\mathcal{H}^{\mathrm{ur}}(G_{\mathbb{Q}_{p^a}})$ associated to $-\mu$ as in Definition 7.1.6. Let $f_p^H = b(f_{-\mu})$ be the image of $f_{-\mu}$ under the twisted transfer map $b : \mathcal{H}^{\mathrm{ur}}(G_{\mathbb{Q}_{p^a}}) \to \mathcal{H}^{\mathrm{ur}}(H_{\mathbb{Q}_p})$ as in §7.2.2. We identify f_p^H with a realization of it in $C_c^\infty(H(\mathbb{Q}_p))$; see Remark 8.4.10 below.

Remark 8.4.10. — Once an element $f_p^H \in \mathcal{H}^{\mathrm{ur}}(H)$ is specified, it still corresponds ambiguously to different functions on $H(\mathbb{Q}_p)$. Namely, for each choice of a hyperspecial subgroup $K_{H,p}$ of $H(\mathbb{Q}_p)$ there is a corresponding $K_{H,p}$-bi-invariant function in $\mathcal{H}(H(\mathbb{Q}_p) /\!\!/ K_{H,p})$. These functions have the same stable orbital integrals, as noted in [**Kot90**, §7]. Indeed, as we saw in §7.1.2, these functions are related to each other under pull-back by inner automorphisms of $H_{\mathbb{Q}_p}$, and these automorphisms do not permute the stable conjugacy classes. The same remark applies to the various canonical constant terms (see Proposition 7.1.10) $(f_p^H)_{M'} \in \mathcal{H}^{\mathrm{ur}}(M')$ for Levi subgroups M' of H defined over $\mathbb{Q}_p$. It follows that the evaluation of ST^H (Definition 8.3.3) at the test function $f^H = f_\infty^H f_p^H f^{H,p,\infty}$ is unaffected by the ambiguity in f_p^H.

Remark 8.4.11. — The function f^H depends on a via the component f_p^H.

8.4.12. — Now suppose M is a standard proper Levi subgroup of G (i.e., one of M_1, M_2, M_{12} as in §1.4) and consider a bi-elliptic endoscopic G-datum for M

$$\mathfrak{e}_{A,B,\mathfrak{p}} = \mathfrak{e}_{A,B,d^+,\delta^+,d^-,\delta^-} = (M', {}^LM', s_M, \eta_M)$$

presented in the explicit form as in §5.5.6. More precisely, the construction in §5.5.6 depends on the choice of a hyperbolic basis as in §5.5.2. Thus we need to fix a hyperbolic basis of $W_1^\perp = V_1 \oplus V/V_1^\perp$ (resp. $W_2^\perp = V_2 \oplus V/V_2^\perp$) when $M \in \{M_2, M_{12}\}$ (resp. $M = M_1$). We always take the hyperbolic basis $\{e_1, e_1'\}$ of $W_1^\perp$ and the hyperbolic basis $\{e_1, e_2, e_2', e_1'\}$ of $W_2^\perp$, where e_i, e_i' are as in §1.4.3.

As in §5.5.6 and Proposition 5.5.7, $\mathfrak{e}_{A,B,\mathfrak{p}}$ induces the endoscopic datum

$$e_{\mathfrak{p}}(M) = (M', {}^LM', s'_M, \eta_M)$$

for M, and the endoscopic datum

$$\mathfrak{e}_{d^++2|A|+4|B|,\delta^+,d^-+2|A^c|+4|B^c|,\delta^-} = (H, {}^LH, s, \eta)$$

for G. Moreover, recall that we have fixed in §5.5.9 an $H(\mathbb{Q})$-conjugacy class of embeddings $M' \hookrightarrow H$ with images Levi subgroups, and in particular we have the diagram (5.5.9.1) commuting up to $\widehat{G}$-conjugation. We now fix such an embedding $M' \hookrightarrow H$ on the nose.

We assume that H satisfies condition (†) in §8.4.1. It follows that M' is unramified at p, and the endoscopic datum $(M', {}^LM', s'_M, \eta_M)$ for M is unramified at p. Also we assume that the parameter $\mathfrak{p}$ is such that the component of s_M in $\widehat{M^{\mathrm{SO}}}$ is not -1, from which it follows that H^+ is non-trivial. Thus the preceding discussion in this section can be applied to $(H, {}^L H, s, \eta)$. We normalize the transfer factors between $(M', {}^L M', s'_M, \eta_M)$ and M as follows.

Away from p and ∞, we normalize the transfer factors by inheriting the normalization between $(H, {}^L H, s, \eta)$ and G fixed in §8.4.7, with respect to our fixed embedding $M' \hookrightarrow H$; see Remark 8.4.13 below. At p, we use the canonical unramified normalization associated to the hyperspecial subgroup of $M(\mathbb{Q}_p)$ determined by K_p (i.e., the image of $P(\mathbb{Q}_p) \cap K_p$ under $P \to M$, where $P \subset G$ is the standard parabolic subgroup such that $M = M_P$; cf. Remark 2.4.5), which is also the same as the normalization inherited from the canonical unramified normalization between $(H, {}^L H, s, \eta)$ and G associated to K_p. For later reference, for each finite place v, we denote the above-mentioned normalization by $(\Delta^M_{M'})^{A,B}_v$ or simply $(\Delta^M_{M'})^{A,B}$. We denote the above-mentioned hyperspecial subgroup of $M(\mathbb{Q}_p)$ by $\mathcal{M}(\mathbb{Z}_p)$. At ∞, we do not yet fix a normalization. In fact, precise knowledge about signs between different normalizations in this case is key to our later computation; this will be investigated in §8.9 below.

Remark 8.4.13. — At each place v of $\mathbb{Q}$, there is a notion of the normalization of the transfer factors between $(M', {}^L M', s'_M, \eta_M)$ and M *inherited* from the normalization of the transfer factors between $(H, {}^L H, s, \eta)$ and G with respect to our fixed $M' \hookrightarrow H$. It is described via a simple formula as in [**Mor10b**, §5.2] or [**Mor11**, §5.1]. Roughly speaking, this means that apart from the difference in Δ_{IV}, the transfer factor between M' and M is equal to the transfer factor between H and G for any G-regular element of $M'(\mathbb{Q}_v) \subset H(\mathbb{Q}_v)$ and any preimage of it in $M(\mathbb{Q}_v) \subset G(\mathbb{Q}_v)$. Here it is crucial that the diagram (5.5.9.1) commutes up to $\widehat{G}$-conjugacy.

An important property is that if the normalizations between H and G at all places satisfy the global product formula, then so do the inherited normalizations between M' and M at all places; this is due to the fact that our choice of $M' \hookrightarrow H$ is global. To see this, one simply notes that the term Δ_{IV} can be ignored from the definition of transfer factor when deciding whether local normalizations satisfy the global product formula.

We now say a few words on the proof of the existence of the inherited normalization. The original source is Kottwitz's unpublished notes, where this result is marked as an easy consequence of the definition of transfer factors in [**LS87**]. Indeed it can be proved similarly as [**Hal93**, Lem. 9.2]. Alternatively, in our particular situation, one can prove this without much difficulty using the explicit formulas for the transfer factors in [**Wal10**].

Proposition 8.4.14. — *Keep the setting of §8.4.12. The function* $(f^{H,p,\infty})_{M'} \in C_c^\infty(M'(\mathbb{A}_f^p))$ *is a Langlands–Shelstad transfer of* $(f^{p,\infty})_M \in C_c^\infty(M(\mathbb{A}_f^p))$ *in the sense of Theorem 8.1.4, with respect to the normalization of transfer factors* $(\Delta^M_{M'})_{A,B}$ *as in §8.4.12.*

Proof. — In view of the Fundamental Lemma we can pass to a local setting over some $\mathbb{Q}_v$ (with $v \neq p, \infty$) instead of the adelic setting. The statement can then be proved similarly as [**Mor10b**, Lem. 6.3.4], with the following two modifications.

Firstly, we replace G_γ and M_γ by G_γ^0 and M_γ^0 in the proof of part (i) of *loc. cit.*.

Secondly, in the proof of part (ii) of *loc. cit.*, Morel cites [**LS90**, Lem. 2.4.A] in order to reduce the proof to checking the matching of orbital integrals for those $\gamma' \in M'(\mathbb{Q}_v)_{\mathrm{ss}}$ that are M-regular, or even G-regular (meaning that all matching elements of $M(\overline{\mathbb{Q}}_v)_{\mathrm{ss}}$ are G-regular).(6) Since M_{der} is not simply connected in our case, we cannot directly apply [**LS90**, Lem. 2.4.A], but this can be circumvented by the following argument. To simplify notation, we understand that all reductive groups and endoscopic data are over $\mathbb{Q}_v$. Suppose we have already established that $\phi \in C_c^\infty(M(\mathbb{Q}_v))$ and $\phi' \in C_c^\infty(M'(\mathbb{Q}_v))$ have matching orbital integrals for all G-regular $\gamma' \in M'(\mathbb{Q}_v)_{\mathrm{ss}}$, and want to deduce the same for all (M, M')-regular $\gamma' \in M'(\mathbb{Q}_v)_{\mathrm{ss}}$. As in §8.1.3, we pick a z-extension $1 \to Z \to M_1 \to M \to 1$, and obtain from it a central extension $1 \to Z \to M_1' \to M' \to 1$ as well as an endoscopic datum $(M_1', {}^L M_1', s'_{M_1}, \eta_{M_1})$ for M_1 such that the diagram analogous to (8.1.3.1) commutes. As in Remark 8.1.5, we identify ϕ with a function $\phi_1 \in C_c^\infty(M_1(\mathbb{Q}_v), 1_Z)$, and identify ϕ' with a function $\phi_1' \in C_c^\infty(M_1'(\mathbb{Q}_v), 1_Z)$, where in both cases 1_Z denotes the trivial character on Z. We say that an element of $M_1'(\mathbb{Q}_v)_{\mathrm{ss}}$ is G-regular if all the matching elements of $M_1(\overline{\mathbb{Q}}_v)_{\mathrm{ss}}$ are preimages of G-regular elements of $M(\overline{\mathbb{Q}}_v)_{\mathrm{ss}}$. Then ϕ_1 and ϕ_1' have matching orbital integrals for all G-regular elements of $M_1'(\mathbb{Q}_v)_{\mathrm{ss}}$. Now note that for any maximal torus $T \subset M_1'$, there is a dense subset of $T(\mathbb{Q}_v)$ consisting of G-regular elements. By this and the proof of [**LS90**, Lem. 2.4.A], ϕ_1 and ϕ_1' have matching orbital integrals for all (M_1, M_1')-regular elements of $M_1'(\mathbb{Q}_v)_{\mathrm{ss}}$. It follows that ϕ and ϕ' have matching orbital integrals for all (M, M')-regular elements of $M'(\mathbb{Q}_v)_{\mathrm{ss}}$, as desired. □

8.5. Statement of the main computation

8.5.1. — Let M be a standard proper Levi subgroup of G. Define

(8.5.1.1)
$$\mathrm{Tr}'_M = (n_M^G)^{-1} \sum_{\substack{\mathfrak{e}_{A,B,\mathfrak{p}} = (M', {}^L M', s_M, \eta_M) \\ \in \dot{\mathscr{E}}_G(M)}} |\mathrm{Out}_G(\mathfrak{e}_{A,B,\mathfrak{p}})|^{-1}\, \tau(G)\tau(H)^{-1} ST^H_{M'}(f^H).$$

Here the summation is over a subset $\dot{\mathscr{E}}_G(M)$ of the set of explicitly presented bi-elliptic endoscopic G-data for M as in §5.5.6 (in other words, $\dot{\mathscr{E}}_G(M)$ is a subset of the parameter set $\mathscr{P}_{r,t} \times' \mathscr{P}_W = \{(A, B, \mathfrak{p})\}$ in the notation of §5.5.6) such that the component of s_M in $\widehat{M^{\mathrm{SO}}}$ is not -1 and such that each isomorphism class in $\mathscr{E}_G(M)$

(6) Note the following typo: In the second line of the second paragraph of the proof of [**Mor10b**, Lem. 6.3.4], “regular in **H**” should be “regular in **M**”.

is represented exactly once. (Clearly the two conditions can be simultaneously met.) For each $(M', {}^L M', s_M, \eta_M) \in \dot{\mathscr{E}}_G(M)$, we let $(H, {}^L H, s, \eta)$ be the induced endoscopic datum for G. More precisely, for $(M', {}^L M', s_M, \eta_M) = \mathfrak{e}_{A,B,d^+,\delta^+,d^-,\delta^-}$, we let

$$(H, {}^L H, s, \eta) := \mathfrak{e}_{d^+ + 2|A| + 4|B|, \delta^+, d^- + 2|A^c| + 4|B^c|, \delta^-}$$

as in Proposition 5.5.7. Note that H^+ is non-trivial by our assumption on s_M. The function f^H is defined in §8.4. We fix $M' \hookrightarrow H$ as in §8.4.12 so as to view M' as a Levi subgroup of H, and define $ST^H_{M'}(f^H)$ as in Definition 8.3.3.

Note that our definition of Tr'_M is independent of the choice of $\dot{\mathscr{E}}_G(M)$. Indeed, one directly checks that the summand associated to $(A, B, \mathfrak{p})$ is equal to that associated to $(A^c, B^c, \mathtt{sw}(\mathfrak{p}))$ (in the case where both parameters satisfy the condition on s_M imposed before). Hence such a summand depends only on the isomorphism class of $\mathfrak{e}_{A,B,\mathfrak{p}}$ in $\mathscr{E}_G(M)$.

Recall that the definition of f^H depends on the fixed integer

$$a \geq a_0(\mathbf{O}(V), \mathbb{V}, \lambda, K, f^\infty, p).$$

Clearly the definitions of both f^H and Tr'_M make sense for all integers $a \geq 1$. We shall henceforth view Tr'_M as a function in $a \in \mathbb{Z}_{\geq 1}$. On the other hand, we have $\mathrm{Tr}_M(f^{p,\infty} dg^{p,\infty}, K, a)$ as in Definition 2.4.3. We abbreviate it as Tr_M, and also view it as a function in $a \in \mathbb{Z}_{\geq 1}$.

Theorem 8.5.2. — *For all large enough a we have $\mathrm{Tr}_M = \mathrm{Tr}'_M$.*

8.5.3. — Note that in the even case and for $M = M_2$, we have $\mathrm{Tr}_M = 0$ since $(M_l)_{\mathbb{R}}$ does not contain elliptic maximal tori (see Remark 2.4.6). In this case, we also know that each M' appearing in (8.5.1.1) is non-cuspidal, and hence $ST^H_{M'} \equiv 0$. Indeed, recall that $M' = M^{\mathrm{GL}} \times M'^{,\mathrm{SO}}$, where $M'^{,\mathrm{SO}}$ is the group in the elliptic endoscopic datum $\mathfrak{e}_{d^+,\delta^+,d^-,\delta^-}(W_1)$ for $M_2^{\mathrm{SO}} = \mathrm{SO}(W_1)$. This ellipticity, together with the fact that M_2^{SO} is not the split SO_2 over $\mathbb{Q}$, implies that neither of $(d^\pm, \delta^\pm)$ is $(2,1)$ in $\mathbb{Z}_{\geq 0} \times (\mathbb{Q}^\times/\mathbb{Q}^{\times,2})$. Hence if M' is cuspidal, then $(M'^{,\mathrm{SO}})_{\mathbb{R}}$ must contain anisotropic maximal tori, and so as in §6.1.1 we have $\delta^\pm = (-1)^{d^\pm/2}$ in $\mathbb{R}^\times/\mathbb{R}^{\times,2}$, from which $\delta = (-1)^{(d^+ + d^-)/2} = (-1)^{d/2-1}$ in $\mathbb{R}^\times/\mathbb{R}^{\times,2}$, contradicting with the fact that $\delta = (-1)^{d/2}$ in $\mathbb{R}^\times/\mathbb{R}^{\times,2}$. Thus in the even case with $M = M_2$ we have already proved the theorem. The proof of the theorem in the remaining cases occupies §§8.6–8.14.

8.6. First simplifications

8.6.1. — We keep the setting of §8.5.1, and assume that we are not in the even case with $M = M_2$, since in that case Theorem 8.5.2 is already proved. As in §1.4.3, we have $M = \mathbb{G}_m^r \times \mathrm{GL}_2^t \times \mathrm{SO}(W)$ for some $r \in \{0,1,2\}, t \in \{0,1\}, W \in \{W_1, W_2\}$. Denote by $\mathscr{E}(M)^{c,\mathrm{ur}}$ the subset of $\mathscr{E}(M)$ consisting of isomorphism classes of endoscopic data whose groups M' are cuspidal and unramified over $\mathbb{Q}_p$. For each isomorphism class in $\mathscr{E}(M)^{c,\mathrm{ur}}$, we fix a representative of the form $\mathfrak{e}_{\mathfrak{p}}(M)$ for some $\mathfrak{p} \in \mathscr{P}_W$, where the notation is as in Definitions 5.4.2, 5.5.4, and §5.5.6. Thus there are *a priori* up to two choices of $\mathfrak{p}$ for each isomorphism class, and we fix one choice. We may and shall also assume that each choice $\mathfrak{p} = (d^+, \delta^+, d^-, \delta^-)$ satisfies $d^+ \geq 2$. In the following, we denote this set of representatives by $\dot{\mathscr{E}}(M)^{c,\mathrm{ur}}$.

Note that M^{SO} is never isomorphic to the split SO_2 over $\mathbb{Q}$. Hence the same argument as in §8.4.1 shows that every element $\mathfrak{e}_{\mathfrak{p}}(M)$ of $\dot{\mathscr{E}}(M)^{c,\mathrm{ur}}$ satisfies the following conditions:

(1) As in §5.5.6, write the group in $\mathfrak{e}_{\mathfrak{p}}(M)$ as $M' = M^{\mathrm{GL}} \times M'^{,\mathrm{SO}}$. Then the $\mathbb{R}$-group $(M'^{,\mathrm{SO}})_{\mathbb{R}}$ contains anisotropic maximal tori.

(2) The localization of $\mathfrak{e}_{\mathfrak{p}}(M)$ over $\mathbb{R}$ is still elliptic as an endoscopic datum over $\mathbb{R}$.

Lemma 8.6.2. — *We have*

$$n_M^G \operatorname{Tr}'_M = \sum_{\mathfrak{e}_{\mathfrak{p}}(M) \in \dot{\mathscr{E}}(M)^{c,\mathrm{ur}}} |\mathrm{Out}_M(\mathfrak{e}_{\mathfrak{p}}(M))|^{-1} \sum_{A,B} \tau(G)\tau(H)^{-1} ST^H_{M'}(f^H).$$

Here the second summation is over the following ranges:

– *In the odd case for $M = M_{12}$, we have $A \in \{\emptyset, \{1\}, \{2\}, \{1,2\}\}, B \in \{\emptyset\}$.*
– *In the even case for $M = M_{12}$, we have $A \in \{\emptyset, \{1,2\}\}, B \in \{\emptyset\}$.*
– *For $M = M_1$, we have $A \in \{\emptyset\}, B \in \{\emptyset, \{1\}\}$.*
– *In the odd case for $M = M_2$, we have $A \in \{\emptyset, \{1\}\}, B \in \{\emptyset\}$.*

For each triple $(e_{\mathfrak{p}}(M) = \mathfrak{e}_{d^+,\delta^+,d^-,\delta^-}(M), A, B)$ appearing in the summation, we set

$$(H, {}^L H, s, \eta) := \mathfrak{e}_{d^+ + 2|A| + 4|B|, \delta^+, d^- + 2|A^c| + 4|B^c|, \delta^-},$$

write M' for the group in $\mathfrak{e}_{\mathfrak{p}}(M)$, and as in §8.5.1 identify M' with a Levi subgroup of H so as to define $ST^H_{M'}(f^H)$.

Proof. — We first note that the formula $\mathfrak{e}_{d^+ + 2|A| + 4|B|, \delta^+, d^- + 2|A^c| + 4|B^c|, \delta^-}$ indeed gives an elliptic endoscopic datum for G, i.e., neither of $(d^+ + 2|A| + 4|B|, \delta^+)$ and $(d^- + 2|A^c| + 4|B^c|, \delta^-)$ is equal to $(2,1) \in \mathbb{Z}_{\geq 0} \times (\mathbb{Q}^\times / \mathbb{Q}^{\times,2})$. Indeed, since M^{SO} is not the split SO_2 over $\mathbb{Q}$, we know that neither of $(d^\pm, \delta^\pm)$ is equal to $(2,1)$, which immediately implies our assertion. Also, we have $d^+ + 2|A| + 4|B| \geq 2$ since we have already assumed that $d^+ \geq 2$ in §8.6.1. Thus $ST^H_{M'}(f^H)$ in the lemma is indeed defined.

It is clear from the definitions that if a term $ST^H_{M'}(f^H)$ on the RHS of (8.5.1.1) is non-zero, then H is cuspidal and unramified over $\mathbb{Q}_p$ (since otherwise $f^H = 0$), and M' is cuspidal (since otherwise $ST^H_{M'} \equiv 0$). Clearly the condition that H is unramified over $\mathbb{Q}_p$ is equivalent to the condition that M' is unramified over $\mathbb{Q}_p$. In the odd case, the cuspidality conditions are automatic. In the even case, suppose we have $(M', {}^L M', s_M, \eta_M) = \mathfrak{e}_{A,B,d^+,\delta^+,d^-,\delta^-} \in \dot{\mathscr{E}}_G(M)$ such that M' is cuspidal. Then as we have mentioned in §8.6.1, $(M'^{,\mathrm{SO}})_{\mathbb{R}}$ contains anisotropic maximal tori, so by the same argument as in §8.5.3 we have $\delta^{\pm} = (-1)^{d^{\pm}/2}$ in $\mathbb{R}^{\times}/\mathbb{R}^{\times,2}$. On the other hand the condition that H is cuspidal is equivalent to $H_{\mathbb{R}}$ having anisotropic maximal tori by the discussion in §8.4.1, and hence is equivalent to the conditions that $\delta^{+} = (-1)^{d^{+}/2+|A|+2|B|}$ and that $\delta^{-} = (-1)^{d^{-}/2+|A^c|+2|B^c|}$ in $\mathbb{R}^{\times}/\mathbb{R}^{\times,2}$. Thus given that M' is cuspidal and given that d is even, H is cuspidal if and only if $|A|$ and $|A^c|$ are both even.

The above discussion shows that in (8.5.1.1), we can replace the summation set $\dot{\mathscr{E}}_G(M)$ by the subset $\dot{\mathscr{E}}_G(M)^{c,\mathrm{ur}}$ consisting of elements $\mathfrak{e}_{A,B,\mathfrak{p}} = (M', {}^L M', s_M, \eta_M)$ such that M' is cuspidal and unramified over $\mathbb{Q}_p$, and such that $|A|$ and $|A^c|$ are even in case d is even. Thus up to re-choosing $\dot{\mathscr{E}}_G(M)$ (which does not affect the definition of Tr'_M), we may assume that whenever $\mathfrak{e}_{A,B,\mathfrak{p}} \in \dot{\mathscr{E}}_G(M)^{c,\mathrm{ur}}$ we have $\mathfrak{e}_{\mathfrak{p}}(M) \in \dot{\mathscr{E}}(M)^{c,\mathrm{ur}}$. We thus have a well-defined map $F : \dot{\mathscr{E}}_G(M)^{c,\mathrm{ur}} \to \dot{\mathscr{E}}(M)^{c,\mathrm{ur}}$ sending each $\mathfrak{e}_{A,B,\mathfrak{p}}$ to $\mathfrak{e}_{\mathfrak{p}}(M)$. For each $\mathfrak{e}_{\mathfrak{p}}(M) \in \dot{\mathscr{E}}(M)^{c,\mathrm{ur}}$, we let $\Gamma(\mathfrak{p})$ denote the set of (A,B) as in the summation range in the current lemma. We divide our analysis into two different cases.

Case 1. Suppose $\mathfrak{p} = (d^+, \delta^+, d^-, \delta^-)$ with $(d^+, \delta^+) \neq (d^-, \delta^-)$. Then one checks that $F^{-1}(\mathfrak{e}_{\mathfrak{p}}(M)) = \{\mathfrak{e}_{A,B,\mathfrak{p}} \mid (A,B) \in \Gamma(\mathfrak{p})\}$. Moreover, for each $\mathfrak{e}_{A,B,\mathfrak{p}} \in F^{-1}(\mathfrak{e}_{\mathfrak{p}}(M))$, we have $|\mathrm{Out}_G(\mathfrak{e}_{A,B,\mathfrak{p}})| = |\mathrm{Out}_M(\mathfrak{e}_{\mathfrak{p}}(M))|$. (See §§5.4.5 and 5.5.8 for the computation of these two groups.) Thus the summand indexed by $\mathfrak{e}_{\mathfrak{p}}(M)$ in the current lemma is equal to the sum over all $\mathfrak{e}_{A,B,\mathfrak{p}} \in F^{-1}(\mathfrak{e}_{\mathfrak{p}}(M))$ in (8.5.1.1).

Case 2. Suppose $\mathfrak{p} = (d^+, \delta^+, d^-, \delta^-)$ with $(d^+, \delta^+) = (d^-, \delta^-)$. Then

$$\left\{\mathfrak{e}_{A,B,\mathfrak{p}} \mid (A,B) \in \Gamma(\mathfrak{p})\right\} = F^{-1}(\mathfrak{e}_{\mathfrak{p}}(M)) \sqcup \left\{\mathfrak{e}_{A^c,B^c,\mathfrak{p}} \mid \mathfrak{e}_{A,B,\mathfrak{p}} \in F^{-1}(\mathfrak{e}_{\mathfrak{p}}(M))\right\}.$$

(The union is disjoint.) Moreover, for each $(A,B) \in \Gamma(\mathfrak{p})$, we have $|\mathrm{Out}_M(\mathfrak{e}_{\mathfrak{p}}(M))| = 2\,|\mathrm{Out}_G(\mathfrak{e}_{A,B,\mathfrak{p}})|$, and we know that the summand $\tau(G)\tau(H)^{-1}ST^H_{M'}(f^H)$ indexed by (A,B) in the current lemma is equal to the term $\tau(G)\tau(H)^{-1}ST^H_{M'}(f^H)$ in (8.5.1.1) arising from either $\mathfrak{e}_{A,B,\mathfrak{p}}$ or $\mathfrak{e}_{A^c,B^c,\mathfrak{p}}$, whichever lies in $\dot{\mathscr{E}}_G(M)$. Thus we again see that the summand indexed by $\mathfrak{e}_{\mathfrak{p}}(M)$ in the current lemma is equal to the sum over all $\mathfrak{e}_{A,B,\mathfrak{p}} \in F^{-1}(\mathfrak{e}_{\mathfrak{p}}(M))$ in (8.5.1.1). The proof of the lemma is complete. □

8.7. Expanding the simplified geometric side of the stable trace formula

Let $(\mathfrak{e}_{\mathfrak{p}}(M), A, B)$ be a summation index as in Lemma 8.6.2. We study the term $ST^H_{M'}(f^H)$ arising from this index.

Definition 8.7.1. — Let $\Sigma(M')$ be a set of representatives in $M'(\mathbb{Q})$ of the stable conjugacy classes in $M'(\mathbb{Q})$ that are $\mathbb{R}$-elliptic.

Lemma 8.7.2. — *We have an expansion*

$$ST^H_{M'}(f^H) = \tau(M') \sum_{\gamma' \in \Sigma(M')} \bar{\iota}^{M'}(\gamma')^{-1} SO_{\gamma'}(f^{H,\infty}_{M'}) S\Phi^H_{M'}(\gamma', f^H_{\infty}). \tag{8.7.2.1}$$

Here $f^{H,\infty} := f^{H,p,\infty} f_p^H$. *Moreover, in (8.7.2.1), only those* γ' *that are* (M, M')*-regular contribute non-trivially.*

Proof. — The first statement follows from the definitions. To show the second statement, suppose $\gamma' \in \Sigma(M')$ is not (M, M')-regular. We show that $S\Phi^H_{M'}(\gamma', f^H_\infty)$ already vanishes. For this, it suffices to show the vanishing of

$$\sum_{\Pi} \Phi^H_{M'}(\gamma'^{-1}, \Theta_\Pi) \operatorname{Tr}(f^H_\infty \mid \Pi),$$

where the summation is over the discrete series L-packets Π for $H_{\mathbb{R}}$. For this it suffices to show the vanishing of

$$\sum_{\varphi_H \in \Phi_H(\varphi_{\mathbb{V}^*})} \det(\omega_*(\varphi_H)) \Phi^H_{M'}(\gamma'^{-1}, \Theta_{\Pi(\varphi_H)}).$$

By [**Mor11**, Prop. 3.2.5, Rem. 3.2.6], the above quantity is zero provided that γ' is not (M, M')-regular. □

8.7.3. — We continue the study of (8.7.2.1). By Lemma 8.7.2, we only need to sum over those $\gamma' \in \Sigma(M')$ that are (M, M')-regular. By Proposition 8.4.14, we may further restrict to those γ' that is an image of a semi-simple element $\gamma_M \in M(\mathbb{A}^p_f)$, and in this case we have

$$SO_{\gamma'}(f^{H,p,\infty}_{M'}) = (\Delta^M_{M'})^{A,B}(\gamma', \gamma_M) O^{s'_M}_{\gamma_M}(f^{p,\infty}_M), \tag{8.7.3.1}$$

where s'_M is given by the endoscopic datum $\mathfrak{e}_\mathfrak{p}(M) = (M', {}^L M', s'_M, \eta_M)$ for M, and $(\Delta^M_{M'})^{A,B}(\gamma', \gamma_M)$ denotes the product of the local transfer factors over finite places $v \neq p$, normalized as in §8.4.12. We remind the reader that s'_M is different from s_M as in $\mathfrak{e}_{\mathfrak{p},A,B} = (M', {}^L M', s_M, \eta_M)$, and s'_M is independent of (A, B). By contrast, the normalization $(\Delta^M_{M'})^{A,B}_v$ of transfer factors between M' and M at v depend on (A, B). Nevertheless, for almost all v, $(\Delta^M_{M'})^{A,B}_v$ is the canonical unramified normalization (associated to the hyperspecial subgroup determined by some reductive model of M over some Zariski open in $\operatorname{Spec} \mathbb{Z}$). Hence for almost all v, $(\Delta^M_{M'})^{A,B}_v$ is independent of (A, B).

Definition 8.7.4. — For each $v \neq p, \infty$, let $\epsilon_v(A, B) \in \mathbb{C}^\times$ be the constant such that $(\Delta^M_{M'})^{A,B}_v = \epsilon_v(A, B)(\Delta^M_{M'})^{\emptyset,\emptyset}_v$. Let

$$\epsilon^{p,\infty}(A, B) = \prod_{v \neq p,\infty} \epsilon_v(A, B),$$

where almost all terms in the product are 1.

Definition 8.7.5. — Let $\Sigma(M')_1$ be the set of $\gamma' \in \Sigma(M')$ such that γ' is (M, M')-regular and is an image of a semi-simple element of $M(\mathbb{A}^p_f)$. For each $\gamma' \in \Sigma(M')_1$, let $\gamma_M \in M(\mathbb{A}^p_f)$ be a semi-simple element such that γ' is an image of γ_M, and define

$$I(\mathfrak{e}_{\mathfrak{p}}(M),\gamma') := \bar{\iota}^{M'}(\gamma')^{-1}(\Delta^{M}_{M'})^{\emptyset,\emptyset}(\gamma',\gamma_M)O^{s'_M}_{\gamma_M}(f^{p,\infty}_M)$$
$$\cdot \sum_{A,B} \epsilon^{p,\infty}(A,B)\tau(G)\tau(H)^{-1}\tau(M')SO_{\gamma'}(f^H_{p,M'})S\Phi^H_{M'}(\gamma',f^H_\infty),$$

where the terms $(\Delta^M_{M'})_{\emptyset,\emptyset}(\gamma',\gamma_M)$ and $O^{s'_M}_{\gamma_M}(f^{p,\infty}_M)$ are the same as in (8.7.3.1) (except that (A,B) is replaced by $(\emptyset,\emptyset)$), and the summation $\sum_{A,B}$ as well as the terms involving H have the same meaning as in Lemma 8.6.2. By (8.7.3.1) we know that this definition is independent of the choice of γ_M. For each (A,B) as above, we also define

$$K(\mathfrak{e}_{\mathfrak{p}}(M),\gamma',A,B) := (-1)^{q(G_{\mathbb{R}})} \sum_{\varphi_H\in\Phi_H(\varphi_{\mathbb{V}^*})} \det(\omega_*(\varphi_H))\Phi^H_{M'}(\gamma'^{-1},\Theta_{\varphi_H}),$$

where $\Theta_{\varphi_H} := \Theta_{\Pi(\varphi_H)}$ is the sum of the characters of the members of the L-packet $\Pi(\varphi_H)$, $\Phi^H_{\mathrm{M}'}(\cdot,\Theta_{\varphi_H})$ is the normalized stable discrete series character as in §4.2.1, and the other notations are as in §8.4.3.

Lemma 8.7.6. — *We have*

$$n^G_M \operatorname{Tr}'_M = \sum_{\mathfrak{e}_{\mathfrak{p}}(M)=(M',{}^LM',s'_M,\eta_M)\in\dot{\mathscr{E}}(M)^{\mathrm{c,ur}}} |\mathrm{Out}_M(\mathfrak{e}_{\mathfrak{p}}(M))|^{-1} \sum_{\gamma'\in\Sigma(M')_1} I(\mathfrak{e}_{\mathfrak{p}}(M),\gamma'),$$

and

$$I(\mathfrak{e}_{\mathfrak{p}}(M),\gamma') = \bar{\iota}^{M'}(\gamma')^{-1}(\Delta^M_{M'})^{\emptyset,\emptyset}(\gamma',\gamma_M)O^{s'_M}_{\gamma_M}(f^{p,\infty}_M)\tau(M)k(M)k(G)^{-1}$$
$$\cdot(-1)^{\dim A_{M'}}\bar{v}((M')^0_{\gamma'})^{-1}\sum_{A,B}\epsilon^{p,\infty}(A,B)SO_{\gamma'}(f^H_{p,M'})K(\mathfrak{e}_{\mathfrak{p}}(M),\gamma',A,B).$$

Here the summation range for $\sum_{A,B}$ is the same as in Lemma 8.6.2.

Proof. — The first identity follows from Lemma 8.6.2, Lemma 8.7.2, §8.7.3, and the definitions. The second identity follows from Corollary 8.2.5, Lemma 8.4.6, and the definitions. □

8.8. Computation of K

8.8.1. — We keep the notation of Definition 8.7.5 and study $K(\mathfrak{e}_{\mathfrak{p}}(M),\gamma',A,B)$. As usual we write $\mathfrak{e}_{\mathfrak{p}}(M) = (M',{}^LM',s'_M,\eta_M)$. We would like to apply [**Mor11**, Prop. 3.2.5] to compute K. First we need some preparations.

By construction $M' = M^{\mathrm{GL}}\times M'^{,\mathrm{SO}}$ and $M'^{,\mathrm{SO}}$ is a product of two special orthogonal groups $M'^{,\mathrm{SO},+}, M'^{,\mathrm{SO},-}$ such that the component of s_M in the dual group of $M'^{,\mathrm{SO},\pm}$ is the scalar matrix ± 1. Fix an elliptic maximal torus $T_{M'}$ in $M'_{\mathbb{R}}$ such that $\gamma'\in T_{M'}(\mathbb{R})$. Then $T_{M'}$ is of the form

$$T_{M'} = T_{M^{\mathrm{GL}}}\times T_{M'^{,\mathrm{SO},+}}\times T_{M'^{,\mathrm{SO},-}}$$

where $T_{M^{\mathrm{GL}}}$ (resp. $T_{M',\mathrm{SO},\pm}$) is an elliptic maximal torus in $M^{\mathrm{GL}}_{\mathbb{R}}$ (resp. $M'^{,\mathrm{SO},\pm}_{\mathbb{R}}$). Moreover, as we have already seen in §8.6.1, the tori $T_{M',\mathrm{SO},\pm}$ are in fact anisotropic over $\mathbb{R}$. Note that when $M = M_{12}$ or M_2, we have necessarily $T_{M^{\mathrm{GL}}} = M^{\mathrm{GL}}$. When $M = M_1$, we have $M^{\mathrm{GL}} = \mathrm{GL}_2$, and $T_{M^{\mathrm{GL}}}$ is $\mathrm{GL}_2(\mathbb{R})$-conjugate to $T^{\mathrm{std}}_{\mathrm{GL}_2}$; cf. §4.1.1.

We then fix an elliptic maximal torus T_M in $M_{\mathbb{R}}$, and an admissible isomorphism $j_M : T_{M'} \xrightarrow{\sim} T_M$. Recall from §1.4 that $M = M^{\mathrm{GL}} \times M^{\mathrm{SO}} = M^{\mathrm{GL}} \times \mathrm{SO}(W)$, where $W = W_2$ if $M = M_1$ or M_{12}, and $W = W_1$ if $M = M_2$. We may and shall assume that T_M is of the form $T_{M^{\mathrm{GL}}} \times T_{M^{\mathrm{SO}}}$, where $T_{M^{\mathrm{SO}}}$ is an elliptic (and in fact anisotropic) maximal torus in $M^{\mathrm{SO}}_{\mathbb{R}}$, and $T_{M^{\mathrm{GL}}}$ is as above. We may and shall also assume that j_M is the product of the identity on $T_{M^{\mathrm{GL}}}$ and an admissible isomorphism

$$j_{M^{\mathrm{SO}}} : T_{M',\mathrm{SO},+} \times T_{M',\mathrm{SO},-} \xrightarrow{\sim} T_{M^{\mathrm{SO}}},$$

where the notion of admissibility is with respect to the endoscopic datum $\mathfrak{e}_{\mathfrak{p}}(W)$ for M^{SO}.

For any choice of a Borel subgroup B_0 of $G_{\mathbb{C}}$ containing $T_{M,\mathbb{C}}$, we get a canonical isomorphism $\mathfrak{d}_{B_0,\mathcal{B}} : \widehat{T}_M \xrightarrow{\sim} \mathcal{T}$ as in §5.6, where $(\mathcal{T},\mathcal{B})$ is the standard Borel pair in $\widehat{G}$ fixed in Definition 5.2.2. Identifying $\mathcal{T}$ with $(\mathbb{C}^{\times})^m$ as in Definition 5.2.2, we have m standard characters on $\mathcal{T}$ forming a basis of $X^*(\mathcal{T})$, and they give rise, via $\mathfrak{d}_{B_0,\mathcal{B}}$, to m cocharacters of $T_{M,\mathbb{C}}$. We denote them (in order) by

$$\tau_{0_1}, \tau_{0_2}, \tau_1, \tau_2, \cdots, \tau_{m-2}, \qquad \text{if } M = M_{12} \text{ or } M_1,$$

and by

$$\tau_0, \tau_1, \tau_2, \cdots, \tau_{m-1}, \qquad \text{if } M = M_2 \text{ (in the odd case)}.$$

We now fix a choice of B_0 such that the resulting cocharacters (just mentioned) satisfy the following conditions, the second of which depends on the choice of j_M.

(1) When $M = M_{12}$, we require that τ_{0_1} and τ_{0_2} are respectively the identity cocharacters of the first $\mathbb{G}_m$ (i.e. $\mathrm{GL}(V_1)$) and the second $\mathbb{G}_m$ (i.e. $\mathrm{GL}(V_2/V_1)$) in $T_{M^{\mathrm{GL}}} = \mathbb{G}_m \times \mathbb{G}_m \subset T_M$. When $M = M_1$, we require that τ_{0_1} and τ_{0_2} are cocharacters of $T_{M^{\mathrm{GL}},\mathbb{C}} \subset T_{M,\mathbb{C}}$, and that they are of the form

$$z \longmapsto g \begin{pmatrix} z & \\ & 1 \end{pmatrix} g^{-1} \qquad \text{and} \qquad z \longmapsto g \begin{pmatrix} 1 & \\ & z \end{pmatrix} g^{-1}$$

for some fixed $g \in M^{\mathrm{GL}}(\mathbb{C})$ conjugating the diagonal torus in $M^{\mathrm{GL}}_{\mathbb{C}} = \mathrm{GL}_{2,\mathbb{C}}$ to $T_{M^{\mathrm{GL}},\mathbb{C}}$. (Clearly this pins down τ_{0_1} and τ_{0_2} up to swapping the two.) When $M = M_2$, we require that τ_0 is the identity cocharacter of $T_{M^{\mathrm{GL}}} = \mathrm{GL}(V_1) = \mathbb{G}_m \subset T_M$.

(2) We require that $j_M^{-1} \circ \tau_i$ is a cocharacter of $T_{M',\mathrm{SO},-,\mathbb{C}}$, for each $1 \le i \le n^-$. Here n^- is the dimension of $T_{M',\mathrm{SO},-,\mathbb{C}}$.

Indeed, the above conditions can be arranged because of the following observations:

– For an arbitrary choice of B_0, the resulting τ's have the following property: The prescribed cocharacter(s) in (1) which we ask τ_{0_1} and τ_{0_2}, or τ_0, to equal, are among the τ's and their inverses. This is because these prescribed cocharacter(s) can be extended to a $\mathbb{Z}$-basis of $X_*(T_M)$ under which the root datum of $(G_{\mathbb{C}}, T_{M,\mathbb{C}})$ becomes the standard type B or type D root datum on $(\mathbb{Z}^m, \mathbb{Z}^m)$.

– By making different choices of B_0, we can arbitrarily permute the order of the τ's and replace an arbitrary number (resp. an even number) of them by their inverses in the odd (resp. even) case. In the even case with $M = M_{12}$ or M_1, we can replace either one or two of τ_{0_1}, τ_{0_2} by their inverses as we wish, since $m \geq 3$. Thus we can always arrange (1).

– Once (1) is satisfied, the cocharacters $\tau_1, \tau_2, \cdots$ form a basis of $X_*(T_{M^{\mathrm{SO}}})$ under which the root datum of $(M^{\mathrm{SO}}_{\mathbb{C}}, T_{M^{\mathrm{SO}},\mathbb{C}})$ becomes the standard type B or type D root datum. Since $j_{M^{\mathrm{SO}}}$ is admissible, exactly n^- of the τ_i's are such that $j_M^{-1} \circ \tau_i$ (equal to $j_{M^{\mathrm{SO}}}^{-1} \circ \tau_i$) is a cocharacter of $T_{M',\mathrm{SO},-,\mathbb{C}}$. We can rechoose B_0 in such a way that τ_{0_1} and τ_{0_2}, or τ_0, are unchanged, but the order of $\tau_1, \tau_2, \cdots$ is permuted so that (2) is satisfied.

8.8.2. — Up to now our discussion has not involved (A, B). We now take them into account, so we have an endoscopic datum $(H, {}^L H, s, \eta)$ for G that is determined by $(\mathfrak{e}_{\mathfrak{p}}(M), A, B)$ as in Lemma 8.6.2 and Definition 8.7.5. Recall from §8.4.2 that we have fixed $(T_H, T_G, j, B_{G,H})$. Similarly as in §8.8.1, the pair $(T_G, B_{G,H})$ determines an ordered m-tuple of cocharacters of $T_{G,\mathbb{C}}$ (via $\mathfrak{d}_{B_{G,H},\mathcal{B}} : \widehat{T}_G \xrightarrow{\sim} \mathcal{T} \cong (\mathbb{C}^\times)^m$). We denote them by

$$\rho_1, \rho_2, \cdots, \rho_m.$$

By the construction of j in §8.4.2 (which uses §6.1.9 and especially Convention 6.1.10), we know that $\left\{ j^{-1} \circ \rho_i \mid 1 \leq i \leq m^- \right\}$ is a basis of $X_*(T_{H^-})$ (where $T_{H^-} := T_H \cap H^-$) under which the root datum of $(H^-_{\mathbb{C}}, T_{H^-,\mathbb{C}})$ becomes the standard type B or D root datum. Similarly, $\left\{ j^{-1} \circ \rho_i \mid m^- + 1 \leq i \leq m \right\}$ is a basis of $X_*(T_{H^+})$ under which the root datum of $(H^+_{\mathbb{C}}, T_{H^+,\mathbb{C}})$ becomes the standard type B or D root datum.

Definition 8.8.3. — Define an isomorphism $i_G(A, B) : T_{M,\mathbb{C}} \xrightarrow{\sim} T_{G,\mathbb{C}}$ as follows. When $M = M_{12}$ (so $B \equiv \emptyset$), let $i_G(A, B)$ map $\tau_{0_1}, \tau_{0_2}, \tau_1, \cdots, \tau_{m-2}$ respectively to

$$\begin{cases} \rho_1, \rho_2, \cdots, \rho_m, & A = \emptyset, \\ \rho_{m^-+1}, \rho_1, \rho_2, \cdots, \rho_{m^-}, \rho_{m^-+2}, \cdots, \rho_m, & A = \{1\}, \\ \rho_1, \rho_{m^-+1}, \rho_2, \cdots, \rho_{m^-}, \rho_{m^-+2}, \cdots, \rho_m, & A = \{2\}, \\ \rho_{m^-+1}, \rho_{m^-+2}, \rho_1, \cdots, \rho_{m^-}, \rho_{m^-+3}, \cdots, \rho_m, & A = \{1, 2\}. \end{cases}$$

(In the even case the parameter A can only assume $\{1, 2\}$ and $\emptyset$, cf. Lemma 8.6.2, and we use only these two cases in the above formula). When $M = M_1$ (so $A \equiv \emptyset$), let $i_G(A, B)$ map $\tau_{0_1}, \tau_{0_2}, \tau_1, \cdots, \tau_{m-2}$ respectively to

$$\begin{cases} \rho_1, \rho_2, \cdots, \rho_m, & B = \emptyset, \\ \rho_{m^-+1}, \rho_{m^-+2}, \rho_1, \cdots, \rho_{m^-}, \rho_{m^-+3}, \cdots, \rho_m, & B = \{1\}. \end{cases}$$

When $M = M_2$ (so $B \equiv \emptyset$), let $i_G(A,B)$ map $\tau_0, \tau_1, \cdots, \tau_{m-1}$ respectively to

$$\begin{cases} \rho_1, \cdots, \rho_m, & A = \emptyset, \\ \rho_{m^-+1}, \rho_1, \cdots, \rho_{m^-}, \rho_{m^-+2}, \cdots, \rho_m, & A = \{1\}. \end{cases}$$

In the following lemma, recall from §8.5.1 that we have identified M' with a Levi subgroup of H.

Lemma 8.8.4. — *Let $i_H(A,B)$ be the unique isomorphism $T_{M',\mathbb{C}} \xrightarrow{\sim} T_{H,\mathbb{C}}$ fitting in the following commutative diagram:*

(8.8.4.1)
$$\begin{array}{ccc} T_{H,\mathbb{C}} & \xrightarrow{j} & T_{G,\mathbb{C}} \\ \uparrow{\scriptstyle i_H(A,B)} & & \uparrow{\scriptstyle i_G(A,B)} \\ T_{M',\mathbb{C}} & \xrightarrow{j_M} & T_{M,\mathbb{C}} \end{array}$$

Then $i_G(A,B)$ (resp. $i_H(A,B)$) is induced by an inner automorphism of $G_{\mathbb{C}}$ (resp. $H_{\mathbb{C}}$).

Proof. — Firstly, the isomorphism $i_G(\emptyset,\emptyset) : T_{M,\mathbb{C}} \xrightarrow{\sim} T_{G,\mathbb{C}}$ is compatible with the two canonical isomorphisms $\mathrm{BRD}(T_{M,\mathbb{C}}, B_0) \cong \mathrm{BRD}(G)$ and $\mathrm{BRD}(T_{G,\mathbb{C}}, B_{G,H}) \cong \mathrm{BRD}(G)$, where $\mathrm{BRD}(G)$ is the canonical based root datum of $G_{\mathbb{C}}$ (see §5.3.1). Hence $i_G(\emptyset,\emptyset)$ is induced by an inner automorphism of $G_{\mathbb{C}}$. For general (A,B), $i_G(A,B)$ differs from $i_G(\emptyset,\emptyset)$ by an automorphism of $T_{G,\mathbb{C}}$ which permutes the order of the ρ_i's. Such an automorphism is in the Weyl group (because under the basis $\{\rho_1, \cdots, \rho_m\}$ of $X_*(T_G)$ the root datum of $(G_{\mathbb{C}}, T_{G,\mathbb{C}})$ becomes the standard type B or D root datum), and is hence still induced by an inner automorphism of $G_{\mathbb{C}}$.

We now prove that $i_H(A,B)$ is induced by an inner automorphism of $H_{\mathbb{C}}$. For brevity, we only illustrate the proof in the special case where $M = M_{12}$ and $(A,B) = (\{1\}, \emptyset)$, the other cases all being similar. Also, we only treat the even case, as the odd case is easier. We freely use the notation of §5.5.9; in particular $M'^{,\mathrm{SO},\pm} = \mathrm{SO}(W^\pm)$, $H^\pm = \mathrm{SO}(V^\pm)$, and $d^\pm = \dim W^\pm$. As in §5.5.9, we have a canonical $\mathrm{SO}(W^+)(\mathbb{C})$-conjugacy class of embeddings

$$\iota_{W^+} : \mathbb{G}_m^{d^+/2} \longrightarrow \mathrm{SO}(W^+)_{\mathbb{C}}$$

and a canonical $\mathrm{SO}(V^+)(\mathbb{C})$-conjugacy class of embeddings

$$\iota_{V^+} : \mathbb{G}_m^{d^+/2+1} \longrightarrow \mathrm{SO}(V^+)_{\mathbb{C}}.$$

(Here, if $\delta^+ = -1$, we identify $\mathrm{U}(1)_{\mathbb{C}}$ with $\mathbb{G}_m$.) By the two conditions satisfied by B_0 in §8.8.1 and the admissibility of $j_{M^{\mathrm{SO}}}$, we know that the embedding[(7)]

$$(j_M^{-1} \circ \tau_{d^-/2+1}, \cdots, j_M^{-1} \circ \tau_{m-2}) : \mathbb{G}_m^{d^+/2} \longrightarrow \mathrm{SO}(W^+)_{\mathbb{C}}$$

is $\mathrm{SO}(W^+)(\mathbb{C})$-conjugate to ι_{W^+}, and that $j_M^{-1} \circ \tau_{0_1}$ is the identity cocharacter of $\mathrm{GL}(V_1)$, namely $\iota_{A,B}^{\mathrm{GL}}$ in the notation of §5.5.9. Thus by the construction in §5.5.9, the embedding

(8.8.4.2)

$$(j_M^{-1} \circ \tau_{0_1}, j_M^{-1} \circ \tau_{d^-/2+1}, \cdots, j_M^{-1} \circ \tau_{m-2}) : \mathbb{G}_m^{d^+/2+1} \longrightarrow \mathrm{GL}(V_1)_{\mathbb{C}} \times \mathrm{SO}(W^+)_{\mathbb{C}}$$

is $\mathrm{SO}(V^+)(\mathbb{C})$-conjugate to ι_{V^+} when we view $\mathrm{GL}(V_1) \times \mathrm{SO}(W^+)$ as a subgroup of $\mathrm{SO}(V^+)$ according to the rule in §5.5.9. On the other hand, the embedding

$$(j^{-1} \circ \rho_{m^-+1}, \cdots, j^{-1} \circ \rho_m) : \mathbb{G}_m^{d^+/2+1} \longrightarrow \mathrm{SO}(V^+)_{\mathbb{C}} \tag{8.8.4.3}$$

is also $\mathrm{SO}(V^+)(\mathbb{C})$-conjugate to ι_{V^+}. Hence (8.8.4.2) and (8.8.4.3) are $\mathrm{SO}(V^+)(\mathbb{C})$-conjugate. Similarly, we know that the embeddings

$$(j_M^{-1} \circ \tau_{0_2}, j_M^{-1} \circ \tau_1, \cdots, j_M^{-1} \circ \tau_{d^-/2}) : \mathbb{G}_m^{d^-/2+1} \longrightarrow \mathrm{GL}(V_2/V_1)_{\mathbb{C}} \times \mathrm{SO}(W^-)_{\mathbb{C}}$$

and

$$(j^{-1} \circ \rho_1, \cdots, j^{-1} \circ \rho_{m^-}) : \mathbb{G}_m^{d^-/2+1} \longrightarrow \mathrm{SO}(V^-)_{\mathbb{C}}$$

are $\mathrm{SO}(V^-)(\mathbb{C})$-conjugate. We conclude that the embeddings

$$(j_M^{-1} \circ \tau_{0_1}, j_M^{-1} \circ \tau_{0_2}, j_M^{-1} \circ \tau_1, \cdots, j_M^{-1} \circ \tau_{m-2}) : \mathbb{G}_m^m \longrightarrow H_{\mathbb{C}}$$

and

$$(j^{-1} \circ \rho_{m^-+1}, j^{-1} \circ \rho_1, \cdots, j^{-1} \circ \rho_{m^-}, j^{-1} \circ \rho_{m^-+2}, \cdots, j^{-1} \circ \rho_m) : \mathbb{G}_m^m \longrightarrow H_{\mathbb{C}}$$

are $H(\mathbb{C})$-conjugate. But these two embeddings have images $T_{M',\mathbb{C}}$ and $T_{H,\mathbb{C}}$ respectively, and if we invert the first and compose with the second we precisely get the isomorphism $i_H(A, B)$. This finishes the proof. □

Definition 8.8.5. — Define the three Borel subgroups:

– B_M, a Borel of $M_{\mathbb{C}}$ containing $T_{M,\mathbb{C}}$, defined to be $B_0 \cap M$.

– B_G, a Borel of $G_{\mathbb{C}}$ containing $T_{G,\mathbb{C}}$, defined to be $i_G(A, B)_* B_0$. This can be different from $B_{G,H}$ fixed in §8.4.2.

– B'_H, a Borel of $H_{\mathbb{C}}$ containing $T_{H,\mathbb{C}}$, defined to be the one induced by (j, B_G). In other words, j carries the B'_H-positive roots on $T_{H,\mathbb{C}}$ to B_G-positive roots on $T_{G,\mathbb{C}}$.

Lemma 8.8.6. — *We have $B'_H = B_H$, where B_H is defined in §8.4.3.*

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(7) Here we use the following notation: If $\mu_1, \cdots, \mu_k$ are cocharacters of a torus T contained in a reductive group R (everything being over $\mathbb{C}$), we write $(\mu_1, \cdots, \mu_k)$ for the homomorphism $\mathbb{G}_m^k \to T \subset R, (z_1, \cdots, z_k) \mapsto \prod_i \mu_i(z_i)$.

Proof. — We use j to identify $T_{H,\mathbb{C}}$ and $T_{G,\mathbb{C}}$. Thus we have an inclusion of root systems

$$\Phi_H := \Phi(H_{\mathbb{C}}, T_{H,\mathbb{C}}) \subset \Phi_G := \Phi(G_{\mathbb{C}}, T_{G,\mathbb{C}}).$$

To prove the lemma, we need to prove that for all $\alpha \in \Phi_H$, it is B_G-positive if and only if it is $B_{G,H}$-positive. We denote the permutation of ρ_i's that appears in Definition 8.8.3 by

$$\rho_{\sigma(1)}, \rho_{\sigma(2)}, \cdots, \rho_{\sigma(m)}, \qquad \sigma \in \mathfrak{S}_m.$$

(For instance, if $A = \{1\}$, then σ sends $1, 2, \cdots, m$ respectively to $m^- + 1, 1, \cdots, m^-, m^- + 2, \cdots, m$.) Let $\{\rho_1^\vee, \cdots, \rho_m^\vee\}$ be the basis of $X^*(T_{G,\mathbb{C}})$ dual to the basis $\{\rho_1, \cdots, \rho_m\}$ of $X_*(T_{G,\mathbb{C}})$. The $B_{G,H}$-positive roots in Φ_G are

$$\begin{cases} \left\{\rho_i^\vee \pm \rho_j^\vee \mid i > j\right\} \cup \left\{\rho_i^\vee \mid i\right\}, & \text{odd case,} \\ \left\{\rho_i^\vee \pm \rho_j^\vee \mid i > j\right\}, & \text{even case.} \end{cases}$$

The B_G-positive roots in Φ_G are

$$\begin{cases} \left\{\rho_{\sigma(i)}^\vee \pm \rho_{\sigma(j)}^\vee \mid i > j\right\} \cup \left\{\rho_i^\vee \mid i\right\}, & \text{odd case,} \\ \left\{\rho_{\sigma(i)}^\vee \pm \rho_{\sigma(j)}^\vee \mid i > j\right\}, & \text{even case.} \end{cases}$$

On the other hand, by the last observation in §8.8.2, we have

$$\Phi_H = \begin{cases} \left\{\pm\rho_i^\vee \pm \rho_j^\vee \mid i, j \le m^-, i \ne j\right\} \cup \left\{\pm\rho_i^\vee \pm \rho_j^\vee \mid i, j > m^-, i \ne j\right\} \cup \left\{\rho_i^\vee \mid i\right\}, \\ \left\{\pm\rho_i^\vee \pm \rho_j^\vee \mid i, j \le m^-, i \ne j\right\} \cup \left\{\pm\rho_i^\vee \pm \rho_j^\vee \mid i, j > m^-, i \ne j\right\}, \end{cases}$$

in the odd and even cases respectively. It remains to check that $\sigma^{-1}|_{\{1,2,\cdots,m^-\}}$ and $\sigma^{-1}|_{\{m^-+1,\cdots,m\}}$ are increasing, which is true. □

8.8.7. — We now transport [**Mor11**, Prop. 3.2.5] to our setting. For any $t \in T_M(\mathbb{R})$, let $\epsilon_R(t) \in \{\pm 1\}$ be -1 to the number of B_0-positive roots α of $(G_{\mathbb{C}}, T_{M,\mathbb{C}})$ such that α is real and $0 < \alpha(t) < 1$. (Compare with the definition in §4.2.2.) Similarly, for $t' \in T_{M'}(\mathbb{R})$, we let $\epsilon_{R_H}(t') \in \{\pm 1\}$ be -1 to the number of $i_H(A,B)_*^{-1}(B'_H)$-positive (or equivalently, $i_H(A,B)_*^{-1}(B_H)$-positive, by Lemma 8.8.6) roots α of $(H_{\mathbb{C}}, T_{M',\mathbb{C}})$ such that α is real and $0 < \alpha(t') < 1$. We set [(8)]

$$\Delta_{j_M,B_M}^{A,B} := (-1)^{q(G_{\mathbb{R}})+q(H_{\mathbb{R}})+q(M_{\mathbb{R}})+q(M'_{\mathbb{R}})} \Delta_{j_M,B_M},$$

where Δ_{j_M,B_M} is Kottwitz's normalization of the archimedean transfer factors between $\mathfrak{e}_{\mathfrak{p}}(M) = (M', {}^L M', s'_M, \eta_M)$ and M associated to (j_M, B_M) (see [**Kot90**, §7],

(8) In [**Mor11**, Prop. 3.2.5], our $\Delta_{j_M,B_M}^{A,B}$ is denoted simply by Δ_{j_M,B_M}. However, this object is not intrinsic to (j_M, B_M), since its definition involves the number $q(H_{\mathbb{R}})$ which depends on (A, B).

cf. §§6.2–6.3). Let $\Theta_{\mathbb{V}^*}$ denote the analogue of $\Theta_{\mathbb{V}}$ in §4.2.1 with $\mathbb{V}$ replaced by $\mathbb{V}^*$. The following result is [**Mor11**, Prop. 3.2.5].

Proposition 8.8.8. — *We have*

$$\begin{aligned}\epsilon_R(j_M(\gamma'^{-1}))\epsilon_{R_H}(\gamma'^{-1})\Delta^{A,B}_{j_M,B_M}(\gamma', j_M(\gamma'))\Phi^G_M(j_M(\gamma')^{-1}, \Theta^H_{\mathbb{V}^*}) \\ = \sum_{\varphi_H \in \Phi_H(\varphi_{\mathbb{V}^*})} \det(\omega'_*(\varphi_H))\Phi^H_{M'}(\gamma'^{-1}, \Theta_{\varphi_H}),\end{aligned}$$

Here the elements $\omega'_*(\varphi_H) \in \Omega$ are the analogues of the elements $\omega_*(\varphi_H) \in \Omega$ in §8.4.3 with $(j, B_{G,H})$ replaced by (j, B_G). The term $\Phi^G_M(\cdot, \Theta^H_{\mathbb{V}^*})$ is given as follows. Only when $M = M_{12}$ and $A = \{1\}$ or $\{2\}$ (which in particular implies that we are in the odd case; see Lemma 8.6.2), it is equal to $\Phi^G_M(\cdot, \Theta_{\mathbb{V}^*})_{\mathrm{eds}}$ (defined as in (4.6.10.2), but with $\mathbb{V}$ replaced by $\mathbb{V}^*$). In all the other cases, it is equal to $\Phi^G_M(\cdot, \Theta_{\mathbb{V}^*})$. □

8.8.9. — For a fixed φ_H as in Proposition 8.8.8, we investigate the relation between $\omega_*(\varphi_H)$ and $\omega'_*(\varphi_H)$. Write $\omega_* := \omega_*(\varphi_H)$ and $\omega'_* := \omega'_*(\varphi_H)$. By definition, φ_H is aligned with $(\omega_*^{-1} \circ j, B_{G,H}, B_H)$ and also aligned with $((\omega'_*)^{-1} \circ j, B_G, B'_H)$. Suppose $\omega_0 \in \Omega(G_{\mathbb{C}}, T_{G,\mathbb{C}})$ measures the difference between B_G and $B_{G,H}$, so that the map $\widehat{T_G} \to \widehat{G}$ determined by B_G and φ_H (namely the first row of the commutative diagram on the bottom of [**Kot92a**, p. 184]) is equal to the composition of $\widehat{\omega_0} : \widehat{T_G} \to \widehat{T_G}$ with the analogous map $\widehat{T_G} \to \widehat{G}$ determined by $B_{G,H}$ and φ_H. By the definition of "being aligned" and by Lemma 8.8.6, we know that the composition

$$\widehat{T} \xrightarrow{\widehat{\omega_0}} \widehat{T} \xrightarrow{\widehat{\omega_*'^{,-1} \circ j}} \widehat{T}_H$$

is equal to the map

$$\widehat{\omega_*^{-1} \circ j} : \widehat{T} \longrightarrow \widehat{T}_H.$$

Hence

$$\omega'_*(\varphi_H) = \omega_*(\varphi_H)\omega_0.$$

In particular,

$$\det(\omega'_*(\varphi_H)) = \det(\omega_*(\varphi_H)) \det(\omega_0). \tag{8.8.9.1}$$

Lemma 8.8.10. — *We have*

$$\det(\omega_0) = \begin{cases} 1, & A = \emptyset, \\ (-1)^{m^-}, & A = \{1\}, \\ (-1)^{m^-+1}, & A = \{2\}, \\ 1, & A = \{1,2\}. \end{cases} \tag{8.8.10.1}$$

(Here the formula works in all cases considered in Lemma 8.6.2. For instance, $A = \{1\}$ could only happen in the odd case either when $M = M_{12}$ or when $M = M_2$.)

Proof. — From the description of the $B_{G,H}$-positive and B_G-positive roots in Φ_G in the proof of Lemma 8.8.6, we see that $\det(\omega_0)$ is equal to the sign of the permutation σ in that proof. Thus (8.8.10.1) follows from direct calculation of this sign. □

Proposition 8.8.11. — *We have*

$$K(\mathfrak{e}_{\mathfrak{p}}(M),\gamma',A,B)=(-1)^{q(G_{\mathbb{R}})}\det(\omega_0)\epsilon_R(j_M(\gamma'^{-1}))\epsilon_{R_H}(\gamma'^{-1})\cdot\Delta^{A,B}_{j_M,B_M}(\gamma',j_M(\gamma'))\Phi^G_M(j_M(\gamma')^{-1},\Theta^H_{\mathbb{V}^*}),$$

where $\det(\omega_0)$ *is given in (8.8.10.1).*

Proof. — This is a consequence of Proposition 8.8.8 and (8.8.9.1). □

8.9. Computation of some signs

We keep the notation of §8.8.

Definition 8.9.1. — Let $\text{♫}(A,B)\in\mathbb{C}^\times$ be the constant such that the normalization $\text{♫}(A,B)\cdot\Delta^{A,B}_{j_M,B_M}$ of transfer factors between $\mathfrak{e}_{\mathfrak{p}}(M)$ and M at ∞ together with the normalizations $(\Delta^M_{M'})^{A,B}_v$ at all finite places (fixed in §8.4.12) satisfy the global product formula. Here $\Delta^{A,B}_{j_M,B_M}$ is defined in §8.8.7.

Lemma 8.9.2. — *The normalization* $\text{♫}(A,B)\Delta^{A,B}_{j_M,B_M}$ *of transfer factors between* $\mathfrak{e}_{\mathfrak{p}}(M)$ *and* M *at* ∞ *is inherited from* $\Delta_{j,B_{G,H}}$ *in the sense of Remark 8.4.13. Let* $\epsilon^{p,\infty}(A,B)$ *be as in Definition 8.7.4. We have*

$$\Delta^{A,B}_{j_M,B_M}\cdot\epsilon^{p,\infty}(A,B)=\Delta^{\emptyset,\emptyset}_{j_M,B_M}\cdot\text{♫}(A,B)^{-1}\,\text{♫}(\emptyset,\emptyset).$$

Proof. — The first assertion follows from the fact that $(\Delta^G_H)_v$ for all v satisfy the global product formula (see §8.4.7), and the fact that inheritance of normalizations respects the global product formula (see Remark 8.4.13). To prove the second assertion, by the definition of $\text{♫}(A,B)$ we must have

$$\text{♫}(A,B)\Delta^{A,B}_{j_M,B_M}\prod_{v\neq\infty}(\Delta^M_{M'})^{A,B}_v=\text{♫}(\emptyset,\emptyset)\Delta^{\emptyset,\emptyset}_{j_M,B_M}\prod_{v\neq\infty}(\Delta^M_{M'})^{\emptyset,\emptyset}_v.$$

But $(\Delta^M_{M'})^{A,B}_p=(\Delta^M_{M'})^{\emptyset,\emptyset}_p$ because they are both the canonical unramified normalization associated to $\mathcal{M}(\mathbb{Z}_p)$. (See §8.4.12 for $\mathcal{M}(\mathbb{Z}_p)$.) Hence

$$\text{♫}(A,B)\Delta^{A,B}_{j_M,B_M}\prod_{v\neq p,\infty}(\Delta^M_{M'})^{A,B}_v=\text{♫}(\emptyset,\emptyset)\Delta^{\emptyset,\emptyset}_{j_M,B_M}\prod_{v\neq p,\infty}(\Delta^M_{M'})^{\emptyset,\emptyset}_v.$$

Our assertion follows from comparing the above equality with Definition 8.7.4. □

8.9.3. — As usual we denote by $W,W^\pm$ the underlying quadratic spaces for $M^{\mathrm{SO}},M'^{,\mathrm{SO},\pm}$, i.e., $M^{\mathrm{SO}}=\mathrm{SO}(W)$, $M'^{,\mathrm{SO},\pm}=\mathrm{SO}(W^\pm)$. Denote by $M^{\mathrm{SO},*}$ the fixed quasi-split inner form of M^{SO} as in §5.5.3. Namely we have $M^{\mathrm{SO},*}=\mathrm{SO}(\underline{W})$, and as in §5.5.3 we have fixed isomorphisms $\phi_{W_{\mathbb{R}}} : W_{\mathbb{C}} \xrightarrow{\sim} \underline{W}_{\mathbb{C}}$ (with respect to $F = \mathbb{R}$ and satisfying the extra condition in Definition 5.1.1) and $\psi_{W_{\mathbb{R}}} : M^{\mathrm{SO}}_{\mathbb{C}} \xrightarrow{\sim} M^{\mathrm{SO},*}_{\mathbb{C}}, g \mapsto \phi_{W_{\mathbb{R}}} g \phi^{-1}_{W_{\mathbb{R}}}$.

By the two conditions noted in §8.6.1, we know that the localization over $\mathbb{R}$ of the endoscopic datum $\mathfrak{e}_{\mathfrak{p}}(W)$ for $M^{\mathrm{SO}}_{\mathbb{R}} = \mathrm{SO}(W_{\mathbb{R}})$ satisfies the hypotheses in §6 (with $V, \underline{V}, V^{\pm}$ there replaced by $W_{\mathbb{R}}, \underline{W}_{\mathbb{R}}, W^{\pm}_{\mathbb{R}}$.) In other words, this is an elliptic endoscopic datum over $\mathbb{R}$, and the group in it contains $\mathbb{R}$-anisotropic maximal tori. Define $\mathrm{ED}(W_{\mathbb{R}})^o, \mathrm{ED}(\underline{W}_{\mathbb{R}})^o, \mathrm{ED}(W^{\pm}_{\mathbb{R}})^o$ as in §6.1.7 and §6.1.9. Inside $\mathrm{ED}(W_{\mathbb{R}})^o$ we have the subset $\mathrm{ED}(W_{\mathbb{R}})^o_{\mathrm{nice}}$ as in Definitions 6.2.12 and 6.3.8. Let $B_{M^{\mathrm{SO}}}$ be the Borel subgroup of $M^{\mathrm{SO}}_{\mathbb{C}}$ given by $B_M \cap M^{\mathrm{SO}}_{\mathbb{C}}$, and let $j_{M^{\mathrm{SO}}} : T_{M',\mathrm{SO},+} \times T_{M',\mathrm{SO},-} \xrightarrow{\sim} T_{M^{\mathrm{SO}}}$ be as in §8.8.1. Thus $(T_{M^{\mathrm{SO}}}, B_{M^{\mathrm{SO}}})$ is a fundamental pair in $M^{\mathrm{SO}}_{\mathbb{R}} = \mathrm{SO}(W_{\mathbb{R}})$.

Lemma 8.9.4. — *There exist $\mathcal{D}_1 \in \mathrm{ED}(W_{\mathbb{R}})^o_{\mathrm{nice}}$ and $\mathcal{D}_2 = (\mathcal{D}_2^+, \mathcal{D}_2^-) \in \mathrm{ED}(W^+_{\mathbb{R}})^o \times \mathrm{ED}(W^-_{\mathbb{R}})^o$ such that the fundamental pair $(T_{M^{\mathrm{SO}}}, B_{M^{\mathrm{SO}}})$ arises from $\mathcal{D}_1$ as in §6.1.6, and $j_{M^{\mathrm{SO}}} = j_{\mathcal{D}_2,\mathcal{D}_1}$ where $j_{\mathcal{D}_2,\mathcal{D}_1}$ is as in §6.1.9.*

Proof. — Firstly, since the signature of $W_{\mathbb{R}}$ is $(d-4,0)$ or $(d-3,1)$, we have $\mathrm{ED}(W_{\mathbb{R}})^o = \mathrm{ED}(W_{\mathbb{R}})^o_{\mathrm{nice}}$. Since all anisotropic maximal tori in $M^{\mathrm{SO}}_{\mathbb{R}}$ are conjugate under $M^{\mathrm{SO}}(\mathbb{R})$, we can find $\mathcal{D}_1 \in \mathrm{ED}(W_{\mathbb{R}})^o$ such that $T_{M^{\mathrm{SO}}} = T_{\mathcal{D}_1}$ (notation as in §6.1.6). By reordering the members of $\mathcal{D}_1$, and in the odd (resp. even) case changing the orientations of an arbitrary (resp. even) number of the members of $\mathcal{D}_1$, we may and shall assume that the fundamental pair $(T_{M^{\mathrm{SO}}}, B_{M^{\mathrm{SO}}})$ arises from $\mathcal{D}_1$. Let m' be the absolute rank of M^{SO}. Using Lemma 6.1.8 and the same argument as in the proof of Lemma 6.1.13, we see that there exist $g \in M^{\mathrm{SO}}(\mathbb{R})$ and $\mathcal{D}_0 \in \mathrm{ED}(\underline{W})^o$ such that $\mathrm{Int}(g) \circ f_{\mathcal{D}_1} = \psi_W^{-1} \circ f_{\mathcal{D}_0}$. (Here $f_{\mathcal{D}_0}$ and $f_{\mathcal{D}_1}$ are as in §6.1.6.) Then by Lemma 6.1.11, the isomorphism

$$(\tau_1, \cdots, \tau_{m'}) : \mathbb{G}^{m'}_{m,\mathbb{C}} \xrightarrow{\sim} T_{M^{\mathrm{SO}},\mathbb{C}}$$

(see §8.8.1 for the τ_i's) is equal to the base change to $\mathbb{C}$ of $f_{\mathcal{D}_1} : \mathrm{U}(1)^{m'} \xrightarrow{\sim} T_{M^{\mathrm{SO}}}$, where we identify $\mathrm{U}(1)_{\mathbb{C}}$ with $\mathbb{G}_{m,\mathbb{C}}$.

To simplify notation below we write $T^{\pm}$ for $T_{M',\mathrm{SO},\pm}$. Since $j_{M^{\mathrm{SO}}}$ is admissible, by condition (2) in §8.8.1 we know that the isomorphisms

$$(j^{-1}_{M^{\mathrm{SO}}} \circ \tau_{n^-+1}, \cdots, j^{-1}_{M^{\mathrm{SO}}} \circ \tau_{m'}) : \mathbb{G}^{m'-n^-}_{m,\mathbb{C}} \xrightarrow{\sim} T^+_{\mathbb{C}} \tag{8.9.4.1}$$

and

$$(j^{-1}_{M^{\mathrm{SO}}} \circ \tau_1, \cdots, j^{-1}_{M^{\mathrm{SO}}} \circ \tau_{n^-}) : \mathbb{G}^{n^-}_{m,\mathbb{C}} \xrightarrow{\sim} T^-_{\mathbb{C}} \tag{8.9.4.2}$$

are induced by the isomorphisms

$$\mathfrak{d}_{B^{\pm},\mathcal{B}^{\pm}} : \widehat{T^{\pm}} \xrightarrow{\sim} \mathcal{T}^{\pm} \cong (\mathbb{C}^{\times})^{m'-n^-} \text{ or } (\mathbb{C}^{\times})^{n^-}$$

associated to some Borel subgroups $B^{\pm}$ of $M'^{,\mathrm{SO},\pm}_{\mathbb{C}}$ containing $T^{\pm}_{\mathbb{C}}$. Here $(\mathcal{T}^{\pm}, \mathcal{B}^{\pm})$ are the standard Borel pairs in the dual groups of $M'^{,\mathrm{SO},\pm}$, and the notation $\mathfrak{d}_{\cdot,\cdot}$ is as in §5.6. By the same argument as before, we can find $\mathcal{D}_2 = (\mathcal{D}_2^+, \mathcal{D}_2^-) \in$

$\mathrm{ED}(W_{\mathbb{R}}^{+})^{o} \times \mathrm{ED}(W_{\mathbb{R}}^{-})^{o}$ such that $\mathcal{D}_2^{\pm}$ gives rise to the fundamental pair $(T^{\pm}, B^{\pm})$. By Lemma 6.1.11, the isomorphisms (8.9.4.1) and (8.9.4.2) are equal to $f_{\mathcal{D}_2^+,\mathbb{C}}$ and $f_{\mathcal{D}_2^-,\mathbb{C}}$ respectively. Combining this with the previously established fact that $(\tau_1, \cdots, \tau_{m'}) = f_{\mathcal{D}_1,\mathbb{C}}$, we conclude that $j_{M^{\mathrm{SO}}} = j_{\mathcal{D}_2,\mathcal{D}_1}$. □

Proposition 8.9.5. — *For (A, B) taking values as in Lemma 8.6.2, we have*

$$♫(\emptyset, \emptyset) = -1,$$

$$♫(A, B)^{-1}\, ♫(\emptyset, \emptyset) = \begin{cases} 1, & \text{if } (A, B) = (\emptyset, \emptyset), \\ -1, & \text{if } A = \{1, 2\} \text{ or } B = \{1\}, \\ (-1)^{m^- + 1}, & \text{in all other cases.} \end{cases}$$

Proof. — In this proof we pass to the local notation over $\mathbb{R}$. For instance, we write M for $M_{\mathbb{R}}$. We use the phrase "Whittaker normalization" when we mean the Whittaker-normalized transfer factors between H and G^* or between H and G, associated to the unique (resp. the type-I) equivalence of Whittaker data for G^* when d is not divisible by 4 (resp. d is divisible by 4); see §6.2.1, Definition 6.2.8, Definition 6.3.4, and Definition 6.3.6. We shall also apply this notion to the transfer factors between $M'^{,\mathrm{SO}}$ and $M^{\mathrm{SO},*}$, and between $M'^{,\mathrm{SO}}$ and M^{SO}. By extending trivially across M^{GL}, we also obtain the "Whittaker normalization" of transfer factors between $M' = M^{\mathrm{GL}} \times M'^{,\mathrm{SO}}$ and $M^* = M^{\mathrm{GL}} \times M^{\mathrm{SO},*}$, and between M' and $M = M^{\mathrm{GL}} \times M^{\mathrm{SO}}$. As in §5.5.3, we view M^* as a Levi subgroup of G^* via (5.5.3.1).

We claim that the Whittaker normalization between M' and M is inherited from the Whittaker normalization between H and G as in Remark 8.4.13.

To prove the claim, first assume d is odd. Then G^* has a unique Whittaker datum (up to equivalence) and a unique $\mathbb{R}$-splitting (up to $G^*(\mathbb{R})$-conjugacy). The same also holds for M^*. Thus the unique Langlands–Shelstad normalization of transfer factors between M' and M^* is inherited from the unique Langlands–Shelstad normalization between H and G^*. (Indeed, one can see this by going through the definitions in [**LS87**]; alternatively, one can see this by using Waldspurger's explicit formula [**Wal10**, §1.10] while noting that the constant η in [**Wal10**, §1.6] attached to the unique splitting of $G^* = \mathrm{SO}(\underline{V})$ is equal to the discriminant δ, and hence equal to the analogous constant for $M^{\mathrm{SO},*} = \mathrm{SO}(\underline{W})$.) Moreover, the local epsilon factor relating the Whittaker normalization and the Langlands–Shelstad normalization (cf. (6.3.11.7)) is 1 in both the (H, G^*)-scenario and the (M', M^*)-scenario. This implies that the Whittaker normalization between M' and M^* is inherited from the Whittaker normalization between H and G^*. Our claim then follows from the three compatibility conditions in §5.5.3.

Second, assume d is even and not divisible by 4. Then by assumption $M = M_1$ or M_{12}, and so $M^{\mathrm{SO}} = \mathrm{SO}(W)$ with $\dim W = d - 4$ again not divisible by 4. Hence we still have uniqueness of Whittaker datum and uniqueness of $\mathbb{R}$-splitting for G^* and

M^*. As in the odd case, the unique Langlands–Shelstad normalization between M' and M^* is inherited from the analogous normalization between H and G^*. (Again, one can see this by using Waldspurger's explicit formula, noting that this time the constant η is equal to 1 for both $G^* = \mathrm{SO}(\underline{V})$ and $M^{\mathrm{SO},*} = \mathrm{SO}(\underline{W})$.) As in the odd case, the epsilon factor is still 1 in both the (H, G^*)-scenario and the (M', M^*)-scenario (since a maximal $\mathbb{R}$-split torus in each of G^*, H, M^*, M' is of the form a direct sum of a split torus and one copy of $\mathrm{U}(1)$). Our claim follows, as in the odd case.

Finally, assume d is divisible by 4. As in the previous case we have $\dim W = d - 4$, and this is divisible by 4. Using Waldspurger's explicit formula [**Wal10**, §1.10], we observe that the normalization between $M' = M^{\mathrm{GL}} \times M'^{,\mathrm{SO}}$ and $M^* = M^{\mathrm{GL}} \times M^{\mathrm{SO},*}$ induced by the Langlands–Shelstad normalization between $M'^{,\mathrm{SO}}$ and $M^{\mathrm{SO},*}$ associated to some $\mathbf{spl}_M \in \mathcal{S}plit(M^{\mathrm{SO},*})$ is inherited from the Langlands–Shelstad normalization between H and G^* associated to some $\mathbf{spl} \in \mathcal{S}plit(G^*)$ provided that $\eta_{\underline{W}}(\mathbf{spl}_M) = \eta_{\underline{V}}(\mathbf{spl})$. Here $\eta_{\underline{V}}(\cdot) : \mathcal{S}plit(G^*) \to \{\pm 1\}$ and $\eta_{\underline{W}}(\cdot) : \mathcal{S}plit(M^{\mathrm{SO},*}) \to \{\pm 1\}$ are as in §6.3.10. We now take $\mathbf{spl}_M$ and $\mathbf{spl}$ such that $\eta_{\underline{W}}(\mathbf{spl}_M) = \eta_{\underline{V}}(\mathbf{spl}) = -1$. By the above observation and by Theorem 6.3.11, we see that the Whittaker normalization between M' and M^* is inherited from the Whittaker normalization between H and G^* times the following constant. The constant is the ratio between the two local epsilon factors appearing in (6.3.11.7) and the analogue of (6.3.11.7) for (M', M). By (6.3.11.8) and a similar computation for (M', M), we see that the two epsilon factors are equal to $(-1)^{m^-}$ and $(-1)^{n^-}$ respectively, where m^- is the absolute rank of H^- and n^- is the absolute rank of $M'^{,\mathrm{SO},-}$. Since we are in the even case, we have $m^- \equiv n^- \mod 2$. Thus the Whittaker normalization between M' and M^* is inherited from the Whittaker normalization between H and G^*, and our claim follows as in the previous two cases.

It follows from the above claim and Lemma 8.9.2 that ♫(A, B) is the product of the following three signs:

(1) the sign between $\Delta^{A,B}_{j_M, B_M}$ and Δ_{j_M, B_M}, namely $(-1)^{q(G)+q(H)+q(M)+q(M')}$;

(2) the sign between Δ_{j_M, B_M} and the Whittaker normalization between M' and M, which is also equal to the sign between $\Delta_{j_{M^{\mathrm{SO}}}, B_{M^{\mathrm{SO}}}}$ and the Whittaker normalization between $M'^{,\mathrm{SO}}$ and M^{SO};

(3) the sign between $\Delta_{j, B_{G,H}}$ and the Whittaker normalization between H and G.

Denote by $m^{\pm}$ (resp. $n^{\pm}$) the absolute ranks of $H^{\pm}$ (resp. $M'^{,\mathrm{SO},\pm}$). Denote by m the absolute rank of G. We divide our computation into cases.

The odd case with $M = M_{12}$: In this case B is always $\emptyset$, and A is any subset of $\{1,2\}$. We have

$$q(G) = \frac{(2m-1)2}{2} = 2m-1, \qquad q(H) = \frac{m^+(m^++1)+m^-(m^-+1)}{2},$$
$$q(M) = 0, \qquad q(M') = \frac{n^+(n^++1)+n^-(n^-+1)}{2}.$$

When $A = \emptyset$, we have $m^+ = n^+$ and $m^- = n^- + 2$. Then

$$\begin{aligned} &q(G)+q(H)+q(M)+q(M') \\ &\qquad = 2m-1+\frac{2m^+(m^++1)+m^-(m^-+1)+(m^--2)(m^--1)}{2} \\ &\qquad \equiv 1+m^+(m^++1)+\frac{2(m^-)^2-2m^-+2}{2} \\ &\qquad \equiv 1+m^+(m^++1)+m^-(m^--1)+1 \\ &\qquad \equiv 0 \mod 2. \end{aligned}$$

When $A = \{1,2\}$, we have $m^+ = n^+ + 2$ and $m^- = n^-$. Observing symmetry we again get

$$q(G)+q(H)+q(M)+q(M') \equiv 0 \mod 2.$$

Now assume $A = \{1\}$ or $\{2\}$. Then $m^+ = n^+ + 1, m^- = n^- + 1$. We have

$$\begin{aligned} &q(G)+q(H)+q(M)+q(M') \\ &\quad = 2m-1+\frac{m^+(m^++1)+m^+(m^+-1)+m^-(m^-+1)+m^-(m^--1)}{2} \\ &\quad \equiv 1+\frac{2(m^+)^2+2(m^-)^2}{2} \\ &\quad \equiv 1+m^++m^- \\ &\quad \equiv m+1 \mod 2. \end{aligned}$$

We conclude that

$$(-1)^{q(G)+q(H)+q(M)+q(M')} = \begin{cases} 1, & A = \{1,2\} \text{ or } \emptyset, \\ (-1)^{m+1}, & A = \{1\} \text{ or } \{2\}. \end{cases}$$

The sign between $\Delta_{j_{M^{\mathrm{SO}}}, B_{M^{\mathrm{SO}}}}$ and the Whittaker normalization is $(-1)^{\lceil n^+/2 \rceil}$ by Lemma 8.9.4 and the $q = 0$ case of Proposition 6.2.20 (1). (We have already noted in §8.9.3 that the results in §6 indeed applies to M^{SO} together with its endoscopic group $M'^{,\mathrm{SO}}$.) The sign between $\Delta_{j,B_{G,H}}$ and the Whittaker normalization is $(-1)^{\lceil m^+/2 \rceil + 1}$ by the $q = 2$ case of Proposition 6.2.20 (1); here the hypothesis $m^+ > 0$ (i.e., H^+ is non-trivial) is guaranteed in §8.5.1, and the hypothesis that $(j, B_{G,H})$ arises from an element of $\mathrm{ED}(V)^o_{\mathrm{nice}}$ and an element of $\mathrm{ED}(V^+)^o \times \mathrm{ED}(V^-)^o$ is guaranteed in §8.4.2.

Thus we have

$$\begin{aligned}
♫(\emptyset,\emptyset) &= (-1)^{\lceil n^+/2\rceil+\lceil m^+/2\rceil+1} = (-1)^{\lceil m^+/2\rceil+\lceil m^+/2\rceil+1} = -1,\\
♫(\{1,2\},\emptyset) &= (-1)^{\lceil n^+/2\rceil+\lceil m^+/2\rceil+1} = (-1)^{\lceil (m^+-2)/2\rceil+\lceil m^+/2\rceil+1} = 1,\\
♫(\{1\},\emptyset) = ♫(\{2\},\emptyset) &= (-1)^{m+1+\lceil n^+/2\rceil+\lceil m^+/2\rceil+1}\\
&= (-1)^{m+1+\lceil (m^+-1)/2\rceil+\lceil m^+/2\rceil+1} = (-1)^{m+m^+} = (-1)^{m^-}.
\end{aligned}$$

This finishes the proof in this case.

The even case with $M = M_{12}$: In this case B is always $\emptyset$, and A is either $\emptyset$ or $\{1,2\}$. Note that $q(G), q(H), q(M), q(M')$ are all even. This is because each of G, H, M, M' is a product of a split torus and one or two cuspidal even special orthogonal group(s), namely some $\mathrm{SO}(a,b)$ with a,b even, for which we have $q(\mathrm{SO}(a,b)) = ab/2 \equiv 0 \mod 2$. It follows that the sign in part (1) is 1.

The sign between $\Delta_{j_{M^{\mathrm{SO}}},B_{M^{\mathrm{SO}}}}$ and the Whittaker normalization is $(-1)^{\lfloor n^-/2\rfloor}$ by Lemma 8.9.4 and the $q=0$ case of Proposition 6.2.20 (2) and Proposition 6.3.9.

Assume it is not the case that m is odd and $m^+ = 1$. Then $\mathcal{D}^H$ which was used to define $(j, B_{G,H})$ in §8.4.2 lies in $\mathrm{ED}(V)^o_{\mathrm{nice}}$. We have $m^+ \geq 2$ since $m^+ > 0$ (see §8.5.1). Applying the ($q = 2, m^+ \geq 2$) case of Proposition 6.2.20 (2) and Proposition 6.3.9, we see that the sign between $\Delta_{j,B_{G,H}}$ and the Whittaker normalization is $(-1)^{\lfloor m^-/2\rfloor}$.

Now assume m is odd and $m^+ = 1$. In this case $\mathcal{D}^H$ used to define $(j, B_{G,H})$ differs from an element of $\mathrm{ED}(V)^o_{\mathrm{nice}}$ by the transposition $(m-1,m) \in \mathfrak{S}_m$. Let $B'_{G,H}$ be the image of $B_{G,H}$ under $(m-1,m)$, viewed as an element of the complex Weyl group. An argument similar to the proof of the second statement of Lemma 6.3.5 shows that

$$\Delta_{j,B_{G,H}} = \langle a_{(m-1,m)}, s\rangle \Delta_{j,B'_{G,H}} = -\Delta_{j,B'_{G,H}}.$$

Hence the sign between $\Delta_{j,B_{G,H}}$ and the Whittaker normalization is -1 times the sign $(-1)^{\lfloor m^-/2\rfloor+1}$ in the ($q = 2, m^+ = 1$) case of Proposition 6.2.20 (2). Namely, it is again $(-1)^{\lfloor m^-/2\rfloor}$.

We conclude that $♫(A,B) = (-1)^{\lfloor n^-/2\rfloor+\lfloor m^-/2\rfloor}$. Specifically,

$$♫(\emptyset,\emptyset) = (-1)^{\lfloor (m^--2)/2\rfloor+\lfloor m^-/2\rfloor} = -1, \quad ♫(\{1,2\},\emptyset) = (-1)^{\lfloor m^-/2\rfloor+\lfloor m^-/2\rfloor} = 1.$$

This finishes the proof in this case.

The odd case with $M = M_1$: In this case A is always $\emptyset$, and B is any subset of $\{1\}$. We have

$$\begin{aligned} q(G) &= \frac{(2m-1)2}{2} = 2m-1, \\ q(H) &= \frac{m^+(m^++1)+m^-(m^-+1)}{2}, \\ q(M') &= q(M'^{,\mathrm{SO}}) + q(M^{\mathrm{GL}}) = \frac{n^+(n^++1)+n^-(n^-+1)}{2} + 1, \\ q(M) &= q(\mathrm{GL}_2) = 1. \end{aligned}$$

When $B = \emptyset$, we have $m^+ = n^+, m^- = n^- + 2$, and so

$$\begin{aligned} &q(G)+q(H)+q(M)+q(M') \\ &\qquad = 2m+1+\frac{2m^+(m^++1)+m^-(m^-+1)+(m^--2)(m^--1)}{2} \\ &\qquad \equiv m^+(m^++1)+\frac{2(m^-)^2-2m^-+2}{2}+1 \\ &\qquad \equiv m^+(m^++1)+m^-(m^--1) \\ &\qquad \equiv 0 \mod 2. \end{aligned}$$

When $B = \{1\}$, we have $m^+ = n^+ + 2, m^- = n^-$, and observing symmetry we again get

$$q(G)+q(H)+q(M)+q(M') \equiv 0 \mod 2.$$

Hence the sign in part (1) is 1.

The sign between $\Delta_{j_{M^{\mathrm{SO}}},B_{M^{\mathrm{SO}}}}$ and the Whittaker normalization is $(-1)^{\lceil n^+/2\rceil}$ by Lemma 8.9.4 and the $q=0$ case of Proposition 6.2.20 (1). The sign between $\Delta_{j,B_{G,H}}$ and the Whittaker normalization is $(-1)^{\lceil m^+/2\rceil+1}$ by the $q=2$ case of Proposition 6.2.20 (1). Thus ♫$(A,B) = (-1)^{\lceil n^+/2\rceil+\lceil m^+/2\rceil+1}$, and specifically

$$♫(\emptyset,\emptyset) = (-1)^{\lceil m^+/2\rceil+\lceil m^+/2\rceil+1} = -1, \quad ♫(\emptyset,\{1\}) = (-1)^{\lceil (m^+-2)/2\rceil+\lceil m^+/2\rceil+1} = 1.$$

This finishes the proof in this case.

The even case with $M = M_1$: As in the previous case, A is always $\emptyset$, and B is any subset of $\{1\}$. Now $q(G), q(H)$ are even, and $q(M), q(M')$ are odd. Hence the sign in part (1) is 1. Similarly as in the even case with $M = M_{12}$ treated before, the sign between $\Delta_{j_{M^{\mathrm{SO}}},B_{M^{\mathrm{SO}}}}$ and the Whittaker normalization is $(-1)^{\lfloor n^-/2\rfloor}$, and the sign between $\Delta_{j,B_{G,H}}$ and the Whittaker normalization is $(-1)^{\lfloor m^-/2\rfloor}$. Thus ♫$(A,B) = (-1)^{\lfloor n^-/2\rfloor+\lfloor m^-/2\rfloor}$, and specifically

$$♫(\emptyset,\emptyset) = (-1)^{\lfloor (m^--2)/2\rfloor+\lfloor m^-/2\rfloor} = -1, \quad ♫(\emptyset,\{1\}) = (-1)^{\lfloor m^-/2\rfloor+\lfloor m^-/2\rfloor} = 1.$$

This finishes the proof in this case.

The odd case with $M = M_2$: In this case B is always $\emptyset$, and A is any subset of $\{1\}$. We have

$$\begin{aligned} q(G) &= \frac{(2m-1)2}{2} = 2m-1, \\ q(H) &= \frac{m^+(m^++1)+m^-(m^-+1)}{2}, \\ q(M) &= \frac{d-3}{2} = m-1, \\ q(M') &= \frac{n^+(n^++1)+n^-(n^-+1)}{2}. \end{aligned}$$

When $A = \emptyset$, we have $m^+ = n^+, m^- = n^- + 1$, and so

$$\begin{aligned} &q(G)+q(H)+q(M)+q(M') \\ &\quad = 3m-2+\frac{2m^+(m^++1)+m^-(m^-+1)+(m^--1)m^-}{2} \\ &\quad \equiv m+m^+(m^++1)+\frac{2(m^-)^2}{2} \\ &\quad \equiv m+m^+(m^++1)+(m^-)^2 \\ &\quad \equiv m+m^- \\ &\quad \equiv m^+ \mod 2. \end{aligned}$$

When $A = \{1\}$, we have $m^+ = n^+ + 1, m^- = n^-$, and a similar computation yields

$$q(G)+q(H)+q(M)+q(M') \equiv m^- \mod 2.$$

We conclude that

$$(-1)^{q(G)+q(H)+q(M)+q(M')} = \begin{cases} (-1)^{m^+}, & A = \emptyset, \\ (-1)^{m^-}, & A = \{1\}. \end{cases}$$

The sign between $\Delta_{j_{M^{\mathrm{SO}}}, B_{M^{\mathrm{SO}}}}$ is $(-1)^{\lfloor n^+/2 \rfloor}$ by Lemma 8.9.4 and the $q = 1$ case of Proposition 6.2.20 (1). The sign between $\Delta_{j, B_{G,H}}$ and the Whittaker normalization is $(-1)^{\lceil m^+/2 \rceil + 1}$ by the $q = 2$ case of Proposition 6.2.20 (1). Thus we have

$$\begin{aligned} ♫(\emptyset, \emptyset) &= (-1)^{m^+ + \lfloor n^+/2 \rfloor + \lceil m^+/2 \rceil + 1} \\ &= (-1)^{m^+ + \lfloor m^+/2 \rfloor + \lceil m^+/2 \rceil + 1} = (-1)^{m^+ + m^+ + 1} = -1, \\ ♫(\{1\}, \emptyset) &= (-1)^{m^- + \lfloor n^+/2 \rfloor + \lceil m^+/2 \rceil + 1} = (-1)^{m^- + \lfloor (m^+-1)/2 \rfloor + \lceil m^+/2 \rceil + 1} = (-1)^{m^-}. \end{aligned}$$

This finishes the proof in this case. □

Definition 8.9.6. — For (A, B) as in Lemma 8.6.2, define the sign

$$☼(A, B) := \begin{cases} -1, & \text{if } 1 \in A \text{ or } 1 \in B, \\ 1, & \text{otherwise.} \end{cases}$$

Suppose g is a function that assigns to each choice of (A,B) an element $g(A,B) \in \mathcal{H}^{\mathrm{ur}}(M'_{\mathbb{Q}_p})$. Define

$$\begin{aligned} J(\mathfrak{e}_{\mathfrak{p}}(M),\gamma',g) &= J(\mathfrak{e}_{\mathfrak{p}}(M),\gamma',(A,B)\mapsto g(A,B)) \\ &:= \sum_{A,B} \text{☼}(A,B) SO_{\gamma'}(g(A,B))\epsilon_R(j_M(\gamma'^{-1}))\epsilon_{R_H}(\gamma'^{-1})\Phi_M^G(j_M(\gamma')^{-1},\Theta_{\mathbb{V}^*}^H), \end{aligned}$$

where the sum is over all choices of (A,B).

Definition 8.9.7. — With notation as in Definition 8.7.5 and §8.8.7, we define

$$\begin{aligned} Q(\mathfrak{e}_{\mathfrak{p}}(M),\gamma') := \bar{\iota}^{M'}(\gamma')^{-1}(\Delta_{M'}^{M})^{\emptyset,\emptyset}(\gamma',\gamma_M) O_{\gamma_M}^{s'_M}(f_M^{p,\infty})\tau(M)k(M)k(G)^{-1} \\ \cdot(-1)^{\dim A_{M'}}\bar{v}({M'}^0_{\gamma'})^{-1}(-1)^{q(G_{\mathbb{R}})}\Delta_{j_M,B_M}^{\emptyset,\emptyset}(\gamma',j_M(\gamma')). \end{aligned}$$

Here, we choose $\gamma_M \in M(\mathbb{A}_f^p)$ as in Definition 8.7.5, which does not affect the definition.

Corollary 8.9.8. — *With J as in Definition 8.9.6 and Q as in Definition 8.9.7, we have*

$$I(\mathfrak{e}_{\mathfrak{p}}(M),\gamma') = Q(\mathfrak{e}_{\mathfrak{p}}(M),\gamma')J(\mathfrak{e}_{\mathfrak{p}}(M),\gamma',(A,B)\mapsto f_{p,M'}^H).$$

Here the mapping $(A,B)\mapsto f_{p,M'}^H$ is defined via the dependence of H on (A,B) as in Lemma 8.6.2.

Proof. — By (8.8.10.1) and Proposition 8.9.5, we have

$$\text{☼}(A,B) = \text{♫}(A,B)^{-1}\,\text{♫}(\emptyset,\emptyset)\det(\omega_0).$$

The corollary then follows from the second equality in Lemma 8.7.6, Proposition 8.8.11, and Lemma 8.9.2. □

8.10. Symmetry of order n_M^G

Definition 8.10.1. — We define a subgroup $\mathfrak{W} \subset \mathrm{Aut}(M^{\mathrm{GL}})$ as follows. When $M = M_{12}$, so that $M^{\mathrm{GL}} = \mathbb{G}_m^2$, we define $\mathfrak{W}$ to be $\{\pm 1\}^2 \rtimes \mathfrak{S}_2$, where each factor $\{\pm 1\}$ acts on each factor $\mathbb{G}_m$ non-trivially and $\mathfrak{S}_2$ acts by swapping the two copies of $\mathbb{G}_m$. When $M = M_1$, so that $M^{\mathrm{GL}} = \mathrm{GL}_2$, we define $\mathfrak{W}$ to be $\mathbb{Z}/2\mathbb{Z}$ with the non-trivial element acting on GL_2 by transpose inverse. When $M = M_2$ in the odd case, so that $M^{\mathrm{GL}} = \mathbb{G}_m$, we define $\mathfrak{W}$ to be equal to $\mathrm{Aut}(M^{\mathrm{GL}}) = \mathrm{Aut}(\mathbb{G}_m) = \mathbb{Z}/2\mathbb{Z}$. When the context is clear we also view $\mathfrak{W}$ as a subgroup of $\mathrm{Aut}(M)$ or $\mathrm{Aut}(M')$, by extending its action on M^{GL} trivially across M^{SO} or $M'^{,\mathrm{SO}}$.

Lemma 8.10.2. — *The natural homomorphism $\mathfrak{W} \to \mathrm{Aut}(A_M)$ is an injection, and its image is equal to the image of $\mathrm{Nor}_G(M)(\mathbb{Q})$ in $\mathrm{Aut}(A_M)$. In particular, $\mathfrak{W}$ is naturally isomorphic to $\mathcal{W}_M^G$ and $|\mathfrak{W}| = n_M^G$ (see Definition 1.1.1 and Remark 1.1.2).*

Proof. — This is straightforward to check. □

8.10.3. — The action of $\mathfrak{W}$ on the set of stable conjugacy classes in $M'(\mathbb{Q})_{\mathrm{ss}}$ preserves the following conditions:

– being $\mathbb{R}$-elliptic,
– being (M, M')-regular,
– being an image of a semi-simple element of $M(\mathbb{A}_f^p)$.

(Indeed, the only non-trivial assertion here is that $\mathfrak{W}$ preserves being $\mathbb{R}$-elliptic in the case $M = M_1$, and this follows from the fact that $M_1^{\mathrm{GL}} = \mathrm{GL}_2$ contains the $\mathbb{R}$-elliptic maximal torus $T_{\mathrm{GL}_2}^{\mathrm{std}}$ which is $\mathfrak{W}$-stable.) Moreover, if $M = M_{12}$ or M_2, then two different elements of $M'(\mathbb{Q})_{\mathrm{ss}}$ in the same $\mathfrak{W}$-orbit are never stably conjugate to each other. Therefore in these cases we may and shall assume that the sets $\Sigma(M')$ and $\Sigma(M')_1$ chosen in Definitions 8.7.1 and 8.7.5 are stable under $\mathfrak{W}$. If $M = M_1$, then two different $\mathbb{R}$-elliptic elements of $M'(\mathbb{Q})$ in the same $\mathfrak{W}$-orbit are either not stably conjugate to each other, or such that their components in $M^{\mathrm{GL}}(\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Q})$ both have determinant 1. (To see this, note that if $g \in \mathrm{GL}_2(\mathbb{Q})_{\mathrm{ss}}$ is stably conjugate to its transpose inverse, then $\det g = \pm 1$, and we have $\det g > 0$ if g is $\mathbb{R}$-elliptic.) Therefore in this case we may and shall assume that $\Sigma(M')_1$ contains a subset $\Sigma(M')_2$ such that $\Sigma(M')_2$ is stable under $\mathfrak{W}$ and the component in $M^{\mathrm{GL}}(\mathbb{Q})$ of every element of $\Sigma(M')_1 - \Sigma(M')_2$ has determinant 1. To unify notation, when $M = M_{12}$ or M_2, we set $\Sigma(M')_2$ to be $\Sigma(M')_1$.

Lemma 8.10.4. — *For $\gamma' \in \Sigma(M')_2$ and $w \in \mathfrak{W}$, we have $Q(\mathfrak{e}_{\mathfrak{p}}(M), \gamma') = Q(\mathfrak{e}_{\mathfrak{p}}(M), w(\gamma'))$. (See Definition 8.9.7 for Q).*

Proof. — By (8.7.3.1), we have

$$Q(\mathfrak{e}_{\mathfrak{p}}(M), \gamma') = C \times SO_{\gamma'}(f_{M'}^{H(\emptyset,\emptyset),p,\infty})\Delta_{j_M,B_M}^{\emptyset,\emptyset}(\gamma', j_M(\gamma')),$$

where C is an expression that is invariant under $\mathfrak{W}$, and $H(\emptyset, \emptyset)$ is the particular choice of H arising from $(A, B) = (\emptyset, \emptyset)$. Note that the subgroup $\mathfrak{W} \subset \mathrm{Aut}(M')$ is contained [9] in the image of the natural map $\mathrm{Nor}_{H(\emptyset,\emptyset)}(M')(\mathbb{Q}) \to \mathrm{Aut}(M')$.

Since w comes from $\mathrm{Nor}_{H(\emptyset,\emptyset)}(M')(\mathbb{Q})$, we have

$$SO_{\gamma'}(f_{M'}^{H(\emptyset,\emptyset),p,\infty}) = SO_{w(\gamma')}(f_{M'}^{H(\emptyset,\emptyset),p,\infty})$$

by exactly the same argument (using Kazhdan density and descent) as in the proof of Lemma 2.5.4. We are left to check

$$\Delta_{j_M,B_M}^{\emptyset,\emptyset}(\gamma', j_M(\gamma')) = \Delta_{j_M,B_M}^{\emptyset,\emptyset}(w(\gamma'), j_M(w(\gamma'))),$$

or equivalently,

$$\Delta_{j_M,B_M}(\gamma', j_M(\gamma')) = \Delta_{j_M,B_M}(w(\gamma'), j_M(w(\gamma'))).$$

(9) Note that this would no longer be not true, for instance, if $H(\emptyset, \emptyset)$ is replaced by the choice of H arising from $(A, B) = (\{1\}, \emptyset)$ when $M = M_{12}$.

The last equality holds because both sides depend only on the common component of γ' and $w(\gamma')$ in $M'^{,\mathrm{SO}}$. More precisely, if we denote this common component by $\gamma'^{,\mathrm{SO}}$, then both sides are equal to

$$\Delta_{j_{M^{\mathrm{SO}}},B_{M^{\mathrm{SO}}}}(\gamma'^{,\mathrm{SO}}, j_{M^{\mathrm{SO}}}(\gamma'^{,\mathrm{SO}})),$$

where $j_{M^{\mathrm{SO}}}$ and $B_{M^{\mathrm{SO}}}$ are as in §8.8.1 and §8.9.3. This finishes the proof. □

Proposition 8.10.5. — *For each $\mathfrak{e} = (M', s'_M, \eta_M) \in \mathscr{E}(M)$, choose a set $\Sigma(M')_1$ as in Definition 8.7.5 and §8.10.3. In view of Lemma 8.10.4, for each $\gamma' \in \Sigma(M')_1$ we write $Q(\mathfrak{e}, \mathfrak{W}\gamma')$ for $Q(\mathfrak{e}, \gamma')$. We have*

$$\begin{aligned} n_M^G \operatorname{Tr}'_M = & \sum_{\mathfrak{e}_{\mathfrak{p}}(M)=(M',{}^LM',s'_M,\eta_M)\in\dot{\mathscr{E}}(M)^{c,\mathrm{ur}}} |\mathrm{Out}_M(\mathfrak{e}_{\mathfrak{p}}(M))|^{-1} \\ & \cdot \Bigg(\sum_{\gamma'\in\Sigma(M')_2} Q(\mathfrak{e}_{\mathfrak{p}}(M),\gamma')\,|\mathfrak{W}|^{-1} \sum_{w\in\mathfrak{W}} J(\mathfrak{e}_{\mathfrak{p}}(M), w(\gamma'), (A,B)\mapsto f^H_{p,M'}) \\ & \quad + \sum_{\gamma'\in\Sigma(M')_1-\Sigma(M')_2} Q(\mathfrak{e}_{\mathfrak{p}}(M),\gamma') J(\mathfrak{e}_{\mathfrak{p}}(M),\gamma',(A,B)\mapsto f^H_{p,M'}) \Bigg). \end{aligned}$$

Proof. — This is a consequence of Lemma 8.7.6, Corollary 8.9.8, Lemma 8.10.4. □

8.11. Computation of J

We compute the term $J(\mathfrak{e}_{\mathfrak{p}}(M), \gamma', (A,B) \mapsto f^H_{p,M'})$ in Corollary 8.9.8 using results from §7.4. We simply write $\mathfrak{e}$ for $\mathfrak{e}_{\mathfrak{p}}(M)$. Recall the functions $\epsilon_R(\cdot) : T_M(\mathbb{R}) \to \{\pm 1\}$ and $\epsilon_{R_H}(\cdot) : T_M(\mathbb{R}) \to \{\pm 1\}$ from §8.8.7. The former depends only on $\mathfrak{e}_{\mathfrak{p}}(M)$, while the latter depends on $\mathfrak{e}_{\mathfrak{p}}(M)$ and (A,B).

Lemma 8.11.1. — *For $M = M_{12}$ in the odd case, we have*

$$\begin{aligned} \epsilon_R(j_M(\gamma'^{-1})) &= \epsilon_{R_H}(\gamma'^{-1})|_{A=\{1,2\}} = \epsilon_{R_H}(\gamma'^{-1})|_{A=\emptyset}, \\ \epsilon_{R_H}(\gamma'^{-1})|_{A=\{1\}} &= \epsilon_{R_H}(\gamma'^{-1})|_{A=\{2\}}. \end{aligned}$$

In all the other cases of M, we have

$$\epsilon_R(j_M(\gamma'^{-1})) = \epsilon_{R_H}(\gamma'^{-1}).$$

Proof. — This follows directly from the definitions. □

8.11.2. — We introduce some notations. Write $p^* := p^{a(d-2)/2}$. Write

$$f^H_{p,M'} = p^* k(A,B) + p^* h \in \mathcal{H}^{\mathrm{ur}}(M'_{\mathbb{Q}_p}),$$

as in Proposition 7.4.2. When $M = M_{12}$, we further write

$$k(A,B) = k_1(A) + k_2(A),$$

where $k_i(A) \in \mathcal{H}^{\mathrm{ur}}(M'_{\mathbb{Q}_p})$ has Satake transform $\nabla_i(A)(\xi_i^a + \xi_i^{-a})$, as in Proposition 7.4.2. Thus we have

$$J(\mathfrak{e}, \gamma', (A,B) \mapsto f^H_{p,M'}) = p^* J(\mathfrak{e}, \gamma', k) + p^* J(\mathfrak{e}, \gamma', h), \tag{8.11.2.1}$$

and when $M = M_{12}$ we further have

$$J(\mathfrak{e}, \gamma', (A,B) \mapsto f^H_{p,M'}) = p^* J(\mathfrak{e}, \gamma', k_1) + p^* J(\mathfrak{e}, \gamma', k_2) + p^* J(\mathfrak{e}, \gamma', h). \tag{8.11.2.2}$$

Here we use the abbreviated notation $J(\mathfrak{e}, \gamma', k) := J(\mathfrak{e}, \gamma', (A,B) \mapsto k(A,B))$, etc. In the following computation We write

$$\begin{aligned} \Phi^G_M &:= \Phi^G_M(j_M(\gamma')^{-1}, \Theta_{\mathbb{V}^*}), \\ \Phi^G_{M,\mathrm{eds}} &:= \Phi^G_M(j_M(\gamma')^{-1}, \Theta_{\mathbb{V}^*})_{\mathrm{eds}}, \\ \epsilon_R \epsilon_{R_H} &:= \epsilon_R(j_M(\gamma'^{-1}))\epsilon_{R_H}(\gamma'^{-1}). \end{aligned}$$

8.11.3. Odd case M_{12}. — With the above notations, it follows from Lemma 8.11.1 and the fact that ϵ_R is independent of (A,B) that we have

(8.11.3.1)

$$\begin{aligned} J(\mathfrak{e}, \gamma', h) &= SO_{\gamma'}(h) \sum_{A,B} ☼(A,B)\epsilon_R\epsilon_{R_H}\Phi^G_M(j_M(\gamma')^{-1}, \Theta^H_{\mathbb{V}^*}) \\ &= SO_{\gamma'}(h)\left[\Phi^G_M - \Phi^G_M - (\epsilon_R\epsilon_{R_H})|_{A=\{1\}}\Phi^G_{M,\mathrm{eds}} + (\epsilon_R\epsilon_{R_H})|_{A=\{2\}}\Phi^G_{M,\mathrm{eds}}\right] \\ &= 0. \end{aligned}$$

Similarly

(8.11.3.2)

$$\begin{aligned} &J(\mathfrak{e}, \gamma', k_1) \\ &\quad = SO_{\gamma'}(k_1(\emptyset))\left[\Phi^G_M + \Phi^G_M + (\epsilon_R\epsilon_{R_H})|_{A=\{1\}}\Phi^G_{M,\mathrm{eds}} + (\epsilon_R\epsilon_{R_H})|_{A=\{2\}}\Phi^G_{M,\mathrm{eds}}\right] \\ &\quad = 2SO_{\gamma'}(k_1(\emptyset))\left[\Phi^G_M + (\epsilon_R\epsilon_{R_H})|_{A=\{1\}}\Phi^G_{M,\mathrm{eds}}\right], \end{aligned}$$

and

$$J(\mathfrak{e}, \gamma', k_2) = 2SO_{\gamma'}(k_2(\emptyset))\left[\Phi^G_M - (\epsilon_R\epsilon_{R_H})|_{A=\{1\}}\Phi^G_{M,\mathrm{eds}}\right]. \tag{8.11.3.3}$$

8.11.4. Even case M_{12}. — With similar computations as above, we get

$$\begin{aligned} J(\mathfrak{e}, \gamma', h) &= 0, \\ J(\mathfrak{e}, \gamma', k_1) &= 2SO_{\gamma'}(k_1(\emptyset))\Phi^G_M, \\ J(\mathfrak{e}, \gamma', k_2) &= 2SO_{\gamma'}(k_2(\emptyset))\Phi^G_M. \end{aligned}$$

8.11.5. Case M_1 and odd case M_2. — Similar computations give

$$J(\mathfrak{e},\gamma',h)=0 \tag{8.11.5.1}$$

$$J(\mathfrak{e},\gamma',k)=2SO_{\gamma'}(k(\emptyset,\emptyset))\Phi^G_M. \tag{8.11.5.2}$$

8.12. Breaking symmetry, case M_{12}

We keep the notation in Proposition 8.10.5 and §8.11.

Definition 8.12.1. — Suppose $M=M_{12}$. We say that an element of $M'(\mathbb{R})$ is *good at* ∞ if its component in $M^{\mathrm{GL}}(\mathbb{R})=\mathbb{R}^\times\times\mathbb{R}^\times$ lies in $(\mathbb{R}_{>0}\times\mathbb{R}_{>0})\cup(\mathbb{R}_{<0}\times\mathbb{R}_{<0})$. We say that an element of $M'(\mathbb{Q}_p)$ is *good at p* if its component in $M^{\mathrm{GL}}(\mathbb{Q}_p)=\mathbb{Q}_p^\times\times\mathbb{Q}_p^\times$ has p-adic valuations $(-a,0)$. Here the first $\mathbb{G}_m$-factor is $\mathrm{GL}(V_1)$ and the second is $\mathrm{GL}(V_2/V_1)$.

Proposition 8.12.2. — *Let $M=M_{12}$. We have*

$$n^G_M\,\mathrm{Tr}'_M=\sum_{\mathfrak{e}_\mathfrak{p}(M)=(M',{}^LM',s'_M,\eta_M)\in\dot{\mathscr{E}}(M)^{c,\mathrm{ur}}}|\mathrm{Out}_M(\mathfrak{e}_\mathfrak{p}(M))|^{-1}\cdot\sum_{\gamma'}Q(\mathfrak{e}_\mathfrak{p}(M),\gamma')4p^*J(\mathfrak{e}_\mathfrak{p}(M),\gamma',k_1),$$

where γ' runs through the elements of $\Sigma(M')_1$ that are good at ∞ and good at p.

Proof. — We start with the formula for $n^G_M\,\mathrm{Tr}'_M$ in Proposition 8.10.5, and recall that in that formula $\Sigma(M')_1=\Sigma(M')_2$ for our $M=M_{12}$. Fix $\mathfrak{e}=\mathfrak{e}_\mathfrak{p}(M)=(M',{}^LM',s'_M,\eta_M)\in\dot{\mathscr{E}}(M)^{c,\mathrm{ur}}$.

We first treat the odd case. Let $w_1:=(-1,1)\in\{\pm1\}^2\subset\mathfrak{W}=\{\pm1\}^2\rtimes\mathfrak{S}_2$, and let w_{12} be the non-trivial element of $\mathfrak{S}_2\subset\mathfrak{W}$. For $\gamma'\in\Sigma(M')_1$, combining the computation of $J(\mathfrak{e},\gamma',k_1)$ and $J(\mathfrak{e},\gamma',k_2)$ in §8.11.3 and the vanishing statement in Proposition 4.6.12, we know that

$$J(\mathfrak{e},\gamma',k_1)=J(\mathfrak{e},\gamma',k_2)=0$$

unless γ' is good at ∞. We also note that being good at ∞ is a property invariant under $\mathfrak{W}$. Now by (8.11.2.2) and (8.11.3.1) we have

$$J(\mathfrak{e},\gamma',(A,B)\mapsto f^H_{p,M'})=p^*J(\mathfrak{e},\gamma',k_1)+p^*J(\mathfrak{e},\gamma',k_2). \tag{8.12.2.1}$$

Therefore, if $\gamma'\in\Sigma(M')_1$ is such that

$$\sum_{w\in\mathfrak{W}}J(\mathfrak{e},w(\gamma'),(A,B)\mapsto f^H_{p,M'})\neq0, \tag{8.12.2.2}$$

then γ' is good at ∞,

Suppose $\gamma' \in \Sigma(M')_1$ is good at ∞. Then by Proposition 4.6.13, (8.11.3.2), and (8.11.3.3), we have

$$J(\mathfrak{e}, \gamma', k_1) = J(\mathfrak{e}, w_{12}(\gamma'), k_2) \tag{8.12.2.3}$$

$$J(\mathfrak{e}, \gamma', k_1) = J(\mathfrak{e}, w_1(\gamma'), k_1) \tag{8.12.2.4}$$

because the functions $k_1(\emptyset)$ and $k_2(\emptyset)$ are pull-backs of each other under w_{12}, and the function $k_1(\emptyset)$ is invariant under w_1. Combining (8.12.2.1) and (8.12.2.3), we obtain

(8.12.2.5)
$$\begin{aligned}\sum_{w\in\mathfrak{W}} J(\mathfrak{e}, w(\gamma'), (A,B)\mapsto f^H_{p,M'}) &= \sum_{w\in\mathfrak{W}} p^* J(\mathfrak{e}, w(\gamma'), k_1) + p^* J(\mathfrak{e}, w_{12}w(\gamma'), k_1) \\ &= 2\sum_{w\in\mathfrak{W}} p^* J(\mathfrak{e}, w(\gamma'), k_1).\end{aligned}$$

Assume (8.12.2.2) holds. Then by (8.12.2.5), there exists $\gamma'' \in \mathfrak{W}\gamma'$ such that $J(\mathfrak{e}, \gamma'', k_1) \neq 0$. By (8.11.3.2), the last condition implies that $SO_{\gamma''}(k_1(\emptyset)) \neq 0$, from which it easily follows that either γ'' or $w_1(\gamma'')$ (but not both) is good at p. Note that in $\mathfrak{W}$, there are either zero or two elements w such that $w(\gamma')$ is good at p. In the latter case, the two elements differ by left multiplication by $w_2 := (1,-1) \in \{\pm 1\}^2 \subset \mathfrak{W}$. Combining this analysis with (8.12.2.4) and (8.12.2.5), we have

(8.12.2.6)
$$\begin{aligned}&\sum_{w\in\mathfrak{W}} J(\mathfrak{e}, w(\gamma'), (A,B)\mapsto f^H_{p,M'}) \\ &\quad= 2p^* \sum_{w\in\mathfrak{W}, w(\gamma')\text{ good at }p} J(\mathfrak{e}, w(\gamma'), k_1) + J(\mathfrak{e}, w_1 w(\gamma'), k_1) \\ &\quad= 4p^* \sum_{w\in\mathfrak{W}, w(\gamma')\text{ good at }p} J(\mathfrak{e}, w(\gamma'), k_1) \\ &\quad= \begin{cases} 0, & \text{if } \nexists \gamma'' \in \mathfrak{W}\gamma' \text{ good at } p, \\ 4p^*\big(J(\mathfrak{e}, \gamma'', k_1) + J(\mathfrak{e}, w_2(\gamma''), k_1)\big), & \text{if } \gamma'' \in \mathfrak{W}\gamma' \text{ is good at } p.\end{cases}\end{aligned}$$

Moreover, if $\gamma'' \in \mathfrak{W}\gamma'$ is good at p, then we have

$$|\mathfrak{W}\gamma'| = \begin{cases} |\mathfrak{W}|, & \text{if } w_2(\gamma'') \neq \gamma'', \\ |\mathfrak{W}|/2, & \text{if } w_2(\gamma'') = \gamma''.\end{cases} \tag{8.12.2.7}$$

Combining the discussion about being good at ∞ at the beginning of the proof, the formulas (8.12.2.6) and (8.12.2.7), and Lemma 8.10.4, we obtain:

$$\sum_{\gamma'\in\Sigma(M')_1} Q(\mathfrak{e},\gamma')\,|\mathfrak{W}|^{-1}\sum_{w\in\mathfrak{W}} J(\mathfrak{e},w(\gamma'),(A,B)\mapsto f^H_{p,M'})$$
$$= \sum_{\substack{\gamma''\in\Sigma(M')_1\\ \gamma''\text{ good at }p,\infty\\ \gamma''\neq w_2\gamma''}} Q(\mathfrak{e},\gamma'')\,|\mathfrak{W}|^{-1}\,|\mathfrak{W}\gamma''|\,4p^*J(\mathfrak{e},\gamma'',k_1)$$
$$+ \sum_{\substack{\gamma''\in\Sigma(M')_1\\ \gamma''\text{ good at }p,\infty\\ \gamma''= w_2\gamma''}} Q(\mathfrak{e},\gamma'')\,|\mathfrak{W}|^{-1}\,|\mathfrak{W}\gamma''|\,8p^*J(\mathfrak{e},\gamma'',k_1)$$
$$= \sum_{\substack{\gamma''\in\Sigma(M')_1\\ \gamma''\text{ good at }p,\infty}} Q(\mathfrak{e},\gamma'')4p^*J(\mathfrak{e},\gamma'',k_1).$$

This together with Proposition 8.10.5 implies the current proposition in the odd case.

The even case is proved in a similar way. The only differences are that we now use the vanishing statement in Proposition 4.6.14 rather than Proposition 4.6.12, and that we simply use the invariance of $\Phi^G_M(\cdot,\Theta_{\mathbb{V}^*})$ under $\mathrm{Nor}_G(M)(\mathbb{R})$ to deduce (8.12.2.3) and (8.12.2.4) . □

8.13. Breaking symmetry, case M_1 and odd case M_2

We keep the notation in Proposition 8.10.5 and §8.11.

Definition 8.13.1. — Suppose $M = M_1$. We say that an element of $M'(\mathbb{Q}_p)$ is *good at p* if its component in $M^{\mathrm{GL}}(\mathbb{Q}_p) = \mathrm{GL}_2(\mathbb{Q}_p)$ has determinant of p-adic valuation $-a$. We say that all elements of $M'(\mathbb{R})$ are *good at* ∞.

Suppose $M = M_2$ in the odd case. We say that an element of $M'(\mathbb{Q}_p)$ is *good at p* if its component in $M^{\mathrm{GL}}(\mathbb{Q}_p) = \mathbb{Q}_p^\times$ has valuation $-a$. We say that an element of $M'(\mathbb{R})$ is *good at* ∞ if its component in $M^{\mathrm{GL}}(\mathbb{R}) = \mathbb{R}^\times$ is positive.

Proposition 8.13.2. — *Suppose $M = M_1$, or $M = M_2$ in the odd case. We have*

$$n^G_M\,\mathrm{Tr}'_M = \sum_{\mathfrak{e}_{\mathfrak{p}}(M)=(M',{}^LM',s'_M,\eta_M)\in\dot{\mathscr{E}}(M)^{c,\mathrm{ur}}} |\mathrm{Out}_M(\mathfrak{e}_{\mathfrak{p}}(M))|^{-1}$$
$$\cdot\sum_{\gamma'} Q(\mathfrak{e}_{\mathfrak{p}}(M),\gamma')2p^*J(\mathfrak{e}_{\mathfrak{p}}(M),\gamma',k),$$

where γ' runs through the elements of $\Sigma(M')_1$ that are good at ∞ and good at p.

Proof. — We start with the formula for $n^G_M\,\mathrm{Tr}'_M$ in Proposition 8.10.5. Fix $\mathfrak{e} = \mathfrak{e}_{\mathfrak{p}}(M) = (M',{}^LM',s'_M,\eta_M) \in \dot{\mathscr{E}}(M)^{c,\mathrm{ur}}$. Let $\gamma' \in \Sigma(M')_1$. Let $w_1 \in \mathfrak{W}$ be the non-trivial element. In view of (8.11.5.2), it follows from the obvious invariance of $k(\emptyset,\emptyset)$ under w_1 and the invariance of $\Phi^G_M(\cdot,\Theta_{\mathbb{V}^*})$ under $\mathrm{Nor}_G(M)(\mathbb{R})$ that we have

$$J(\mathfrak{e},\gamma',k) = J(\mathfrak{e},w_1(\gamma'),k). \tag{8.13.2.1}$$

By (8.11.2.1) and (8.11.5.1) we have

$$J(\mathfrak{e},\gamma',(A,B)\mapsto f^H_{p,M'}) = p^* J(\mathfrak{e},\gamma',k). \tag{8.13.2.2}$$

If this is non-zero, then $SO_{\gamma'}(k(\emptyset,\emptyset)) \neq 0$ by (8.11.5.2), and it easily follows that either γ' or $w_1\gamma'$ is good at p. This implies that $\gamma' \in \Sigma(M')_2$ (as $a \geq 1$). Thus

$$\sum_{\gamma'\in\Sigma(M')_1-\Sigma(M')_2} Q(\mathfrak{e},\gamma')J(\mathfrak{e},\gamma',(A,B)\mapsto f^H_{p,M'}) = 0. \tag{8.13.2.3}$$

Now suppose $\gamma' \in \Sigma(M')_2$. By (8.13.2.1) and (8.13.2.2) we have

$$\sum_{w\in\mathfrak{W}} J(\mathfrak{e},w(\gamma'),(A,B)\mapsto f^H_{p,M'}) = 2p^*J(\mathfrak{e},\gamma',k) = 2p^*J(\mathfrak{e},w_1(\gamma'),k).$$

Suppose this is non-zero. Then one of γ' and $w_1(\gamma')$ is good at p, by the same argument as before. Also, by (8.11.5.2), we have $\Phi^G_M(j_M(\gamma')^{-1},\Theta_{\mathbb{V}^*}) \neq 0$. By the vanishing statement in Proposition 4.5.2, the last condition implies that γ' (and hence also $w_1(\gamma')$) is good at ∞ when $M = M_2$. Note that at most one of γ' and $w_1(\gamma')$ can be good at p. Hence

$$\begin{aligned}
&\sum_{\gamma'\in\Sigma(M')_2} Q(\mathfrak{e},\gamma')\,|\mathfrak{W}|^{-1}\sum_{w\in\mathfrak{W}} J(\mathfrak{e},w(\gamma'),(A,B)\mapsto f^H_{p,M'}) \\
&= \sum_{\substack{\gamma'\in\Sigma(M')_2\\ \gamma' \text{ good at } p,\infty}} Q(\mathfrak{e},\gamma')2^{-1}2p^*J(\mathfrak{e},\gamma',k) \\
&\quad+ \sum_{\substack{\gamma'\in\Sigma(M')_2\\ w_1(\gamma') \text{ good at } p,\infty}} Q(\mathfrak{e},\gamma')2^{-1}2p^*J(\mathfrak{e},w_1(\gamma'),k) \\
&= \sum_{\substack{\gamma'\in\Sigma(M')_2\\ \gamma' \text{ good at } p,\infty}} Q(\mathfrak{e},\gamma')2^{-1}2p^*J(\mathfrak{e},\gamma',k) \\
&\quad+ \sum_{\substack{\gamma'\in\Sigma(M')_2\\ \gamma' \text{ good at } p,\infty}} Q(\mathfrak{e},\gamma')2^{-1}2p^*J(\mathfrak{e},\gamma',k) \\
&= \sum_{\substack{\gamma'\in\Sigma(M')_2\\ \gamma' \text{ good at } p,\infty}} Q(\mathfrak{e},\gamma')2p^*J(\mathfrak{e},\gamma',k).
\end{aligned} \tag{8.13.2.4}$$

Here for the second equality, we made the substitution $\gamma' \mapsto w_1(\gamma')$ in the second summation and used Lemma 8.10.4. The proposition follows from Proposition 8.10.5, (8.13.2.3), and (8.13.2.4). □

8.14. Main computation

We keep letting M denote one of M_1, M_2, M_{12}, and excluding M_2 in the even case.

Proposition 8.14.1. — *Let $\mathscr{C}_M = 1$ for $M = M_{12}$ or M_1, and let $\mathscr{C}_M = 2$ for $M = M_2$ (in the odd case). When $a \in \mathbb{Z}_{>0}$ is large enough (for a fixed $f^{p,\infty}$), we have*

(8.14.1.1)

$$\mathscr{C}_M \operatorname{Tr}'_M = 4p^* \sum_{\mathfrak{e}_\mathfrak{p}(M)=(M',{}^LM',s'_M,\eta_M)\in\dot{\mathscr{E}}(M)^{c,\mathrm{ur}}} (-1)^{\dim A_{M'}+q(G_\mathbb{R})} \left|\mathrm{Out}_M(\mathfrak{e}_\mathfrak{p}(M))\right|^{-1} \cdot \sum_{\gamma'} Q(\mathfrak{e}_\mathfrak{p}(M),\gamma') SO_{\gamma'}(k_1(\emptyset)) L_M(j_M(\gamma')),$$

where γ' runs through the elements of $\Sigma(M')_1$ that are good at ∞ and good at p. Here we understand that $k_1(\emptyset) := k(\emptyset,\emptyset)$ when $M = M_1$ or M_2; see §8.11.2 for k_1 and k. Moreover, $(-1)^{\dim A_{M'}}$ depends only on M, and is 1 if $M = M_{12}$ and -1 other wise.[10]

Proof. — The claim about $(-1)^{\dim A_{M'}}$ is straightforward. To prove (8.14.1.1), we first treat the odd case with $M = M_{12}$. By Lemma 8.10.2, we have $n^G_M = 8$. Then by Proposition 8.12.2 and (8.11.3.2), we have

(8.14.1.2)

$$\begin{aligned} 8 \operatorname{Tr}'_M &= \sum_{\mathfrak{e}=(M',{}^LM',s'_M,\eta_M)\in\dot{\mathscr{E}}(M)^{c,\mathrm{ur}}} \left|\mathrm{Out}_M(\mathfrak{e})\right|^{-1} \sum_{\gamma'} Q(\mathfrak{e},\gamma') 4p^* J(\mathfrak{e},\gamma',k_1) \\ &= \sum_{\mathfrak{e}} \left|\mathrm{Out}_M(\mathfrak{e})\right|^{-1} \sum_{\gamma'} Q(\mathfrak{e},\gamma') 8p^* SO_{\gamma'}(k_1(\emptyset)) \cdot \\ &\cdot \left[\Phi^G_M(j_M(\gamma')^{-1},\Theta_{\mathbb{V}^*}) + \epsilon_R(j_M(\gamma'^{-1}))\epsilon_{R_H}(\gamma'^{-1})|_{A=\{1\}} \Phi^G_M(j_M(\gamma'^{-1}),\Theta_{\mathbb{V}^*})_{\mathrm{eds}}\right], \end{aligned}$$

where γ' runs through the elements in $\Sigma(M')_1$ that are good at ∞ and good at p. Suppose that γ' contributes non-trivially to the above sum. Then $Q(\mathfrak{e},\gamma') \neq 0$. From Definition 8.9.7, we have

$$O^{s'_M}_{\gamma_M}(f^{p,\infty}_M) \neq 0,$$

where γ_M is as in that definition. Therefore the component of γ_M in $M^{\mathrm{GL}}(\mathbb{A}^p_f)$ lies in a compact subset that depends only on $f^{p,\infty}$ and not on a. Because γ' is an image of γ_M, the component of γ' in $M^{\mathrm{GL}}(\mathbb{Q})$ is equal to the component of γ_M in $M^{\mathrm{GL}}(\mathbb{A}^p_f)$. When a is large enough, this observation together with the assumption that γ' is good at p implies that the real absolute value of the component of γ' in the first $\mathbb{G}_m$ is strictly smaller than the ±1-st power of that of the second. In other words, $j_M(\gamma')$ is

[10] This dichotomy is to be compared with the dichotomy of behaviors of signs in Propositions 4.6.12 and 4.6.14 for M_{12} on the one hand, and in Propositions 4.4.2 and 4.5.2 for M_1 and M_2 on the other hand.

in the range $x_1 < -|x_2|$ considered in Propositions 4.6.12 and 4.6.14. Observe

$$\Phi^G_M(j_M(\gamma')^{-1}, \Theta_{\mathbb{V}^*}) = \Phi^G_M(j_M(\gamma'), \Theta_{\mathbb{V}}),$$
$$\Phi^G_M(j_M(\gamma')^{-1}, \Theta_{\mathbb{V}^*})_{\mathrm{eds}} = \Phi^G_M(j_M(\gamma'), \Theta_{\mathbb{V}})_{\mathrm{eds}},$$

and

$$\epsilon_R(j_M(\gamma'^{-1}))\epsilon_{R_H}(\gamma'^{-1})|_{A=\{1\}} = 1$$

for $j_M(\gamma'^{-1})$ in the range mentioned above. Therefore by Proposition 4.6.12, the sum in the bracket in (8.14.1.2) is $4(-1)^{q(G_{\mathbb{R}})}L_M(j_M(\gamma'))$. Substituting this into (8.14.1.2), dividing both sides by 8, and inserting the sign $(-1)^{\dim A_{M'}} = 1$ on the right hand side, we obtain the desired (8.14.1.1).

The even case M_{12}, odd and even case M_1, and odd case M_2, are proved in a similar way, by applying the corresponding computation in §8.11 and Propositions 8.13.2, 4.6.14, 4.4.2, and 4.5.2. (The number n^M_G can again be computed using Lemma 8.10.2, and is seen to be $8, 2, 2$ for M_{12}, M_1, M_2.) We only add the following details: When $M = M_1$, we only know that the component of γ' in $M^{\mathrm{GL}}(\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Q})$ is (stably) conjugate to the component of γ_M in $M^{\mathrm{GL}}(\mathbb{A}^p_f) = \mathrm{GL}_2(\mathbb{A}^p_f)$ (as opposed to knowing that they are equal), but this already implies that they have equal determinant. Again from the assumption that a is large and γ' is good at p, we deduce that $j_M(\gamma')$ is in the range $\det < 1$ considered in Proposition 4.4.2. When $M = M_2$ in the odd case, we deduce that $j_M(\gamma')$ is in the range $0 < a < 1$ considered in Proposition 4.5.2 in the same way as when $M = M_{12}$. Finally, we note that the constant n^G_M appearing in Proposition 8.13.2 is the same for M_1 and M_2 (equal to 2), but in Propositions 4.4.2 there is an extra factor 2 on the right hand side compared to Proposition 4.5.2. This is why in the current proposition we have $\mathscr{C}_{M_1} = 1$ and $\mathscr{C}_{M_2} = 2$. □

8.14.2. — We now plug the definition of $Q(\mathfrak{e}, \gamma')$ (Definition 8.9.7) into the formula (8.14.1.1), and obtain:

$$\mathscr{C}_M \operatorname{Tr}'_M = 4p^*\tau(M)k(M)k(G)^{-1} \sum_{\mathfrak{e}=(M', {}^L M', s'_M, \eta_M)\in\dot{\mathscr{E}}(M)^{c,\mathrm{ur}}} |\mathrm{Out}_M(\mathfrak{e})|^{-1}$$
$$\cdot \sum_{\gamma'} SO_{\gamma'}(k_1(\emptyset)) L_M(j_M(\gamma'))\bar{\iota}^{M'}(\gamma')^{-1}(\Delta^M_{M'})^{\emptyset,\emptyset}(\gamma', \gamma_M)$$
$$\cdot O^{s'_M}_{\gamma_M}(f^{p,\infty}_M)\bar{v}(M'^0_{\gamma'})^{-1}\Delta^{\emptyset,\emptyset}_{j_M, B_M}(\gamma', j_M(\gamma')). \tag{8.14.2.1}$$

(The sign $(-1)^{\dim A_{M'}}$ appears both in the definition of $Q(\mathfrak{e}, \gamma')$ and in (8.14.1.1), and hence it gets canceled in the above.) Observe that when $\gamma' \in \Sigma(M')_1$ is good at p, we have

$$SO_{\gamma'}(k_1(\emptyset)) = SO_{\gamma'}(k_a \otimes 1_{M'^{,\mathrm{SO}},p}), \tag{8.14.2.2}$$

where, in the notation of Proposition 7.4.2, $k_a \in \mathcal{H}^{\mathrm{ur}}(M^{\mathrm{GL}}_{\mathbb{Q}_p})$ is given by

$$\text{(8.14.2.3)} \qquad \begin{cases} -\xi_1^{-a}, & M = M_{12} \text{ or } M_2 \text{ (so that } M^{\mathrm{GL}} = \mathbb{G}_m^2 \text{ or } \mathbb{G}_m \text{ resp.)}, \\ -\zeta_1^{-a} - \zeta_2^{-a}, & M = M_1 \text{ (so that } M^{\mathrm{GL}} = \mathrm{GL}_2), \end{cases}$$

and $1_{M',\mathrm{SO},p}$ denotes the unit element of $\mathcal{H}^{\mathrm{ur}}(M'^{,\mathrm{SO}}_{\mathbb{Q}_p})$. (Thus k_a differs from $k(\emptyset,\emptyset)$ in that we throw away the *positive* powers of the variables ξ_i, ζ_i, as well as powers of ξ_2 when $M = M_{12}$.) Conversely, if the right hand side of (8.14.2.2) is non-zero, then γ' is necessarily good at p. Thus after making the substitution (8.14.2.2) inside (8.14.2.1), we no longer need to impose the condition of being good at p in the summation over γ'.

Let $\gamma'_{p,\mathrm{GL}}$ (resp. $\gamma'_{p,\mathrm{SO}}$) be the component of γ' in $M^{\mathrm{GL}}(\mathbb{Q}_p)$ (resp. $M'^{,\mathrm{SO}}(\mathbb{Q}_p)$). Then we can rewrite (8.14.2.2) as

$$\text{(8.14.2.4)} \qquad SO_{\gamma'}(k_1(\emptyset)) = SO_{\gamma'_{p,\mathrm{GL}}}(k_a) SO_{\gamma'_{p,\mathrm{SO}}}(1_{M',\mathrm{SO},p}).$$

Since γ' is (M, M')-regular (being in $\Sigma(M')_1$), $\gamma'_{p,\mathrm{SO}}$ is $(M^{\mathrm{SO}}, M'^{,\mathrm{SO}})$-regular. By the Fundamental Lemma (Theorem 8.1.4 (2)), we know that

$$SO_{\gamma'_{p,\mathrm{SO}}}(1_{M',\mathrm{SO},p}) \neq 0$$

only if $\gamma'_{p,\mathrm{SO}}$ is an image of a semi-simple element $\gamma_{p,\mathrm{SO}} \in M^{\mathrm{SO}}(\mathbb{Q}_p)$, and in this case we have

$$\text{(8.14.2.5)} \qquad SO_{\gamma'_{p,\mathrm{SO}}}(1_{M',\mathrm{SO}}) = \Delta^{M^{\mathrm{SO}}}_{M',\mathrm{SO}}(\gamma'_{p,\mathrm{SO}}, \gamma_{p,\mathrm{SO}}) O^{s^{\mathrm{SO}}}_{\gamma_{p,\mathrm{SO}}}(1_{M^{\mathrm{SO}},p}),$$

where $\Delta^{M^{\mathrm{SO}}}_{M',\mathrm{SO}}$ is the canonical unramified normalization of transfer factors at p associated to the hyperspecial subgroup $M^{\mathrm{SO}}(\mathbb{Q}_p) \cap \mathcal{M}(\mathbb{Z}_p) \subset M^{\mathrm{SO}}(\mathbb{Q}_p)$, and $1_{M^{\mathrm{SO}},p}$ denotes the unit element of $\mathcal{H}(M^{\mathrm{SO}}(\mathbb{Q}_p) /\!\!/ (M^{\mathrm{SO}}(\mathbb{Q}_p) \cap \mathcal{M}(\mathbb{Z}_p)))$.

When $\gamma'_{p,\mathrm{SO}}$ is an image of $\gamma_{p,\mathrm{SO}} \in M^{\mathrm{SO}}(\mathbb{Q}_p)$ as above, note that $\gamma' = \gamma'_{p,\mathrm{GL}}\gamma'_{p,\mathrm{SO}}$ is an image of $\gamma'_{p,\mathrm{GL}}\gamma_{p,\mathrm{SO}} \in M(\mathbb{Q}_p)$, and for the canonical unramified normalizations of transfer factors we have

$$\text{(8.14.2.6)} \qquad \Delta^{M^{\mathrm{SO}}}_{M',\mathrm{SO}}(\gamma'_{p,\mathrm{SO}}, \gamma_{p,\mathrm{SO}}) = \Delta^{M}_{M'}(\gamma', \gamma'_{p,\mathrm{GL}}\gamma_{p,\mathrm{SO}}).$$

From (8.14.2.1) (8.14.2.4) (8.14.2.5) (8.14.2.6), we obtain

$$\text{(8.14.2.7)} \quad \mathscr{C}_M \operatorname{Tr}'_M = 4p^* \tau(M) k(M) k(G)^{-1} \sum_{\mathfrak{e} = (M', {}^L M', s'_M, \eta_M) \in \dot{\mathscr{E}}(M)^{c,\mathrm{ur}}} |\mathrm{Out}_M(\mathfrak{e})|^{-1}$$
$$\cdot \sum_{\gamma'} \bar{\iota}^{M'}(\gamma')^{-1} \bar{v}(M'^{0}_{\gamma'})^{-1} SO_{\gamma'_{p,\mathrm{GL}}}(k_a) L_M(j_M(\gamma')) O^{s'_M}_{\gamma_M}(f^{p,\infty}_M) O^{s^{\mathrm{SO}}}_{\gamma_{p,\mathrm{SO}}}(1_{M^{\mathrm{SO}},p})$$
$$\cdot (\Delta^{M}_{M'})^{\emptyset,\emptyset}(\gamma', \gamma_M) \Delta^{\emptyset,\emptyset}_{j_M, B_M}(\gamma', j_M(\gamma')) \Delta^{M}_{M'}(\gamma', \gamma'_{p,\mathrm{GL}}\gamma_{p,\mathrm{SO}}),$$

where γ' runs through the elements of $\Sigma(M')_1$ that are good at ∞, and for each γ' we choose $\gamma_M \in M(\mathbb{A}^p_f)$ and $\gamma_{p,\mathrm{SO}} \in M^{\mathrm{SO}}(\mathbb{Q}_p)$ such that γ' is an image of γ_M (over

$\mathbb{A}_f^p$) and an image of $\gamma'_{p,\mathrm{GL}}\gamma_{p,\mathrm{SO}}$ (over $\mathbb{Q}_p$). Here we no longer need the condition that γ' is good at p, as we have already seen.

Lemma 8.14.3. — *If $\gamma' \in M'(\mathbb{Q})_{\mathrm{ss}}$ is $\mathbb{R}$-elliptic and is an image from $M(\mathbb{A}_f)_{\mathrm{ss}}$, then it is an image from $M(\mathbb{Q})_{\mathrm{ss}}$. Moreover, if $\gamma_\infty \in M(\mathbb{R})_{\mathrm{ss}}$ is a prescribed elliptic element of which γ' is an image, then γ' is an image of some $\gamma \in M(\mathbb{Q})_{\mathrm{ss}}$ such that γ is conjugate to γ_∞ in $M(\mathbb{R})$.*

Proof. — We recall a construction from [**Lab99**] in our setting. Let $\gamma^* \in M^*(\mathbb{Q})_{\mathrm{ss}}$ be such that γ' is an image of it. By hypothesis γ' is an image from $M(\mathbb{A}_f)_{\mathrm{ss}}$, and note that γ' is also an image from $M(\mathbb{R})_{\mathrm{ss}}$ since it is $\mathbb{R}$-elliptic. Thus let $\gamma_{\mathbb{A}} \in M(\mathbb{A})_{\mathrm{ss}}$ be such that γ' is an image of it. When γ_∞ is prescribed as in the statement of the lemma, we take $\gamma_{\mathbb{A}}$ such that its archimedean component is γ_∞. From γ^* and $\gamma_{\mathbb{A}}$, Labesse constructs a non-empty subset

$$\mathrm{obs}_{\gamma^*}(\gamma_{\mathbb{A}}) \subset \mathfrak{E}(I^*, M^*; \mathbb{A}/\mathbb{Q}) := \mathbf{H}^0_{\mathrm{ab}}(\mathbb{A}/\mathbb{Q}, I^*\backslash M^*)/\,\mathbf{H}^0_{\mathrm{ab}}(\mathbb{A}, M^*),$$

generalizing the construction of Kottwitz in [**Kot86**]; see [**Lab99**, §2.6], with $L = M, H = M^*$. By [**Lab99**, Thm. 2.6.3], the condition that $1 \in \mathrm{obs}_{\gamma^*}(\gamma_{\mathbb{A}})$ would imply the existence of an element of $M(\mathbb{Q})_{\mathrm{ss}}$ that is conjugate to $\gamma_{\mathbb{A}} \in M(\mathbb{A})$, and the current lemma would follow. Thus it suffices to prove that $1 \in \mathrm{obs}_{\gamma^*}(\gamma_{\mathbb{A}})$ for a suitable choice of $\gamma_{\mathbb{A}}$.

Note that to prove the lemma we may always modify $\gamma_{\mathbb{A}}$ by replacing its v-adic component with another element stably conjugate to it over $\mathbb{Q}_v$, for some finite place v. We claim that after such a modification we can achieve $1 \in \mathrm{obs}_{\gamma^*}(\gamma_{\mathbb{A}})$. In fact, we know that $\mathfrak{E}(I^*, M^*; \mathbb{A}/\mathbb{Q})$ is isomorphic to the Pontryagin dual group $\mathfrak{K}(I^*/\mathbb{Q})^D$ of the finite abelian group $\mathfrak{K}(I^*/\mathbb{Q})$ (for $I^* \subset M^*$) considered in [**Kot86**, §4.6]; cf. [**KSZ**, Cor. 1.7.4]. The same argument as the second paragraph of [**Kot90**, p. 188] implies that the natural map $\mathfrak{K}(I^*/\mathbb{Q}_v)^D \to \mathfrak{K}(I^*/\mathbb{Q})^D$ is a surjection for some finite place v. On the other hand,

$$\mathfrak{K}(I^*/\mathbb{Q}_v)^D \cong \mathfrak{E}(I^*, M^*; \mathbb{Q}_v) \cong \mathfrak{D}(I^*, M^*; \mathbb{Q}_v) = \ker(\mathbf{H}^1(\mathbb{Q}_v, I^*) \to \mathbf{H}^1(\mathbb{Q}_v, M^*)).$$

From the construction of Labesse we know that if we twist $\gamma_{\mathbb{A}}$ within its stable conjugacy class by a class $c \in \mathfrak{D}(I^*, M^*; \mathbb{Q}_v)$, then $\mathrm{obs}_{\gamma^*}(\gamma_{\mathbb{A}})$ gets shifted by the image of c in the abelian group $\mathfrak{E}(I^*, M^*, \mathbb{A}/\mathbb{Q})$. The claim follows. □

8.14.4. — By Lemma 8.14.3, we may assume that each γ' in (8.14.2.7) is an image of some $\gamma \in M(\mathbb{Q})_{\mathrm{ss}}$, and that γ is conjugate to $j_M(\gamma')$ in $M(\mathbb{R})$. Note that we have $L_M(j_M(\gamma')) = L_M(\gamma)$. (In fact $L_M(\cdot)$ depends only on $\mathbb{C}$-conjugacy classes.) We may and shall take γ_M and $\gamma'_{p,\mathrm{GL}}\gamma_{p,\mathrm{SO}}$ to be localizations of γ in $M(\mathbb{A}_f^p)$ and $M(\mathbb{Q}_p)$ respectively.

We have seen that ♫$(\emptyset, \emptyset) = -1$ in Proposition 8.9.5. Therefore with the above assumptions on γ_M and $\gamma'_{p,\mathrm{GL}}\gamma_{p,\mathrm{SO}}$, the product of the three transfer factors in the third line of (8.14.2.7) becomes -1. We summarize the above discussion in the following proposition.

Proposition 8.14.5. — *When $a \in \mathbb{Z}_{>0}$ is large enough (for a fixed $f^{p,\infty}$), we have*

(8.14.5.1)

$$\mathscr{C}_M \operatorname{Tr}'_M = -4p^*\tau(M)k(M)k(G)^{-1} \sum_{\mathfrak{e}=(M',{}^LM',s'_M,\eta_M)\in\dot{\mathscr{E}}(M)^{\mathrm{ur},c}} |\mathrm{Out}_M(\mathfrak{e})|^{-1}$$
$$\cdot \sum_{\gamma'} \bar{\iota}^M(\gamma)^{-1}\bar{v}(M^0_\gamma)^{-1} SO_{\gamma_{\mathrm{GL}}}(k_a) L_M(\gamma) O^{s'_M}_{\gamma}(f^{p,\infty}_M) O^{s^{\mathrm{SO}}}_{\gamma_{\mathrm{SO}}}(1_{M^{\mathrm{SO}},p})$$
$$\cdot \Delta^{\emptyset,\emptyset}_{j_M,B_M}(\gamma', j_M(\gamma'))\Delta^{\emptyset,\emptyset}_{j_M,B_M}(\gamma',\gamma)^{-1},$$

where γ' runs through the elements of $\Sigma(M')_1$ that are good at ∞, and such that γ' is an image of some $\gamma \in M(\mathbb{Q})_{\mathrm{ss}}$. For each γ', we fix a corresponding γ, and use γ_{GL} and γ_{SO} to denote the (localizations over $\mathbb{Q}_p$ of) the components of γ in M^{GL} and M^{SO} respectively. □

Definition 8.14.6. — For any reductive group I over $\mathbb{R}$ that contains elliptic maximal tori, let $\mathscr{D}(I)$ be the cardinality of $\mathfrak{D}(T, I; \mathbb{R}) = \ker(\mathbf{H}^1(\mathbb{R}, T) \to \mathbf{H}^1(\mathbb{R}, I))$, where T is any elliptic maximal torus in I.

Lemma 8.14.7. — *Let I and T be as in Definition 8.14.6.*

(1) *We have $\mathscr{D}(I) = |\Omega_{\mathbb{C}}(I, T)/\Omega_{\mathbb{R}}(I, T)|$. In particular $\mathscr{D}(I)$ is independent of the choice of T.*

(2) *If $\mathscr{D}(I) = 1$, then any two elliptic elements of $I(\mathbb{R})$ that are stably conjugate to each other are conjugate under $I(\mathbb{R})$.*

(3) *If $\mathscr{D}(I) = 1$, then for any elliptic element $x \in I(\mathbb{R})$, we have $\mathscr{D}(I^0_x) = 1$.*

Proof. — Statement (1) follows from [**Lab11**, Prop. 6.4.2], and the fact that all elliptic maximal tori are conjugate under $I(\mathbb{R})$. For (2), it suffices to prove that for any (connected) reductive subgroup J of I containing an elliptic maximal torus T in I, we have $\mathfrak{D}(J, I; \mathbb{R}) = 1$. But this follows from [**Kot86**, Lem. 10.2], which says that $\mathbf{H}^1(\mathbb{R}, T)$ surjects onto $\mathbf{H}^1(\mathbb{R}, J)$. Finally, (3) follows from the fact that I^0_x contains a maximal torus which is elliptic in both I^0_x and I. □

Lemma 8.14.8. — *We have $\mathscr{D}(M_{\mathbb{R}}) = 1$.*

Proof. — If $M = M_1$ or M_{12}, then $M_{\mathbb{R}}$ is a product of copies of GL_2 or $\mathbb{G}_m$ and an anisotropic group, so $\mathscr{D}(M) = 1$. Now suppose $M = M_2$ in the odd case. Write n for $d-2$, and recall that $n \geq 3$. We have $M_{\mathbb{R}} \cong \mathbb{G}_m \times \mathrm{SO}(n-1, 1)$, so $\mathscr{D}(M_{\mathbb{R}}) = \mathscr{D}(\mathrm{SO}(n-1, 1))$. To compute $\mathscr{D}(\mathrm{SO}(n-1, 1))$, consider an elliptic (anisotropic) maximal torus $T \cong \mathrm{U}(1)^{(n-1)/2}$ in $\mathrm{SO}(n-1, 1)$, which is inside the maximal compact subgroup $\mathrm{S}(\mathrm{O}(n-1)\times\mathrm{O}(1))$ of $\mathrm{SO}(n-1, 1)$. It is well known (see for instance [**AT18**, Prop. 6.16]) that we have

$$|\Omega_{\mathbb{R}}(\mathrm{SO}(n-1,1),T)| = \left|\mathrm{Nor}_{\mathrm{S(O}(n-1)\times\mathrm{O}(1))(\mathbb{C})}(T(\mathbb{C}))/T(\mathbb{C})\right|.$$

On the other hand one can directly check that as subgroups of $\mathrm{Aut}(T_{\mathbb{C}})$ we have

$$\mathrm{Nor}_{\mathrm{S(O}(n-1)\times\mathrm{O}(1))(\mathbb{C})}(T(\mathbb{C}))/T(\mathbb{C}) = \Omega_{\mathbb{C}}(\mathrm{SO}(n-1,1),T) \cong \{\pm 1\}^{(n-1)/2} \rtimes \mathfrak{S}_{(n-1)/2}.$$

It then follows from Lemma 8.14.7 (1) that $\mathscr{D}(\mathrm{SO}(n-1,1)) = 1$. □

Proposition 8.14.9. — *Keep the setting and notation of Proposition 8.14.5. We have*

(8.14.9.1)

$$\mathscr{C}_M \operatorname{Tr}'_M = -4p^*\tau(M)k(M)k(G)^{-1}\sum_{\gamma_0}\sum_{\kappa}\bar{\iota}^M(\gamma_0)^{-1}\bar{v}(I_0)^{-1}SO_{\gamma_{0,\mathrm{GL}}}(k_a)L_M(\gamma_0) \cdot O^{\kappa}_{\gamma_0}(f^{p,\infty}_M)O^{\kappa^{\mathrm{SO}}}_{\gamma_{0,\mathrm{SO}}}(1_{M^{\mathrm{SO}},p}),$$

where

– γ_0 *runs through a fixed set of representatives of the stable conjugacy classes in* $M(\mathbb{Q})$ *that are elliptic over* $\mathbb{R}$ *and good at* ∞. *We let* $\gamma_{0,\mathrm{GL}}$ *and* $\gamma_{0,\mathrm{SO}}$ *denote the (localizations over* $\mathbb{Q}_p$ *of) the components of* γ_0 *in* M^{GL} *and* M^{SO} *respectively.*

– $I_0 := M^0_{\gamma_0}$.

– κ *runs through* $\mathfrak{K}(I_0/\mathbb{Q}) = \mathfrak{E}(I_0, M; \mathbb{A}/\mathbb{Q})^D$.

Proof. — By Lemma 8.14.7 (2) and Lemma 8.14.8, every γ in (8.14.5.1) is conjugate to $j_M(\gamma')$ over $\mathbb{R}$. Hence the quotient of the two transfer factors at the end of (8.14.5.1) is equal to 1. Thus we have

$$\mathscr{C}_M \operatorname{Tr}'_M = -4p^*\tau(M)k(M)k(G)^{-1}\sum_{\mathfrak{e}=(M',{}^LM',s'_M,\eta_M)\in\dot{\mathscr{E}}(M)^{\mathrm{ur},c}}|\mathrm{Out}_M(\mathfrak{e})|^{-1} \cdot \sum_{\gamma'}\bar{\iota}^M(\gamma)^{-1}\bar{v}(M^0_\gamma)^{-1}SO_{\gamma_{\mathrm{GL}}}(k_a)L_M(\gamma)O^{s'_M}_{\gamma}(f^{p,\infty}_M)O^{s^{\mathrm{SO}}}_{\gamma_{\mathrm{SO}}}(1_{M^{\mathrm{SO}},p}).$$

This implies (8.14.9.1) by the usual conversion from summation over $(\mathfrak{e},\gamma')$ to summation over (γ_0,κ) in the theory of stabilization (see [**Lab04**, Cor. IV.3.6] and [**KSZ**, §8.3]). □

8.14.10. — Now by Fourier analysis on the finite abelian groups $\mathfrak{K}(I_0/\mathbb{Q})^D = \mathfrak{E}(I_0, M; \mathbb{A}/\mathbb{Q})$, (cf. [**Kot86**, p. 395], [**Kot90**, p. 174], [**KSZ**, §8.1]), from Proposition 8.14.9 we deduce

(8.14.10.1)

$$\mathscr{C}_M \operatorname{Tr}'_M = -4p^*\tau(M)k(M)k(G)^{-1}\sum_{\gamma_0,\gamma_1}\bar{\iota}^M(\gamma_0)^{-1}\bar{v}(I_0)^{-1}e(I_{0,\mathbb{R}})SO_{\gamma_{0,\mathrm{GL}}}(k_a) \cdot L_M(\gamma_0)O_{\gamma_1}(f^{p,\infty}_M)O_{\gamma_{1,\mathrm{SO}}}(1_{M^{\mathrm{SO}},p})\left[\tau(M)^{-1}\tau(I_0)\left|\ker(\ker^1(\mathbb{Q},I_0)\to\ker^1(\mathbb{Q},M))\right|\right],$$

where

– γ_0 runs through a fixed set of representatives of the stable conjugacy classes in $M(\mathbb{Q})$ that are elliptic over $\mathbb{R}$ and good at ∞.

– $I_0 := M^0_{\gamma_0}$.

– γ_1 runs through the subset of $\mathfrak{D}(I_0, M; \mathbb{A}) := \ker(\mathbf{H}^1(\mathbb{A}, I_0) \to \mathbf{H}^1(\mathbb{A}, M))$ consisting of elements whose images in $\mathfrak{E}(I_0, M; \mathbb{A}/\mathbb{Q})$ are trivial. Each such γ_1 determines a conjugacy class in $M(\mathbb{A})$ which we also denote by γ_1. We let $\gamma_{1,\mathrm{SO}}$ be the component of γ_1 in $M^{\mathrm{SO}}(\mathbb{Q}_p)$.

– The number $\left[\tau(M)^{-1}\tau(I_0)\left|\ker(\ker^1(\mathbb{Q}, I_0) \to \ker^1(\mathbb{Q}, M))\right|\right]$ is none other than the cardinality of $\mathfrak{K}(I_0/\mathbb{Q})$. (This can be shown by combining [**Kot86**, §9] and Weil's conjecture on the Tamagawa number proved by Kottwitz [**Kot88**], cf. [**Kot90**, §4].)

8.14.11. — The last major operation to be applied to (8.14.10.1) is the Base Change Fundamental Lemma, which relates $\mathrm{SO}_{\gamma_{0,\mathrm{GL}}}(k_a)$ to the twisted orbital integrals in Kottwitz's point counting formula. We only need this result for $\mathbb{G}_m$, in which case it is trivial, and for GL_2, in which case it was initially proved by Langlands [**Lan80**]. For an account of the theory for GL_n see [**AC89**] and for the proof in the general case see [**Clo90a, Lab90**].

Observe that the function $k_a \in \mathcal{H}^{\mathrm{ur}}(M^{\mathrm{GL}}_{\mathbb{Q}_p})$ defined in (8.14.2.3) is equal to the image under the base change map (see §7.2.5)

$$\mathcal{H}^{\mathrm{ur}}(M^{\mathrm{GL}}_{\mathbb{Q}_{p^a}}) \longrightarrow \mathcal{H}^{\mathrm{ur}}(M^{\mathrm{GL}}_{\mathbb{Q}_p})$$

of the element $p^{-a/2}\phi_a^{M_h}$, resp. $-\phi_a^{M_h}$, resp. $-\phi_a^{M_h} \otimes 1$, where $\phi_a^{M_h}$ is as in Definition 2.3.9, when $M = M_1$, resp. M_2, resp. M_{12}. Here when $M = M_{12}$ we have $M^{\mathrm{GL}} = M_h \times \mathbb{G}_m$, and we write $-\phi_a^{M_h} \otimes 1$ corresponding to this decomposition, where 1 is the unit of $\mathcal{H}^{\mathrm{ur}}(\mathbb{G}_{m,\mathbb{Q}_{p^a}})$. By the Base Change Fundamental Lemma, we have, for any semi-simple conjugacy class (which is the same as stable conjugacy class) $\gamma_{0,\mathrm{GL}}$ in $M^{\mathrm{GL}}(\mathbb{Q})$, the following identity:

$$SO_{\gamma_{0,\mathrm{GL}}}(k_a) = \begin{cases} -\sum_\delta e(\delta)TO_\delta(\phi_a^{M_h}), & \text{if } M = M_2, \\ -p^{-a/2}\sum_\delta e(\delta)TO_\delta(\phi_a^{M_h}), & \text{if } M = M_1, \\ -\sum_\delta e(\delta)TO_\delta(\phi_a^{M_h})1_{\mathbb{Z}_p^\times}(y), & \text{if } M = M_{12}, \end{cases} \tag{8.14.11.1}$$

where δ runs through the σ-conjugacy classes in $M_h(\mathbb{Q}_{p^a})$ such that it has norm the M_h-component of γ_{GL}, $e(\delta)$ denotes the Kottwitz sign of the twisted centralizer of δ (a reductive group over $\mathbb{Q}_p$), and in the last case we write $\gamma_{0,\mathrm{GL}} = (x, y) \in M_h \times \mathbb{G}_m$. (In fact, by [**AC89**] or direct verification, the above summation over δ is either empty or over a singleton.)

The next lemma is sometimes called "pre-stabilization" in the literature.

Lemma 8.14.12. — *Let $F(x,y)$ be a $\mathbb{C}$-valued function on the set of compatible pairs (x,y) of a stable conjugacy class x in $M(\mathbb{Q})$ and a conjugacy class y in $M(\mathbb{A})$. Then we have*

$$\sum_{\gamma} \iota^M(\gamma)^{-1} F(\gamma,\gamma) = \sum_{\gamma_0,\gamma_1} \bar{\iota}^M(\gamma_0)^{-1} \left|\ker(\ker^1(\mathbb{Q}, I_0) \to \ker^1(\mathbb{Q}, M))\right| F(\gamma_0,\gamma_1),$$

where on the LHS γ runs through the conjugacy classes in $M(\mathbb{Q})$ which are $\mathbb{R}$-elliptic, and on the RHS γ_0 runs through an arbitrary set of representatives of the stable conjugacy classes in $M(\mathbb{Q})$ that are $\mathbb{R}$-elliptic, and γ_1 runs through the subset of $\mathfrak{D}(I_0, M; \mathbb{A})$ consisting of elements whose images in $\mathfrak{E}(I_0, M; \mathbb{A}/\mathbb{Q})$ are trivial. Here we have denoted $I_0 := M^0_{\gamma_0}$. Moreover, if we restrict the summation on the LHS to only those γ good at ∞, and restrict the summation on the RHS to only those γ_0 good at ∞, we still get an equality.

Proof. — The multiplicity of a $M(\mathbb{Q})$-conjugacy class γ appearing in the set $\mathfrak{D}(I_0, M; \mathbb{Q})$ is equal to $\bar{\iota}^M(\gamma_0) \cdot \iota^M(\gamma)^{-1}$. The fibers of the map $\mathfrak{D}(I_0, M; \mathbb{Q}) \to \mathfrak{D}(I_0, M, \mathbb{A})$ all have size

$$\left|\ker(\ker^1(\mathbb{Q}, I_0) \to \ker^1(\mathbb{Q}, M))\right|.$$

The lemma then easily follows. □

We are now ready to prove Theorem 8.5.2.

Proof of Theorem 8.5.2. — By (8.14.10.1) and Lemma 8.14.12, we have

$$(8.14.12.1) \quad \mathscr{C}_M \operatorname{Tr}'_M = -4p^* k(M) k(G)^{-1} \sum_{\gamma} \iota^M(\gamma)^{-1} \left[\bar{v}(M^0_\gamma)^{-1} e(M^0_{\gamma,\mathbb{R}}) \tau(M^0_\gamma)\right] \cdot SO_{\gamma_{\mathrm{GL}}}(k_a) L_M(\gamma) O_\gamma(f^{p,\infty}_M) O_{\gamma_{\mathrm{SO}}}(1_{M^{\mathrm{SO}},p}),$$

where γ runs through conjugacy classes in $M(\mathbb{Q})$ that are elliptic over $\mathbb{R}$ and good at ∞. By Harder's formula (see [**GKM97**, §7.10]), we have

$$\chi(M^0_\gamma) = \bar{v}(M^0_\gamma)^{-1} e(M^0_{\gamma,\mathbb{R}}) \mathscr{D}(M^0_{\gamma,\mathbb{R}}) \tau(M^0_\gamma).$$

By Lemma 8.14.7 (3) and Lemma 8.14.8, $D(M^0_{\gamma,\mathbb{R}}) = 1$. Hence the product in the bracket in (8.14.12.1) is equal to $\chi(M^0_\gamma)$, and therefore

$$(8.14.12.2) \quad \mathscr{C}_M \operatorname{Tr}'_M = -4p^* k(M) k(G)^{-1} \sum_{\gamma} \iota^M(\gamma)^{-1} \chi(M^0_\gamma) \cdot SO_{\gamma_{\mathrm{GL}}}(k_a) L_M(\gamma) O_\gamma(f^{p,\infty}_M) O_{\gamma_{\mathrm{SO}}}(1_{M^{\mathrm{SO}},p}).$$

Denote

$$p^{**} := \begin{cases} p^*, & \text{if } M = M_2 \text{ or } M_{12}, \\ p^{-a/2} p^*, & \text{if } M = M_1. \end{cases}$$

By (8.14.11.1) and (8.14.12.2) we have

(8.14.12.3) $$\mathscr{C}_M \operatorname{Tr}'_M = 4p^{**}k(M)k(G)^{-1} \cdot \sum_{\gamma,\delta} \iota^M(\gamma)^{-1}\chi(M^0_\gamma)e(\delta)TO_\delta(\phi_a^{M_h})L_M(\gamma)O_\gamma(f_M^{p,\infty})O_{\gamma_L}(1_{M_l(\mathbb{Z}_p)}),$$

where γ_L denotes the component of γ in M_l under the decomposition $M = M_h \times M_l$ (which only differs from the decomposition $M = M^{\mathrm{GL}} \times M^{\mathrm{SO}}$ when $M = M_{12}$), and $1_{M_l(\mathbb{Z}_p)}$ is as in Definition 2.4.3. To finish the proof we divide into different cases.

Case $M = M_{12}$**.**

In Definition 2.4.3, δ runs through those elements of $\mathbb{Q}_{p^a}^\times$ with norm γ_0 and such that the Kottwitz invariant of δ in $\pi_1(M_h)_{\Gamma_p} = X_*(\mathbb{G}_m) = \mathbb{Z}$ is equal to the image of $-\mu$. The last condition is equivalent to requiring that $v_p(\delta) = -1$, which is a necessary (and also sufficient) condition for $TO_\delta(\phi_a^{M_h}) \neq 0$. Hence we may drop this condition in the summation in Definition 2.4.3. Every term $c(\gamma_0, \gamma, \delta)$ is easily computed to be 2^{-1} (with $c_1 = \mathrm{vol}(\mathbb{G}_m(\mathbb{R})/\mathbb{G}_m(\mathbb{R})^0)^{-1} = 2^{-1}, c_2 = 1$). On the other hand, in (8.14.12.3) every term $e(\delta)$ is 1. Comparing Definition 2.4.3 and (8.14.12.3), we see that it suffices to prove that

(8.14.12.4) $$2^{-1}\delta^{1/2}_{P_{12}(\mathbb{Q}_p)}(\gamma_h) = 4p^*k(M)k(G)^{-1}\chi(M_{h,\gamma_h})$$

for $\gamma = \gamma_h\gamma_L$ contributing to (8.14.12.3). (Here γ_h and γ_L denote the components of γ in $M_h(\mathbb{Q})$ and $M_l(\mathbb{Q})$.) We have $\chi(M_{h,\gamma_h}) = \chi(\mathbb{G}_m) = 2^{-1}$ by Harder's formula, and we have $k(M) = 2^{m-3}, k(G) = 2^{m-1}$ by Propositions 8.2.3 and 8.2.4. Moreover, if $\gamma = \gamma_h\gamma_L$ contributes then $v_p(\gamma_h) = -a$ (because δ should exist) , and therefore in the odd case

$$\delta_{P_{12}(\mathbb{Q}_p)}(\gamma_h) = \prod_{\alpha\in\Phi^+-\Phi_M^+} |\alpha(\gamma_h)|_p = |\gamma_h|_p^{2m-1} = p^{(d-2)a} = (p^*)^2,$$

where the contributing roots are $\epsilon_1, \epsilon_1 \pm \epsilon_j, j \geq 2$. Similarly, in the even case,

$$\delta_{P_{12}(\mathbb{Q}_p)}(\gamma_h) = \prod_{\alpha\in\Phi^+-\Phi_M^+} |\alpha(\gamma_h)|_p = |\gamma_h|_p^{2m-2} = p^{(d-2)a} = (p^*)^2,$$

where the contributing roots are $\epsilon_1 \pm \epsilon_j, j \geq 2$. The equality (8.14.12.4) follows, and the proof is finished in this case.

Case $M = M_1$**.**

First we claim that if $\gamma_0 \in \mathrm{GL}_2(\mathbb{Q})$ is semi-simple and $\mathbb{R}$-elliptic, then

$$c_2(\gamma_0) = \tau(\mathrm{GL}_{2,\gamma_0}) = 1.$$

In particular, we have [(11)],

$$c(\gamma_0) = c_1(\gamma_0)c_2(\gamma_0) = \mathrm{vol}(A_{\mathrm{GL}_2}(\mathbb{R})^0\backslash\overline{\mathrm{GL}_{2,\gamma_0}}(\mathbb{R}))^{-1}.$$

[(11)] This equality also follows from the formula for c on p. 174 of [**Kot90**], the fact that $\tau(\mathrm{GL}_2) = 1$, and Lemma 2.3.5

We prove the claim. Write I_0 for GL_{2,γ_0}. If $I_0 = \mathrm{GL}_2$, then $\tau(I_0) = 1$ by Proposition 8.2.4, and $c_2(\gamma_0) = 1$ by definition. Otherwise $I_0 = T$ is a maximal torus in GL_2 that is elliptic over $\mathbb{R}$. Observe that $T = \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$ for some imaginary quadratic field F, so $\mathbf{H}^1(\mathbb{Q}, T) = 0$ by Shapiro's lemma and Hilbert 90. Hence $c_2(\gamma_0) = 1$. Now by [**Kot84b**, (5.1.1)] and Weil's conjecture on Tamagawa numbers proved in [**Kot88**], we have

$$\tau(T)c_2(\gamma_0) = \tau(T)\left|\ker^1(\mathbb{Q}, T)\right| = \tau(T)\left|\ker^1(\Gamma, \widehat{T})\right| = \left|\pi_0(\widehat{T}^{\Gamma})\right|.$$

Thus to show $\tau(T) = 1$ it suffices to show that $\widehat{T}^{\Gamma}$ is connected. We have seen in the proof of Lemma 2.3.5 that $\widehat{T}^{\Gamma_\infty} \subset Z(\widehat{\mathrm{GL}}_2)$. On the other hand $Z(\widehat{\mathrm{GL}}_2) \subset \widehat{T}^{\Gamma}$. Hence $\widehat{T}^{\Gamma} = Z(\widehat{\mathrm{GL}}_2) = \mathbb{C}^{\times}$, which is connected as desired. The claim is proved.

We continue to consider such $\gamma_0 \in \mathrm{GL}_2(\mathbb{Q})$ as in the claim, and write I_0 for GL_{2,γ_0}. By Harder's formula we have

$$\chi(I_0) = e(I_{0,\mathbb{R}})\bar{v}^{-1}(I_0)\tau(I_0)\left|\mathscr{D}(I_{0,\mathbb{R}})\right|.$$

Since $I_{0,\mathbb{R}}$ is either $\mathrm{GL}_{2,\mathbb{R}}$ or an elliptic maximal torus in $\mathrm{GL}_{2,\mathbb{R}}$, we have $e(I_{0,\mathbb{R}}) = |\mathscr{D}(I_{0,\mathbb{R}})| = 1$. Hence

$$\chi(I_0) = e(\overline{I_0})\,\mathrm{vol}(A_{\mathrm{GL}_2}(\mathbb{R})^0\backslash\overline{I_0})^{-1}\tau(I_0)$$

where $\overline{I_0}$ is the inner form over $\mathbb{R}$ of $I_{0,\mathbb{R}}$ that is anisotropic modulo center.

If $\delta \in G(\mathbb{Q}_{p^a})$ has norm stably conjugate to some $\gamma_0 \in \mathrm{GL}_2(\mathbb{Q})$ and γ_0 is good at p (i.e., its determinant has valuation $-a$), then we have $e(\delta) = e(\overline{I_0})$, where $\overline{I_0}$ is defined in terms of γ_0 as above. In fact, this follows from the existence of the (global) inner form I of I_0 as in §2.3.6, the product formula for the Kottwitz signs for I, and the observation that for any place finite $v \neq p$, $e(I_{0,\mathbb{Q}_v}) = 1$ since $I_{0,\mathbb{Q}_v}$ is either a torus or $\mathrm{GL}_{2,\mathbb{Q}_v}$.

From the discussion so far we deduce that for δ and γ_0 as in the last paragraph we have

$$c(\gamma_0) = e(\delta)\chi(I_0).$$

Moreover, if $\delta \in M_h(\mathbb{Q}_{p^a})$ is such that $TO_\delta(\phi_a^{M_h}) \neq 0$, then necessarily $v_p(\det\delta) = -1$, and it follows easily that the Kottwitz invariant of δ in $\pi_1(M_h)_{\Gamma_p} \cong \mathbb{Z}$ is equal to the image of $-\mu$. It remains to show that

$$\delta_{P_1(\mathbb{Q}_p)}(\gamma_h)^{1/2} = 4p^{**}k(M)k(G)^{-1},$$

for any $\gamma = \gamma_h\gamma_L$ contributing to (8.14.12.3). We have $k(M) = 2^{m-3}, k(G) = 2^{m-1}$ by Propositions 8.2.3 and 8.2.4. For $\gamma = \gamma_h\gamma_L$ contributing, we have $v_p(\det\gamma_h) = -a$ (because δ should exist), and therefore in the odd case

$$\delta_{P_1(\mathbb{Q}_p)}(\gamma_h) = \prod_{\alpha\in\Phi^+-\Phi^+_M} |\alpha(\gamma_h)|_p = |\det(\gamma_h)|_p^{2m-2} = p^{(d-3)a} = (p^{**})^2,$$

where the contributing roots are $\epsilon_1, \epsilon_2, \epsilon_1+\epsilon_2, \epsilon_1 \pm \epsilon_j, \epsilon_2 \pm \epsilon_j, j \geq 3$. In the even case, the contributing roots are $\epsilon_1+\epsilon_2, \epsilon_1 \pm \epsilon_j, \epsilon_2 \pm \epsilon_j, j \geq 3$, and $|\det(\gamma_h)|_p^{2m-2}$ is replaced by $|\det(\gamma_h)|_p^{2m-3}$, which is still equal to $(p^{**})^2$. The proof is finished in this case.

Case $M = M_2$ (odd case).

Similarly to the case $M = M_{12}$, we reduce the proof to proving the following equality:

$$2^{-1}\delta^{1/2}_{P_{12}(\mathbb{Q}_p)}(\gamma) = 2^{-1}4p^* k(M)k(G)^{-1}\chi(M_{h,\gamma_h}).$$

The extra factor 2^{-1} on the RHS in comparison to (8.14.12.4) appears due to the fact that $\mathscr{C}_M = 2$ for $M = M_2$. We have $\chi(M_{h,\gamma_h}) = \chi(\mathbb{G}_m) = 2^{-1}$ by Harder's formula, and $k(M) = 2^{m-2}, k(G) = 2^{m-1}$ by Propositions 8.2.3 and 8.2.4. Also as in the M_{12} case, if $\gamma = \gamma_h\gamma_L$ contributes then

$$\delta_{P_2(\mathbb{Q}_p)}(\gamma_h) = \prod_{\alpha\in\Phi^+-\Phi^+_M} |\alpha(\gamma_h)|_p = |\gamma_h|_p^{2m-1} = p^{(d-2)a} = (p^*)^2,$$

where the contributing roots are $\epsilon_1, \epsilon_1 \pm \epsilon_j, j \geq 2$. The proof is finished in this case. $\square$

At this point we have completed the proof of Theorem 8.5.2. In the next two sections we prove vanishing results that are complementary to Theorem 8.5.2.

8.15. A vanishing result, odd case

8.15.1. — Assume we are in the odd case. Consider a Levi subgroup M^* of $G^* = \mathrm{SO}(\underline{V})$ of the form considered in §5.5. Thus we fix $r, t \in \mathbb{Z}_{\geq 0}$, a non-degenerate subspace $\underline{W}$ of $\underline{V}$ of codimension $2(r+2t)$, a hyperbolic basis $\mathbb{B}_{\underline{W}^\perp}$ of $\underline{W}^\perp$, an embedding

$$\mathbb{G}_m^r \times \mathrm{GL}_2^t \xrightarrow{\sim} M^{*,\mathrm{GL}} \subset \mathrm{SO}(\underline{W}^\perp)$$

as in (5.5.2.1), and obtain M^* as $M^* = M^{*,\mathrm{GL}} \times \mathrm{SO}(\underline{W}) \subset G^*$. We write $M^{*,\mathrm{SO}}$ for $\mathrm{SO}(\underline{W})$. As in §5.5.6 and Proposition 5.5.7, isomorphism classes in $\mathscr{E}_{G^*}(M^*)$ have explicit representatives $\mathfrak{e}_{A,B,\mathfrak{p}}$ for parameters $(A, B, \mathfrak{p}) \in \mathscr{P}_{r,t} \times' \mathscr{P}_{\underline{W}}$. In complete analogy with §8.5.1, we fix $\dot{\mathscr{E}}_{G^*}(M^*)$ to be a subset of these $\mathfrak{e}_{A,B,\mathfrak{p}} = (M', {}^LM', s_{M^*}, \eta_{M^*})$ such that the component of s_{M^*} in $\widehat{M^{*,\mathrm{SO}}}$ is not -1 and such that each isomorphism class in $\mathscr{E}_{G^*}(M^*)$ is represented exactly once. For each $\mathfrak{e}_{A,B,\mathfrak{p}} = \mathfrak{e}_{A,B,d^+,\delta^+,d^-,\delta^-} = (M', {}^LM', s_{M^*}, \eta_{M^*}) \in \dot{\mathscr{E}}_{G^*}(M^*)$, we let

$$(H, {}^LH, s, \eta) := \mathfrak{e}_{d^+ +2|A|+4|B|,\delta^+,d^- +2|A^c|+4|B^c|,\delta^-}$$

which is the induced elliptic endoscopic datum for G^* as in Proposition 5.5.7. We also view $(H, {}^LH, s, \eta)$ as an elliptic endoscopic datum for G. Since H^+ is non-trivial by our assumption on s_{M^*}, the function f^H is defined as in §8.4. Moreover, as in §8.4, we have the fixed pair $(j : T_H \to T_G, B_{G,H})$, and a normalization for transfer factors between H and G at all finite places. We fix $M' \hookrightarrow H$ as in §5.5.9 so as to view M' as a Levi subgroup of H, and define $ST^H_{M'}(f^H)$ as in Definition 8.3.3. In analogy with

(8.5.1.1), we define

(8.15.1.1)
$$\mathrm{Tr}'_{M^*} := (n^{G^*}_{M^*})^{-1} \sum_{\substack{\mathfrak{e}=(M',{}^LM',s_{M^*},\eta_{M^*})\\ \in\dot{\mathscr{E}}_{G^*}(M^*)}} |\mathrm{Out}_{G^*}(\mathfrak{e})|^{-1}\,\tau(G)\tau(H)^{-1}ST^H_{M'}(f^H).$$

Theorem 8.15.2. — *Assume that M^* does not transfer to G. Then* $\mathrm{Tr}'_{M^*}=0$.

Proof. — By hypothesis at least one of the following conditions holds:

$$rt>0 \qquad \text{or} \qquad r\geq 3 \qquad \text{or} \qquad t\geq 2.$$

Let $\mathscr{E}(M^*)^{c,\mathrm{ur}}$ be the subset of $\mathscr{E}(M^*)$ consisting of isomorphism classes of endoscopic data whose groups are cuspidal over $\mathbb{Q}$ (which is automatic in the odd case) and unramified over $\mathbb{Q}_p$. Define a set $\dot{\mathscr{E}}(M^*)^{c,\mathrm{ur}}$ of representatives of $\mathscr{E}(M^*)^{c,\mathrm{ur}}$ in exactly the same way as in §8.6.1. Thus $\dot{\mathscr{E}}(M^*)^{c,\mathrm{ur}}$ consists of $\mathfrak{e}_{\mathfrak{p}}(M^*)$ for certain $\mathfrak{p}=(d^+,\delta^+,d^-,\delta^-)\in\mathscr{P}_{\underline{W}}$, which all satisfy that $d^+\geq 2$. Then the same arguments as in §§8.6–8.7 yield a decomposition of Tr'_{M^*} into a sum as follows. The indexing set for the sum is the set of pairs $(\mathfrak{e},\gamma')$, where $\mathfrak{e}=(M',{}^LM',s'_{M^*},\eta_{M^*})$ runs through $\dot{\mathscr{E}}(M^*)^{c,\mathrm{ur}}$, and for each fixed $\mathfrak{e}$, γ' runs through a set of representatives in $M'(\mathbb{Q})$ of the semi-simple $\mathbb{R}$-elliptic (M^*,M')-regular stable conjugacy classes. For each $(\mathfrak{e},\gamma')$, the summand is a complex number times

(8.15.2.1)
$$\sum_{A,B} SO_{\gamma'}((f^{H,p,\infty})_{M'})SO_{\gamma'}(f^H_{p,M'}) \sum_{\varphi_H\in\Phi_H(\varphi_{\mathbb{V}^*})} \det(\omega_*(\varphi_H))\Phi^H_{M'}(\gamma'^{-1},\Theta_{\varphi_H})$$

where:

– The first summation is over all subsets A of $[r]$ (recall that this is our short-hand notation for $\{1,2,\cdots,r\}$) and all subsets B of $[t]$.

– For each (A,B), we define $(H,{}^LH,s,\eta)$ with respect to $\mathfrak{e}$ and (A,B), and view M' as a Levi subgroup of H, as explained in §8.15.1.

We now fix (A,B) and analyze the terms $SO_{\gamma'}((f^{H,p,\infty})_{M'})$ and $SO_{\gamma'}(f^H_{p,M'})$. If there is one finite place $v\neq p$ such that $M^*_{\mathbb{Q}_v}$ does not transfer to $G_{\mathbb{Q}_v}$, then $SO_{\gamma'}((f^{H,p,\infty})_{M'})=0$ by the proof of [**Mor10b**, Lem. 6.3.5 (ii)]. In this case (8.15.2.1) is zero for all $(\mathfrak{e},\gamma')$, and the theorem is already proved. Thus we assume that $M^*_{\mathbb{Q}_v}$ transfers to a Levi subgroup M_v of $G_{\mathbb{Q}_v}$ at each finite place $v\neq p$. In this case, the localization at v of $\mathfrak{e}$ can be viewed as an endoscopic datum for M_v, and there is a normalization $(\Delta^{M_v}_{M'})^{A,B}_v$ of transfer factors between M' and M_v inherited from the normalization $(\Delta^G_H)_v$ of transfer factors between H and G at v fixed in §8.4.7. For almost all v, $(\Delta^{M_v}_{M'})^{A,B}_v$ is the canonical unramified normalization (associated to the hyperspecial subgroup of $M_v(\mathbb{Q}_v)$ determined by the hyperspecial subgroup of $G(\mathbb{Q}_v)$ determined by some reductive model of G over some Zariski open in $\operatorname{Spec}\mathbb{Z}$), and is hence independent of (A,B). Define

$$\epsilon^{p,\infty}(A,B) := \prod_{v\neq p,\infty} \frac{(\Delta_{M'}^{M_v})_v^{A,B}}{(\Delta_{M'}^{M_v})_v^{\emptyset,\emptyset}},$$

which is a finite product. Then as an analogue of Proposition 8.4.14, $SO_{\gamma'}((f^{H,p,\infty})_{M'})$ is equal to $\epsilon^{p,\infty}(A,B)$ times a number independent of (A,B).

By Proposition 7.4.2, we know that $SO_{\gamma'}(f^H_{p,M'})$ is a linear combination of $\nabla_i(A)$, $\nabla_j(B)$, and 1 (where $i\in[r]$ and $j\in[t]$) with coefficients independent of (A,B). We conclude that (8.15.2.1) is a linear combination of the following $r+t+1$ expressions:

$$R_i := \sum_{A,B} \nabla_i(A)\epsilon^{p,\infty}(A,B) \sum_{\varphi_H\in\Phi_H(\varphi_{\mathbb{V}^*})} \det(\omega_*(\varphi_H))\Phi^H_{M'}(\gamma'^{-1},\Theta_{\varphi_H}), \qquad 1\le i\le r,$$

$$T_j := \sum_{A,B} \nabla_j(B)\epsilon^{p,\infty}(A,B) \sum_{\varphi_H\in\Phi_H(\varphi_{\mathbb{V}^*})} \det(\omega_*(\varphi_H))\Phi^H_{M'}(\gamma'^{-1},\Theta_{\varphi_H}), \qquad 1\le j\le t,$$

$$S := \sum_{A,B} \epsilon^{p,\infty}(A,B) \sum_{\varphi_H\in\Phi_H(\varphi_{\mathbb{V}^*})} \det(\omega_*(\varphi_H))\Phi^H_{M'}(\gamma'^{-1},\Theta_{\varphi_H}).$$

We shall show that these $r+t+1$ expressions are all zero, which will prove the theorem.

We first seek to compute the term $\sum_{\varphi_H\in\Phi_H(\varphi_{\mathbb{V}^*})}\det(\omega_*(\varphi_H))\Phi^H_{M'}(\gamma'^{-1},\Theta_{\varphi_H})$ for each fixed (A,B), in a way similar to §8.8. Fix an elliptic maximal torus $T_{M'}$ of $M'_{\mathbb{R}}$ such that $\gamma'\in T_{M'}(\mathbb{R})$. As usual we have $M' = M^{*,\mathrm{GL}}\times M'^{,\mathrm{SO}}$, so necessarily $T_{M'}$ is a direct product of (1) the direct factor $\mathbb{G}_m^r$ of $M^{*,\mathrm{GL}}$, (2) an elliptic maximal torus in the direct factor GL_2^t of $M^{*,\mathrm{GL}}$, and (3) an elliptic (anisotropic) maximal torus $T_{M',\mathrm{SO}} = T_{M',\mathrm{SO},+}\times T_{M',\mathrm{SO},-}$ in $M'^{,\mathrm{SO}} = M'^{,\mathrm{SO},+}\times M'^{,\mathrm{SO},-}$. We denote the product of (1) and (2) by $T_{M^*,\mathrm{GL}}$. Note that all of R_i, T_j, S can be viewed as continuous functions in γ' varying in $T_{M'}(\mathbb{R})$ (cf. §4.2.1). Hence we may and shall assume the following condition:

(†) The r components of γ' in $\mathbb{G}_m^r\subset M^{*,\mathrm{GL}}$ are distinct from each other and distinct from the inverse of each other.

Let r' be the number such that exactly r' among the r components of γ' in $\mathbb{G}_m^r$ are positive.

Fix an elliptic maximal torus T_{M^*} in $M^*_{\mathbb{R}}$ of the form $T_{M^*,\mathrm{GL}}\times T_{M^*,\mathrm{SO}}$, where $T_{M^*,\mathrm{GL}}$ is as above and $T_{M^*,\mathrm{SO}}$ is an elliptic (anisotropic) maximal torus in $M^{*,\mathrm{SO}}$. Fix an admissible isomorphism $j_{M^*}: T_{M'}\xrightarrow{\sim} T_{M^*}$ of the form $\mathrm{id}_{T_{M^*,\mathrm{GL}}}\times j_{M^*,\mathrm{SO}}$, where $j_{M^*,\mathrm{SO}}$ is an admissible isomorphism $T_{M',\mathrm{SO}}\xrightarrow{\sim} T_{M^*,\mathrm{SO}}$. As in §8.8.1, for any choice of Borel subgroup B_0 of $G^*_{\mathbb{C}}$ containing $T_{M^*,\mathbb{C}}$, we obtain m cocharacters of $T_{M^*,\mathbb{C}}$ forming a basis of $X_*(T_{M^*})$. We denote them by

$$\tau_{0_1},\cdots,\tau_{0_{r+2t}},\tau_1,\cdots,\tau_{m-r-2t}.$$

Since we are in the odd case, by making different choices of B_0 we can arbitrarily permute the τ's and replace an arbitrary number of them by their inverses. By similar arguments as in §8.8.1, we can choose B_0 such that the following conditions are satisfied. (Here condition **C** depends on the assumption (†) above.)

A : For each $1 \leq i \leq r$, τ_{0_i} is a cocharacter of the direct factor $\mathbb{G}_m^r$ of $M^{*,\mathrm{GL}}$. Moreover, there is a permutation $\delta \in \mathfrak{S}_r$ such that for each $1 \leq i \leq r$, τ_{0_i} is either the identity cocharacter or the inverse of the identity cocharacter of the $\delta(i)$-th copy of $\mathbb{G}_m$.

B : For each $1 \leq j \leq t$, $\tau_{0_{r+2j-1}}$ and $\tau_{0_{r+2j}}$ are cocharacters of the j-th copy of GL_2 in $M^{*,\mathrm{GL}}$. Moreover, these two are simultaneously GL_2-conjugate to the following cocharacters of GL_2:

$$z \longmapsto \begin{pmatrix} z & \\ & 1 \end{pmatrix} \qquad \text{and} \qquad z \longmapsto \begin{pmatrix} 1 & \\ & z \end{pmatrix}.$$

C : Let $\{\epsilon_1, \cdots, \epsilon_r\}$ be the basis of $X^*(\mathbb{G}_m^r)$ dual to the basis $\{\tau_{0_1}, \cdots, \tau_{0_r}\}$ of $X_*(\mathbb{G}_m^r)$. We also view each ϵ_i as a character on T_{M^*}, via the projection from T_{M^*} to the direct factor $\mathbb{G}_m^r$ of $T_{M^{*,\mathrm{GL}}}$. For each $1 \leq i \leq r$, we require that

(8.15.2.2) $$\epsilon_i(\gamma') > 0 \text{ if and only if } i \leq r'.$$

For all $1 \leq i < j \leq r'$, or $r'+1 \leq i < j \leq r$, we require that

(8.15.2.3) $$\frac{\epsilon_i(\gamma'^{-1})}{\epsilon_j(\gamma'^{-1})} \in]0,1[,$$

and

(8.15.2.4) $$\left|\epsilon_i(\gamma'^{-1})\right| < 1.$$

D : Let n^- be the dimension of $T_{M',\mathrm{SO},-,\mathbb{C}}$. For each $1 \leq i \leq n^-$, $j_{M^*}^{-1} \circ \tau_i$ is a cocharacter of $T_{M',\mathrm{SO},-,\mathbb{C}}$.

The pair $(j : T_H \xrightarrow{\sim} T_G, B_{G,H})$ fixed in §8.4 can be transferred to a pair $(\underline{j} : T_H \xrightarrow{\sim} T_{G^*}, B_{G^*,H})$ as follows. We fix an anisotropic maximal torus T_{G^*} in $G^*_{\mathbb{R}}$ and an isomorphism $\nu : T_G \xrightarrow{\sim} T_{G^*}$ coming from any inner twisting $G_{\mathbb{C}} \xrightarrow{\sim} G^*_{\mathbb{C}}$ in the canonical $G^*(\mathbb{C})$-conjugacy class of such inner twistings. Then we define $\underline{j} := \nu \circ j$, and define $B_{G^*,H}$ to be the Borel subgroup of $G^*_{\mathbb{C}}$ containing T_{G^*} such that ν relates all $B_{G,H}$-positive roots on $T_{G,\mathbb{C}}$ with $B_{G^*,H}$-positive roots on $T_{G^*,\mathbb{C}}$. From $(\underline{j}, B_{G^*,H})$, we obtain an ordered m-tuple of cocharacters of $T_{G^*,\mathbb{C}}$

$$\underline{\rho}_1, \cdots, \underline{\rho}_m$$

similarly as in §8.8.2. Define an isomorphism $i_{G^*}(A,B) : T_{M^*,\mathbb{C}} \xrightarrow{\sim} T_{G^*,\mathbb{C}}$ by the following rule. Write $m^{\pm}$ for the absolute ranks of $H^{\pm}$, and $n^{\pm}$ for the absolute ranks of $M^{\prime,\mathrm{SO},\pm}$. Thus we have

$$m^+ = n^+ + |A| + 2\,|B|\,,$$
$$m^- = n^- + |A^c| + 2\,|B^c|\,.$$

Let $\sigma \in \mathfrak{S}_m$ be the unique permutation such that σ^{-1} is increasing on $\{1, 2, \cdots, m^-\}$ and on $\{m^- + 1, m^- + 2, \cdots, m\}$, and

$$\sigma^{-1}(\{1, \cdots, m^-\}) \\ = A^c \cup \{r + 2j - 1, r + 2j \mid j \in B^c\} \cup \{r + 2t + 1, \cdots, r + 2t + n^-\}.$$

We then require that $i_{G^*}(A, B)$ sends $\tau_{0_1}, \cdots, \tau_{0_{r+2t}}, \tau_1, \cdots, \tau_{m-r-2t}$ respectively to $\underline{\rho}_{\sigma(1)}, \cdots, \underline{\rho}_{\sigma(m)}$. Our $i_{G^*}(A, B)$ is a direct analogue of $i_G(A, B)$ in Definition 8.8.3, an it enjoys similar properties as in Lemmas 8.8.4 and 8.8.6, with j and j_M replaced by $\underline{j}$ and j_{M^*}. Let $B_{M^*} := B_0 \cap M^*$, and let

$$\Delta^{A,B}_{j_{M^*}, B_{M^*}} := (-1)^{q(G_{\mathbb{R}}) + q(H_{\mathbb{R}}) + q(M^*_{\mathbb{R}}) + q(M'_{\mathbb{R}})} \Delta_{j_{M^*}, B_{M^*}}. \tag{8.15.2.5}$$

By [**Mor11**, Prop. 3.2.5] (cf. Proposition 8.8.8) and similar arguments as in §8.8.9, and the proof of Lemma 8.8.10, we have

$$\sum_{\varphi_H \in \Phi_H(\varphi_{\mathbb{V}^*})} \det(\omega_*(\varphi_H)) \Phi^H_{M'}(\gamma'^{-1}, \Theta_{\varphi_H}) \\ = \operatorname{sgn}(\sigma) \epsilon_R(j_{M^*}(\gamma'^{-1})) \epsilon_{R_H}(\gamma'^{-1}) \Delta^{A,B}_{j_{M^*}, B_{M^*}}(\gamma', j_{M^*}(\gamma')) \Phi^{G^*}_{M^*}(j_{M^*}(\gamma'^{-1}), \Theta^H_{\mathbb{V}^*}). \tag{8.15.2.6}$$

Here,

– σ is the permutation as above, used to define $i_{G^*}(A, B)$.

– R is the set of real roots of $(G^*_{\mathbb{C}}, T_{M^*,\mathbb{C}})$, and $\epsilon_R(t)$ is -1 to the number of B_0-positive roots α in R such that $0 < \alpha(t) < 1$.

– R_H is the set of real roots of $(H_{\mathbb{C}}, T_{M',\mathbb{C}})$, and $\epsilon_{R_H}(t')$ is -1 to the number of $\alpha \in R_H$ such that $0 < \alpha(t') < 1$ and such that $\alpha \circ (j_{M^*})^{-1} \circ i_{G_*}(A, B)^{-1} \circ \underline{j} \in X^*(T_H)$ is a B_H-positive root.

– $\Phi^{G^*}_{M^*}(\cdot, \Theta^H_{\mathbb{V}^*})$ is defined analogously as $\Phi^G_M(\cdot, \Theta)_{\mathrm{eds}}$ in (4.6.10.1) and (4.6.10.2), with the role of $\mathbb{V}$ played by $\mathbb{V}^*$, and the role of R_{eds} in (4.6.10.1) played by the root system

$$R_{H,\gamma'} := \{\alpha \in R_H \mid \alpha(\gamma') > 0\}.$$

We analyze how the terms on the right hand side of (8.15.2.6) depend on (A, B). We observe that $\epsilon_R(j_{M^*}(\gamma'^{-1}))$ is independent of (A, B), while R_H and $R_{H,\gamma'}$ as above depend only on A, not on B. To simplify notation we denote

$$\Phi(\gamma', A) := \Phi^{G^*}_{M^*}(j_{M^*}(\gamma'^{-1}), \Theta^H_{\mathbb{V}^*}). \tag{8.15.2.7}$$

and

$$R_A := R_H, \quad R_{A,\gamma'} := R_{H,\gamma'}. \tag{8.15.2.8}$$

We claim that $\epsilon_{R_H}(\gamma'^{-1})$ is independent of (A, B). Indeed, the roots $\alpha \in R_H$ such that $\alpha \circ (j_{M^*})^{-1} \circ i_{G_*}(A, B)^{-1} \circ \underline{j}$ are B_H-positive are exactly $\epsilon_i + \epsilon_j$ and

$\epsilon_i - \epsilon_j$ where $i < j$ and i, j simultaneously belong to one of A and A^c, together with ϵ_i for all $i \in [r]$, together with certain characters of the direct factor $T_{M^*,\mathrm{GL}} \cap \mathrm{GL}_2^t$ of T_{M^*} constituting a set independent of (A, B). Among them, those satisfying $0 < \alpha(\gamma'^{-1}) < 1$ are, by (8.15.2.2) (8.15.2.3) (8.15.2.4), exactly $\epsilon_i + \epsilon_j$ and $\epsilon_i - \epsilon_j$ where $i < j$ and i, j simultaneously belong to one of the four sets $\{u \in A \mid u \leq r'\}, \{u \in A^c \mid u \leq r'\}, \{u \in A \mid u > r'\}, \{u \in A^c \mid u > r'\}$, together with certain other roots constituting a set independent of (A, B). The total number of $\epsilon_i + \epsilon_j$ and $\epsilon_i - \epsilon_j$ where $i < j$ and i, j simultaneously belong to one of the four sets as above is obviously even. Our claim follows.

In the rest of the proof, we write "Const." for any quantity that is independent of (A, B). By (8.15.2.5), (8.15.2.6), and the above analysis, we have

(8.15.2.9)
$$\sum_{\varphi_H \in \Phi_H(\varphi_{\mathbb{V}^*})} \det(\omega_*(\varphi_H)) \Phi^H_{M'}(\gamma'^{-1}, \Theta_{\varphi_H}) = \mathrm{Const.}\, \mathrm{sgn}(\sigma)(-1)^{q(H_{\mathbb{R}})} \Phi(\gamma', A).$$

We now simplify $\mathrm{sgn}(\sigma)$ and $(-1)^{q(H_{\mathbb{R}})}$. Define $\omega_0(A)$ to be the sign of the element $\sigma_A \in \mathfrak{S}_r$ which sends $\{1, 2, \cdots, |A^c|\}$ increasingly to A^c and sends $\{|A^c| + 1, \cdots, r\}$ increasingly to A. If we view σ_A as an element of $\mathfrak{S}_m$, then $\sigma_A^{-1} \circ \sigma^{-1}$ sends $\{1, \cdots, m^-\}$ increasingly to

$$\{1, \cdots, |A^c|\} \cup \{r + 2j - 1, r + 2j \mid j \in B^c\} \cup \left\{r + 2t + 1, \cdots, r + 2t + n^-\right\},$$

and sends $\{m^- + 1, \cdots, m\}$ increasingly to

$$\{|A^c| + 1, \cdots, r\} \cup \{r + 2j - 1, r + 2j \mid j \in B^c\} \cup \left\{r + 2t + n^- + 1, \cdots, m\right\}.$$

From this, one sees that the sign of $\sigma_A^{-1} \circ \sigma^{-1}$ is $(-1)^{|A|n^-}$ (since all n^- elements of $\{r + 2t + 1, \cdots, r + 2t + n^-\}$ are greater than all $|A|$ elements of $\{|A^c| + 1, \cdots, r\}$). Hence we have

$$\mathrm{sgn}(\sigma) = \omega_0(A)(-1)^{|A|n^-}. \tag{8.15.2.10}$$

As for $(-1)^{q(H_{\mathbb{R}})}$, we compute

$$\begin{aligned} &2q(H_{\mathbb{R}}) = m^+(m^+ + 1) + m^-(m^- + 1) \\ &= (n^+ + |A| + 2\,|B|)(n^+ + |A| + 2\,|B| + 1) + (n^- + |A^c| + 2\,|B^c|)(n^- + |A^c| + 2\,|B^c| + 1), \end{aligned}$$

and so

$$q(H_{\mathbb{R}}) \equiv \mathrm{Const.} + (m + 1)(|A| + 2\,|B|) \equiv \mathrm{Const.} + (m + 1)\,|A| \mod 2. \tag{8.15.2.11}$$

Plugging (8.15.2.10) and (8.15.2.11) into (8.15.2.9), we get

$$\sum_{\varphi_H \in \Phi_H(\varphi_{\mathbb{V}^*})} \det(\omega_*(\varphi_H)) \Phi^H_{M'}(\gamma'^{-1}, \Theta_{\varphi_H}) = \mathrm{Const.}\, \omega_0(A)(-1)^{|A|(n^- + m + 1)} \Phi(\gamma', A).$$

Hence

$$R_i = \text{Const.} \sum_{A,B} \nabla_i(A)\epsilon^{p,\infty}(A,B)\omega_0(A)(-1)^{|A|(n^-+m+1)}\Phi(\gamma',A), \tag{8.15.2.12}$$

$$T_j = \text{Const.} \sum_{A,B} \nabla_j(B)\epsilon^{p,\infty}(A,B)\omega_0(A)(-1)^{|A|(n^-+m+1)}\Phi(\gamma',A), \tag{8.15.2.13}$$

$$S = \text{Const.} \sum_{A,B} \epsilon^{p,\infty}(A,B)\omega_0(A)(-1)^{|A|(n^-+m+1)}\Phi(\gamma',A). \tag{8.15.2.14}$$

We now compute $\epsilon^{p,\infty}(A,B)$. Let $(H, {}^L H, s, \eta)$ be determined by (A,B). For each place v, as explained in Remark 5.1.4, the choice of $\phi_{V_{\mathbb{Q}_v}} : V_{\mathbb{Q}_v} \otimes \overline{\mathbb{Q}}_v \xrightarrow{\sim} \underline{V}_{\mathbb{Q}_v} \otimes \overline{\mathbb{Q}}_v$ and the resulting pure inner twist $(\psi_{V_{\mathbb{Q}_v}}, u_{V_{\mathbb{Q}_v}})$ allows us to pass between normalizations of transfer factors between H and G and between H and G^* at v. Hence we obtain from $(\Delta_H^G)_v$ a normalization $(\Delta_H^{G^*})_v$ of transfer factors between H and G^* at v, and then inherit from the latter a normalization $(\Delta_{M'}^{M^*})_v^{A,B}$ of transfer factors between M' and M^* at v. For each finite $v \neq p$, we have

$$\frac{(\Delta_{M'}^{M^*})_v^{A,B}}{(\Delta_{M'}^{M^*})_v^{\emptyset,\emptyset}} = \frac{(\Delta_{M'}^{M_v})_v^{A,B}}{(\Delta_{M'}^{M_v})_v^{\emptyset,\emptyset}},$$

and so

$$\epsilon^{p,\infty}(A,B) = \prod_{v \neq p,\infty} \frac{(\Delta_{M'}^{M^*})_v^{A,B}}{(\Delta_{M'}^{M^*})_v^{\emptyset,\emptyset}}.$$

Recall that the normalizations $(\Delta_H^G)_v$ for all places v satisfy the global product formula. We claim that $(\Delta_{M'}^{M^*})_v^{A,B}$ for all v also satisfy the global product formula, for which we provide an argument that also works in the even case. Recall from Remarks 5.1.3 and 5.1.4 that for each v we have the freedom of changing $\phi_{V_{\mathbb{Q}_v}} : V_{\mathbb{Q}_v} \otimes_{\mathbb{Q}_v} \overline{\mathbb{Q}}_v \xrightarrow{\sim} \underline{V}_{\mathbb{Q}_v} \otimes_{\mathbb{Q}_v} \overline{\mathbb{Q}}_v$ by composing it with an element of $G^*(\overline{\mathbb{Q}}_v)$. Also recall the compatibility condition (1) imposed in §5.3.3. Thus for the sake of proving the claim, we may replace each $\phi_{V_{\mathbb{Q}_v}}$ by the isomorphism $V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_v \xrightarrow{\sim} \underline{V} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_v$ induced by the global $\phi_V : V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \xrightarrow{\sim} \underline{V}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$. Then one sees that $(\Delta_H^{G^*})_v$ for all v satisfy the global product formula, since the local cocycles $u_{V_{\mathbb{Q}_v}} : \rho \mapsto {}^\rho\phi_{V_{\mathbb{Q}_v}} \phi_{V_{\mathbb{Q}_v}}^{-1}$ come from the global cocycle $u_V : \rho \mapsto {}^\rho\phi_V \phi_V^{-1}$. Therefore the inherited normalizations $(\Delta_{M'}^{M^*})_v^{A,B}$ also satisfy the global product formula.

By our claim, the product $\prod_v (\Delta_{M'}^{M^*})_v^{A,B}$ over all places v is independent of (A,B). Hence

$$\epsilon^{p,\infty}(A,B) = \prod_{v \in \{p,\infty\}} \frac{(\Delta_{M'}^{M^*})_v^{\emptyset,\emptyset}}{(\Delta_{M'}^{M^*})_v^{A,B}}.$$

Now $V_{\mathbb{Q}_p}$ is quasi-split by our assumption that $G_{\mathbb{Q}_p}$ is unramified (in particular split) and by Proposition 1.2.8. Hence there exists $g \in G^*(\overline{\mathbb{Q}}_p)$ such that $g \circ \phi_{V_{\mathbb{Q}_p}}$ is defined over $\mathbb{Q}_p$. (Clearly we can find $g' \in \mathrm{O}(\underline{V})(\overline{\mathbb{Q}}_p)$ such that $g' \circ \phi_{V_{\mathbb{Q}_p}}$ is defined over $\mathbb{Q}_p$. We can then construct g by left multiplying g' by any element of $\mathrm{O}(\underline{V})(\mathbb{Q}_p)$

of determinant -1, which exists.) It then follows that $(\Delta_{M'}^{M^*})_p^{A,B}$ is the canonical unramified normalization associated to a hyperspecial subgroup of $M^*(\mathbb{Q}_p)$ that is independent of (A,B).[12] Hence $(\Delta_{M'}^{M^*})_p^{A,B}$ is independent of (A,B). We conclude that

$$\epsilon^{p,\infty}(A,B) = \frac{(\Delta_{M'}^{M^*})_\infty^{\emptyset,\emptyset}}{(\Delta_{M'}^{M^*})_\infty^{A,B}}.$$

By the same argument as in the proof of Proposition 8.9.5 (see the "claim" in that proof), the Whittaker normalization between M' and M^* at ∞ is inherited from the Whittaker normalization between H and G^* at ∞. The former is independent of (A,B). Hence $\epsilon(A,B)$ is up to a non-zero multiplicative constant equal to the ratio of the Whittaker normalization between H and G^* at ∞ to the normalization $(\Delta_H^{G^*})_\infty$. This ratio is the same as the ratio of the Whittaker normalization between H and G to $(\Delta_H^G)_\infty = \Delta_{j,B_{G,H}}$, which is equal to

$$(-1)^{\lceil m^+/2\rceil+1} = (-1)^{\lceil \frac{n^+ + |A| + 2|B|}{2}\rceil + 1}$$

as shown in the proof of Proposition 8.9.5. When n^+ is even, the above is equal to Const.$(-1)^{\lceil |A|/2\rceil + |B|}$. When n^+ is odd, the above is equal to Const.$(-1)^{\lfloor |A|/2\rfloor + |B|}$. In both cases, taking into account the equality $m = n^+ + n^- + r + 2t$, we obtain:

$$\epsilon^{p,\infty}(A,B)(-1)^{|A|(n^- + m + 1)} = \text{Const.}(-1)^{r|A| + \lfloor |A|/2\rfloor + |B|}.$$

Plugging this into (8.15.2.12), (8.15.2.13), and (8.15.2.14), we obtain

$$R_i = \text{Const.}\sum_{A,B} \nabla_i(A)\omega_0(A)(-1)^{r|A|+\lfloor |A|/2\rfloor + |B|}\Phi(\gamma', A), \tag{8.15.2.15}$$

$$T_j = \text{Const.}\sum_{A,B} \nabla_j(B)\omega_0(A)(-1)^{r|A|+\lfloor |A|/2\rfloor + |B|}\Phi(\gamma', A), \tag{8.15.2.16}$$

$$S = \text{Const.}\sum_{A,B} \omega_0(A)(-1)^{r|A|+\lfloor |A|/2\rfloor + |B|}\Phi(\gamma', A), \tag{8.15.2.17}$$

where A runs through subsets of $[r]$ and B runs through subsets of $[t]$. We need to show that the right hand sides are all zero. This we accomplish in the next proposition. □

Proposition 8.15.3. — *Assume $rt > 0$, or $r \geq 3$, or $t \geq 2$. The right hand sides of (8.15.2.15) (8.15.2.16) (8.15.2.17) are all zero.*

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(12) In the current odd case, all hyperspecial subgroups of $M^*(\mathbb{Q}_p)$ are conjugate under $M^*(\mathbb{Q}_p)$, so the canonical unramified normalizations associated to all hyperspecial subgroups are actually equal to each other. This is no longer true in the even case. Nevertheless, the statement in the text remains true in the even case, as long as there exists $g \in G^*(\overline{\mathbb{Q}}_p)$ such that $g \circ \phi_{V_{\mathbb{Q}_p}}$ is defined over $\mathbb{Q}_p$.

Proof. — We first treat the case $t \geq 2$, which is the easiest. In this case we have the elementary combinatorial identities

$$\sum_{B \subset [t]} (-1)^{|B|} = 0 \tag{8.15.3.1}$$

and

(8.15.3.2)

$$\begin{aligned}\sum_{B \subset [t]} \nabla_j(B)(-1)^{|B|} &= \sum_{k=0}^{t} (-1)^k \Big[\# \{B \mid |B| = k, j \in B\} - \# \{B \mid |B| = k, j \notin B\}\Big] \\ &= \sum_{k=0}^{t} (-1)^k \left[\binom{t-1}{k-1} - \binom{t-1}{k}\right] \\ &= -2 \sum_{k=0}^{t-1} (-1)^k \binom{t-1}{k} = 0.\end{aligned}$$

(Note that for $t = 1$, (8.15.3.1) still holds, but $\sum_B \nabla_1(B)(-1)^{|B|} = -2$.) Hence we have $R_i = T_j = S = 0$ in this case, and the proof is finished.

Before treating the other cases, we observe that

$$\omega_0(A)\omega_0(A^c) = (-1)^{|A||A^c|},$$

from which

$$\omega_0(A)(-1)^{r|A| + \lfloor |A|/2 \rfloor} \omega_0(A^c)(-1)^{r|A^c| + \lfloor |A^c|/2 \rfloor} = (-1)^{\lceil r/2 \rceil}. \tag{8.15.3.3}$$

Now suppose $rt > 0$ and $r \in \{1, 2\}$. Again (8.15.3.1) holds, so $R_i = S = 0$. To show $T_j = 0$, observe that $\Phi(\gamma', A) = \Phi(\gamma', A^c)$, so it suffices to show that (8.15.3.3) is -1, which is indeed true for $r = 1, 2$.

Finally we treat the case $r \geq 3$, which is the most complicated. We need a computation that is similar to [**Mor11**, pp. 1698-1699], applying the result of Herb [**Her79**]. In the following we will view γ' and B as being fixed, and let A vary.

We have

$$A_{M'} = A_{M^*} = \mathbb{G}_m^r \times \mathbb{G}_m^t,$$

where the factor $\mathbb{G}_m^r$ is the canonical copy of $\mathbb{G}_m^r$ in $M^{*,\mathrm{GL}} = \mathbb{G}_m^r \times \mathrm{GL}_2^t$, and the factor $\mathbb{G}_m^t$ is the product of the centers of the t copies of GL_2 in $M^{*,\mathrm{GL}}$. Let $\epsilon_1, \cdots, \epsilon_r \in X^*(\mathbb{G}_m^r)$ be as in condition **C** satisfied by B_0 in the proof of Theorem 8.15.2. Let $\{\alpha_1, \cdots, \alpha_t\}$ be the standard basis of $X^*(\mathbb{G}_m^t)$. Define

$$\begin{aligned} I^+ &:= \{i \in [r] \mid \epsilon_i(\gamma') > 0\}, & I^- &:= [r] - I^+, \\ A^+ &:= A \cap I^+, & A^- &:= A \cap I^-, \\ A^{c,+} &:= A^c \cap I^+, & A^{c,-} &:= A^c \cap I^-. \end{aligned}$$

By (8.15.2.2), we know that $I^+ = [r']$.

Let $R_{A,\gamma'} = R_{H,\gamma'}$ be the real root system involved in the definition of $\Phi(\gamma', A)$; see (8.15.2.7) and (8.15.2.8). Then $R_{A,\gamma'}$ is of type

$$\mathsf{B}_{|A^+|} \times \mathsf{B}_{|A^{c,+}|} \times \mathsf{D}_{|A^-|} \times \mathsf{D}_{|A^{c,-}|} \times \mathsf{A}_1^{\times t}, \tag{8.15.3.4}$$

where $\mathsf{B}_{|A^+|}$ consists of the roots

$$\epsilon_i,\ \epsilon_i \pm \epsilon_j,\ i,j \in A^+, i \neq j$$

and $\mathsf{D}_{|A^-|}$ consists of the roots

$$\epsilon_i \pm \epsilon_j,\ i,j \in A^-, i \neq j,$$

and similarly for $\mathsf{B}_{|A^{c,+}|}$ and $\mathsf{D}_{|A^{c,-}|}$. The part $\mathsf{A}_1^{\times t}$ consists of the roots[(13)]

$$\pm 2\alpha_1, \cdots, \pm 2\alpha_t.$$

By (8.15.3.4), we see that the Weyl group of $R_{A,\gamma'}$ contains -1 if and only if $|A^-|$ and $|A^{c,-}|$ (and *a fortiori* $|I^-|$) are even, if and only if $\gamma' \in H(\mathbb{R})^0$. These conditions are necessary for $\Phi(\gamma', A)$ to be non-zero. Assume that these conditions are satisfied. Then

$$\Phi(\gamma', A) = \sum_{\omega \in \Omega} C(\gamma', \omega) n_A(\gamma', \omega B_0),$$

where Ω is the complex Weyl group of G^*, the coefficients $C(\gamma', \omega)$ are independent of A, and

$$n_A(\gamma', \omega B_0) := \bar{c}_{R_{A,\gamma'}}(x, \wp(\omega\lambda_{B_0} + \omega\rho_{B_0})),$$

with notations explained below:

– $x \in X_*(A_{M^*})_{\mathbb{R}}$ is characterized by the condition

$$j_{M^*}(\gamma'^{-1}) \in \exp(x) T_{M^*}(\mathbb{R})_1 \subset T_{M^*}(\mathbb{R}), \tag{8.15.3.5}$$

where $T_{M^*}(\mathbb{R})_1$ is the maximal compact subgroup of $T_{M^*}(\mathbb{R})$.

– $\wp : X^*(T_{M^*})_{\mathbb{R}} \to X^*(A_{M^*})_{\mathbb{R}}$ is the natural restriction map.

– ρ_{B_0} is the half sum of the B_0-positive (absolute) roots in $X^*(T_{M^*})$, and $\lambda_{B_0} \in X^*(T_{M^*})$ is the B_0-highest weight of $\mathbb{V}^*$.

– $\bar{c}_{R_{A,\gamma'}}(\cdot,\cdot)$ is the function associated to the root system $R_{A,\gamma'} \subset X^*(A_{M^*})_{\mathbb{R}}$ as in (4.2.4.1).

We note that

$$\chi := \wp(\omega\lambda_{B_0} + \omega\rho_{B_0}) \in X^*(A_{M^*})_{\mathbb{R}}$$

is independent of A. In the following we will use only this property of χ.

Thus to show that $R_i = T_j = S = 0$, it suffices to show that the following quantities are zero, where the summations are over $A \subset [r]$ such that $|A^-|$ and $|A^{c,-}|$ are both

(13) This follows from the following argument: Let ϵ_1, ϵ_2 denote the two standard characters on the diagonal torus in GL_2, and identify them with two characters on an elliptic maximal torus in $\mathrm{GL}_{2,\mathbb{R}}$. Then with respect to the real structure of the latter, $\pm(\epsilon_1 + \epsilon_2)$ are the only real characters among $\epsilon_1, \epsilon_2, \epsilon_1 \pm \epsilon_2, -\epsilon_1 \pm \epsilon_2$.

even:

$$M_i := \sum_A \nabla_i(A)\omega_0(A)(-1)^{r|A|+\lfloor |A|/2\rfloor}\bar{c}_{R_{A,\gamma'}}(x,\chi), \qquad 1 \le i \le r \tag{8.15.3.6}$$

$$N := \sum_A \omega_0(A)(-1)^{r|A|+\lfloor |A|/2\rfloor}\bar{c}_{R_{A,\gamma'}}(x,\chi). \tag{8.15.3.7}$$

More precisely, the vanishing of M_i implies the vanishing of R_i, and the vanishing of N implies the vanishing of T_j and S. We show the vanishing of M_i and N (for $r \ge 3$) in the next proposition. □

Proposition 8.15.4. — *Let $x \in X_*(A_{M^*})_{\mathbb{R}}$ be characterized by the condition (8.15.3.5), where $\gamma' \in T_{M'}(\mathbb{R})$ satisfies the conditions (8.15.2.2), (8.15.2.3), and (8.15.2.4). Let $\chi \in X^*(A_{M^*})_{\mathbb{R}}$ be an element independent of A. When $r \ge 3$, the quantities M_i and N in (8.15.3.6) and (8.15.3.7) are zero.*

8.15.5. — In the proof of Proposition 8.15.4 we need to apply Herb's formula for $\bar{c}_{R_{A,\gamma'}}$, which we now recall. We will follow the notation and definitions of [**Mor11**, pp. 1698-1699]. Note that in *loc. cit.* root systems of types C and D are considered, whereas we need to consider root systems of types B and D. Nevertheless the formulas for type B and type C root systems are identical; see [**Her79**].

For $a, b \in \mathbb{R}$, we define

$$c_1(a) := \begin{cases} 1, & \text{if } a > 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$c_{2,\mathsf{B}}(a,b) := \begin{cases} 1, & \text{if } 0 < a < b \text{ or } 0 < -b < a, \\ 0, & \text{otherwise.} \end{cases}$$

$$c_{2,\mathsf{D}}(a,b) := \begin{cases} 1, & \text{if } a > |b|, \\ 0, & \text{otherwise.} \end{cases}$$

Our $c_{2,\mathsf{B}}$ is equal $c_{2,\mathsf{C}}$ in [**Mor11**].

Let I be a finite set. We will denote an unordered partition p of I by $p = \{I_z \mid z \in Z\}$, where Z is the indexing set, and $I = \coprod_{z\in Z} I_z$. Let $\mathcal{P}^0_{\le 2}(I)$ be the set of unordered partitions $\{I_z \mid z \in Z\}$ of I such that all I_z have cardinality 2 or 1 and at most one I_z has cardinality 1. If I is equipped with a total order $\le$, we can define a sign function

$$\epsilon : \mathcal{P}^0_{\le 2}(I) \longrightarrow \{\pm 1\} \tag{8.15.5.1}$$

as follows. Given $p \in \mathcal{P}^0_{\le 2}(I)$, we enumerate the elements of p as $I_1, \cdots, I_k$, and let σ be the unique bijection $I \xrightarrow{\sim} I$ satisfying the following conditions:

- For all $i, j \in [k]$ with $i < j$, and for all $s \in \sigma(I_i)$ and $s' \in \sigma(I_j)$, we have $s < s'$.
- If $i \in [k]$ is such that $|I_i| = 2$, then σ is increasing on I_i.

With respect to the total order on I, the permutation σ of I has a well-defined sign. We define $\epsilon(p)$ to be that sign. This definition does not depend on the enumeration of the elements of p.

For $\mu \in \mathbb{R}^r$ and J a subset of $[r]$ of cardinality 1 or 2, we make the following definitions. If $J = \{s\}$, define

$$c_{J,\mathsf{B}}(\mu) := c_1(\mu_s).$$

If $J = \{s_1, s_2\}$ with $s_1 < s_2$, define

$$c_{J,\mathsf{B}}(\mu) := c_{2,\mathsf{B}}(\mu_{s_1}, \mu_{s_2}),$$
$$c_{J,\mathsf{D}}(\mu) := c_{2,\mathsf{D}}(\mu_{s_1}, \mu_{s_2}).$$

Now for $I \subset [r]$ and $p = \{I_z \mid z \in Z\} \in \mathcal{P}^0_{\leq 2}(I)$, define

$$c_{\mathsf{B}}(p, \mu) := \prod_{z \in Z} c_{I_z,\mathsf{B}}(\mu).$$

If in addition $|I|$ is even, define

$$c_{\mathsf{D}}(p, \mu) := \prod_{z \in Z} c_{I_z,\mathsf{D}}(\mu).$$

Let $\chi \in X^*(A_{M^*})_{\mathbb{R}}$ and let μ be its projection to $X^*(\mathbb{G}_m^r)_{\mathbb{R}}$. We identify $X^*(\mathbb{G}_m^r)_{\mathbb{R}}$ with $\mathbb{R}^r$ using the basis $\{\epsilon_1, \cdots, \epsilon_r\}$ fixed in the proof of Theorem 8.15.2 (as opposed to the standard basis), and view μ as an element of $\mathbb{R}^r$. Let x be as in the statement of the Proposition 8.15.4. Then Herb's formula states that

$$(8.15.5.2) \quad \bar{c}_{R_{A,\gamma'}}(x, \chi) = \text{Const.} \sum_{p_1^+ \in \mathcal{P}^0_{\leq 2}(A^+)} \sum_{p_1^- \in \mathcal{P}^0_{\leq 2}(A^-)} \sum_{p_2^+ \in \mathcal{P}^0_{\leq 2}(A^{c,+})} \sum_{p_2^- \in \mathcal{P}^0_{\leq 2}(A^{c,-})} \epsilon(p_1^+)\epsilon(p_1^-)\epsilon(p_2^+)\epsilon(p_2^-) c_{\mathsf{B}}(p_1^+, \mu) c_{\mathsf{B}}(p_2^+, \mu) c_{\mathsf{D}}(p_1^-, \mu) c_{\mathsf{D}}(p_2^-, \mu),$$

where Const. is independent of A.

Remark 8.15.6. — To compare (8.15.5.2) with the formula on p. 1699 of [**Mor11**], note that the root system considered in *loc. cit.* is of type $\mathsf{C}_{|A_1^-|} \times \mathsf{C}_{|A_1^+|} \times \mathsf{D}_{|A_2^-|} \times \mathsf{D}_{|A_2^+|} \times \mathsf{A}_1^{\times t}$, whereas our root system is $\mathsf{B}_{|A^+|} \times \mathsf{B}_{|A^{c,+}|} \times \mathsf{D}_{|A^-|} \times \mathsf{D}_{|A^{c,-}|} \times \mathsf{A}_1^{\times t}$. Our γ'^{-1} plays the same role as γ_M in *loc. cit.*.

Proof of Proposition 8.15.4. — We divide the proof into two cases according to the parity of r .

The case where $r \geq 3$ is odd.

Since $|A^-|$ and $|A^{c,-}|$ must be even, we know that $|A^+|$ and $|A^{c,+}|$ must have different parity. In particular I^+ has odd cardinality. Write $|I^+| = 2k - 1$ with $k \geq 1$, and write $|I^-| = 2l$ with $l \geq 0$.

For $p_1^+ \in \mathcal{P}^0_{\leq 2}(A^+)$ and $p_2^+ \in \mathcal{P}^0_{\leq 2}(A^{c,+})$, we have $p^+ := p_1^+ \cup p_2^+ \in \mathcal{P}^0_{\leq 2}(I^+)$. Also for $p_1^- \in \mathcal{P}^0_{\leq 2}(A^-)$ and $p_2^- \in \mathcal{P}^0_{\leq 2}(A^{c,-})$, we have $p^- := p_1^- \cup p_2^- \in \mathcal{P}^0_{\leq 2}(I^-)$. We also have

$$\omega_0(A)\epsilon(p_1^+)\epsilon(p_1^-)\epsilon(p_2^+)\epsilon(p_2^-) = \epsilon(p^+)\epsilon(p^-).$$

In this way we have "encoded" the quadruple $(p_1^+, p_2^+, p_1^-, p_2^-)$ and the left hand side of the above equality into (p^+, p^-).

Conversely, we explain how to recover (p_1^+, p_2^+) from p^+ with extra data, and recover (p_1^-, p_2^-) from p^- with extra data. Given $p^+ \in \mathcal{P}^0_{\leq 2}(I^+)$, write $p^+ = p^+(2) \sqcup p^+(1)$, where $p^+(2)$ consists of the cardinality-2 members of p^+ and $p^+(1)$ consists of the singleton member of p^+. (Note that $|p^+(2)| = k-1$ and $|p^+(1)| = 1$.) To recover (p_1^+, p_2^+) is the same as to recover the subset A^+ of I^+. For that it suffices to specify a subset U of $p^+(2)$ and a subset V of $p^+(1)$ such that $A^+ = \bigcup_{I \in U \cup V} I$. Thus we have established a bijection from the set of (A^+, p_1^+, p_2^+) to the set of (p^+, U, V). Under this bijection, we have $|A^+| = 2|U| + |V|$. For a fixed $i \in I^+$, we can also encode the function $A^+ \mapsto \nabla_i(A^+)$ into a function in the variables p^+, U, and V as follows. Define

$$\nabla_i(p^+, U, V) := \begin{cases} 1, & \text{if } i \in I \text{ for some } I \in U \cup V, \\ -1, & \text{otherwise.} \end{cases}$$

Then we have $\nabla_i(A^+) = \nabla_i(p^+, U, V)$ if (A^+, p_1^+, p_2^+) corresponds to (p^+, U, V) as above.

Similarly, given $p^- \in \mathcal{P}^0_{\leq 2}(I^-)$, to recover (p_1^-, p_2^-) or equivalently A^-, it suffices to specify a subset W of p^- such that $A^- = \bigcup_{I \in W} I$. This again establishes a bijection from the set of (A^-, p_1^-, p_2^-) to the set of (p^-, W). We have $|A^-| = 2|W|$. For a fixed $i \in I^-$, define

$$\nabla_i(p^-, W) := \begin{cases} 1, & \text{if } i \in I \text{ for some } I \in W, \\ -1, & \text{otherwise.} \end{cases}$$

Then we have $\nabla_i(A^-) = \nabla_i(p^-, W)$.

In conclusion, we may change the summation index $(p_1^+, p_1^-, p_2^+, p_2^-)$ in (8.15.5.2) into the new summation index (p^+, p^-, U, V, W), and obtain

$$\begin{aligned} N = {} & \text{Const.} \sum_{p^+ \in \mathcal{P}^0_{\leq 2}(I^+)} \sum_{p^- \in \mathcal{P}^0_{\leq 2}(I^-)} \epsilon(p^+)\epsilon(p^-) c_{\mathsf{B}}(p^+, \mu) c_{\mathsf{D}}(p^-, \mu) \\ & \cdot \sum_{U \subset p^+(2), V \subset p^+(1), W \subset p^-} (-1)^{r(2|U|+|V|+2|W|) + \lfloor (2|U|+|V|+2|W|)/2 \rfloor} \\ = {} & \text{Const.} \sum_{U \subset [k-1], V \subset [1], W \subset [l]} (-1)^{|U|+|V|+|W|}, \end{aligned}$$

and for $i \in [r]$

$$M_i = \text{Const.} \sum_{p^+ \in \mathcal{P}^0_{\leq 2}(I^+)} \sum_{p^- \in \mathcal{P}^0_{\leq 2}(I^-)} \epsilon(p^+)\epsilon(p^-) c_{\mathsf{B}}(p^+, \mu) c_{\mathsf{D}}(p^-, \mu) \cdot \sum_{U \subset p^+(2), V \subset p^+(1), W \subset p^-} (-1)^{|U|+|V|+|W|} \nabla_i(p^+, p^-, U, V, W),$$

where

$$\nabla_i(p^+, p^-, U, V, W) := \begin{cases} \nabla_i(p^+, U, V), & \text{if } i \in I^+, \\ \nabla_i(p^-, W), & \text{if } i \in I^-. \end{cases}$$

Note that

$$\sum_{V \subset [1]} (-1)^{|V|} = 0. \tag{8.15.6.1}$$

Hence $N = 0$ as desired. To show $M_i = 0$, it suffices to prove that for each fixed $p^+ \in \mathcal{P}^0_{\leq 2}(I^+)$ and $p^- \in \mathcal{P}^0_{\leq 2}(I^-)$, the quantity

$$L := \sum_{U \subset p^+(2), V \subset p^+(1), W \subset p^-} (-1)^{|U|+|V|+|W|} \nabla_i(p^+, p^-, U, V, W)$$

is zero. By definition, depending on the relative position of (p^+, p^-, i), the term $\nabla_i(p^+, p^-, U, V, W)$ is either independent of V, or independent of (U, W). In the first case, we know $L = 0$ because of (8.15.6.1). In the second case, unless $k = 1$ and $l = 0$, we have either

$$\sum_{U \subset p^+(2)} (-1)^{|U|} = \sum_{U \subset [k-1]} (-1)^{|U|} = 0$$

or

$$\sum_{W \subset p^-} (-1)^{|W|} = \sum_{W \subset [l]} (-1)^{|W|} = 0,$$

and therefore $L = 0$. But if $k = 1$ and $l = 0$, then $r = |I^+| + |I^-| = 2k - 1 + 2l = 1$, a contradiction. Thus $L = 0$ as desired. The proof of the proposition for odd $r \geq 3$ is complete.

The case where $r \geq 3$ is even.

Now $|I^+|$ and $|I^-|$ are both even. Write $|I^+| = 2k$ and $|I^-| = 2l$, with $k, l \geq 0$ and $k + l = r/2 \geq 2$.

We need some combinatorial preparations. For a finite set I of even cardinality, we define $\mathcal{P}'(I)$ to be the set of unordered partitions $p = \{I_z \mid z \in Z\}$ of I equipped with a marked element of p such that exactly two members of p are singletons, all the other members of p have cardinality 2, and the marked element of p is one of the two singleton members. When I is equipped with a total order $\leq$, we define a map

$$\epsilon : \mathcal{P}'(I) \longrightarrow \{\pm 1\}$$

as follows. Given $p \in \mathcal{P}'(I)$, we can merge the two singletons in p into a cardinality-2 set and obtain an element $p_0 \in \mathcal{P}^0_{\leq 2}(I)$. Then we define $\epsilon(p)$ to be $\epsilon(p_0)$ if the marked singleton in p is greater than the other singleton in p, and define $\epsilon(p)$ to be $-\epsilon(p_0)$ otherwise. Here $\epsilon(p_0)$ is as in (8.15.5.1). If I is a subset of $[r]$ and $p \in \mathcal{P}'(I)$, we define

$$c_{\mathsf{B}}(p,\mu) := \prod_{z\in Z} c_{I_z}(\mu),$$

where $\{I_z \mid z \in Z\}$ is the partition of I underlying p.

We now seek to change the summation index in (8.15.5.2) in a similar manner as in the previous case with odd r. If $|A^+|$ is odd then so is $|A^{c,+}|$. In this case $k \geq 1$, and from each $p_1^+ \in \mathcal{P}^0_{\leq 2}(A^+)$ and $p_2^+ \in \mathcal{P}^0_{\leq 2}(A^{c,+})$, we obtain an element $p^+ := p_1^+ \cup p_2^+ \in \mathcal{P}'(I^+)$, where the marked singleton in p^+ is defined to be the singleton in p_1^+. Conversely, suppose $k \geq 1$ and suppose $p^+ \in \mathcal{P}'(I^+)$. Write $p^+ = p^+(2) \sqcup \left\{ I^u_{p^+}, I^m_{p^+} \right\}$, where $p^+(2)$ consists of the cardinality-2 members of p^+, and we denote by $I^u_{p^+}$ and $I^m_{p^+}$ the unmarked and marked singleton members of p^+ respectively. (Note that $|p^+(2)| = k-1$.) Then we can recover A^+ from p^+ together with a subset U of $p^+(2)$ such that $A^+ = \bigcup_{I\in U} I \cup I^m_{p^+}$. We have $|A^+| = 2\,|U| + 1$. For $i \in I^+$, define

$$\nabla_i(p^+,U) := \begin{cases} 1, & \text{if } i \in I \text{ for some } I \in U \text{ or } i \in I^m_{p^+}, \\ -1, & \text{otherwise.} \end{cases}$$

Then we have $\nabla_i(A^+) = \nabla_i(p^+,U)$.

If $|A^+|$ is even, then so is $|A^{c,+}|$. From each $p_1^+ \in \mathcal{P}^0_{\leq 2}(A^+)$ and $p_2^+ \in \mathcal{P}^0_{\leq 2}(A^{c,+})$, we obtain $p^+ := p_1^+ \cup p_2^+ \in \mathcal{P}^0_{\leq 2}(I^+)$. Conversely, given $p^+ \in \mathcal{P}^0_{\leq 2}(I^+)$, to recover A^+ it suffices to specify a subset U of p^+ such that $A^+ = \bigcup_{I\in U} I$. We have $|A^+| = 2\,|U|$. For $i \in I^+$, define

$$\nabla_i(p^+,U) := \begin{cases} 1, & \text{if } i \in I \text{ for some } I \in U, \\ -1, & \text{otherwise.} \end{cases}$$

Then $\nabla_i(A^+) = \nabla_i(p^+,U)$.

Similarly, since $|A^-|$ and $|A^{c,-}|$ are always even, from $p_1^- \in \mathcal{P}^0_{\leq 2}(A^-)$ and $p_2^- \in \mathcal{P}^0_{\leq 2}(A^{c,-})$ we obtain an element $p^- := p_1^- \cup p_2^- \in \mathcal{P}^0_{\leq 2}(I^-)$, and conversely, given $p^- \in \mathcal{P}^0_{\leq 2}(I^-)$, to recover A^- it suffices to specify a subset W of p^- such that $A^- = \bigcup_{I\in W} I$. We have $|A^-| = 2\,|W|$. For $i \in I^-$, define

$$\nabla_i(p^-,W) := \begin{cases} 1, & \text{if } i \in I \text{ for some } I \in W, \\ -1, & \text{otherwise.} \end{cases}$$

Then we have $\nabla_i(A^-) = \nabla_i(p^-,W)$.

For both parities of $|A^+|$, we have

$$\omega_0(A)\epsilon(p_1^+)\epsilon(p_2^+)\epsilon(p_1^-)\epsilon(p_2^-) = \epsilon(p^+)\epsilon(p^-).$$

We now split

$$N = \sum_{A\subset[r],|A^-|\text{ even}} \omega_0(A)(-1)^{r|A|+\lfloor |A|/2\rfloor}\bar{c}_{R_{A,\gamma'}}(x,\chi)$$

as $N = N_{(1)} + N_{(2)}$, where $N_{(1)}$ (resp. $N_{(2)}$) is the sum of the terms indexed by A such that $|A^+|$ is odd (resp. even). Similarly, for $i \in [r]$, we split

$$M_i = \sum_{A\subset[r],|A^-|\text{ even}} \nabla_i(A)\omega_0(A)(-1)^{r|A|+\lfloor |A|/2\rfloor}\bar{c}_{R_{A,\gamma'}}(x,\chi)$$

as $M_i = M_{i,(1)} + M_{i,(2)}$. We shall prove that $N_{(1)} = N_{(2)} = M_{i,(1)} = M_{i,(2)} = 0$. Note that when dealing with $N_{(1)}$ and $M_{i,(1)}$ we may assume that $k \geq 1$, since otherwise they are obviously zero.

The above discussion shows that

$$\begin{aligned} N_{(1)} = \text{Const.} &\sum_{p^+\in\mathcal{P}'(I^+)}\ \sum_{p^-\in\mathcal{P}^0_{\leq 2}(I^-)} \epsilon(p^+)\epsilon(p^-)c_{\mathsf{B}}(p^+,\mu)c_{\mathsf{D}}(p^-,\mu) \\ &\cdot \sum_{U\subset p^+(2),W\subset p^-} (-1)^{r(2|U|+1+2|W|)+\lfloor(2|U|+1+2|W|)/2\rfloor} \\ &= \text{Const.} \sum_{U\subset[k-1],W\subset[l]} (-1)^{|U|+|W|}. \end{aligned}$$

This is zero because by $k + l \geq 2$ we have either $l \geq 1$ or $k - 1 \geq 1$. Also,

$$\begin{aligned} N_{(2)} = \text{Const.} &\sum_{p^+\in\mathcal{P}^0_{\leq 2}(I^+)}\ \sum_{p^-\in\mathcal{P}^0_{\leq 2}(I^-)} \epsilon(p^+)\epsilon(p^-)c_{\mathsf{B}}(p^+,\mu)c_{\mathsf{D}}(p^-,\mu) \\ &\cdot \sum_{U\subset p^+,W\subset p^-} (-1)^{r(2|U|+2|W|)+\lfloor(2|U|+2|W|)/2\rfloor} \\ &= \text{Const.} \sum_{U\subset[k],W\subset[l]} (-1)^{|U|+|W|}, \end{aligned}$$

which is zero because $kl > 0$.

Similarly, we have

$$\begin{aligned} M_{i,(1)} = \text{Const.} &\sum_{p^+\in\mathcal{P}'(I^+)}\ \sum_{p^-\in\mathcal{P}^0_{\leq 2}(I^-)} \epsilon(p^+)\epsilon(p^-)c_{\mathsf{B}}(p^+,\mu)c_{\mathsf{D}}(p^-,\mu) \\ &\cdot \sum_{U\subset p^+(2),W\subset p^-} (-1)^{|U|+|W|}\nabla_i(p^+,p^-,U,W), \end{aligned} \tag{8.15.6.2}$$

and

$$\begin{aligned} M_{i,(2)} = \text{Const.} &\sum_{p^+\in\mathcal{P}^0_{\leq 2}(I^+)}\ \sum_{p^-\in\mathcal{P}^0_{\leq 2}(I^-)} \epsilon(p^+)\epsilon(p^-)c_{\mathsf{B}}(p^+,\mu)c_{\mathsf{D}}(p^-,\mu) \\ &\cdot \sum_{U\subset p^+,W\subset p^-} (-1)^{|U|+|W|}\nabla_i(p^+,p^-,U,W), \end{aligned} \tag{8.15.6.3}$$

where

$$\nabla_i(p^+,p^-,U,W) := \begin{cases} \nabla_i(p^+,U), & \text{if } i \in I^+, \\ \nabla_i(p^-,W), & \text{if } i \in I^-. \end{cases}$$

(Here the formula for $M_{i,(1)}$ presupposes that $k \geq 1$; otherwise we already know that $M_{i,(1)} = 0$.) In the rest of the proof we show that $M_{i,(1)} = M_{i,(2)} = 0$. We introduce two auxiliary definitions. For $q^+ \in \mathcal{P}'(I^+), p^+ \in \mathcal{P}^0_{\leq 2}(I^+), p^- \in \mathcal{P}^0_{\leq 2}(I^-)$, let

$$L_{i,(1)}(q^+,p^-) := \sum_{U \subset q^+(2), W \subset p^-} (-1)^{|U|+|W|} \nabla_i(q^+,p^-,U,W),$$

$$L_{i,(2)}(p^+,p^-) := \sum_{U \subset p^+, W \subset p^-} (-1)^{|U|+|W|} \nabla_i(p^+,p^-,U,W).$$

We first show that $M_{i,(1)} = 0$. We may assume that $k \geq 1$. If $i \in I^-$, then the function $\mathcal{P}'(I^+) \times \mathcal{P}^0_{\leq 2}(I^-) \ni (p^+,p^-) \mapsto L_{i,(1)}(p^+,p^-)$ is constant with respect to the variable p^+ . Hence by (8.15.6.2) we have

$$M_{i,(1)} = \text{Const.} \sum_{p^+ \in \mathcal{P}'(I^+)} \epsilon(p^+) c_{\mathsf{B}}(p^+,\mu).$$

This is zero because on $\mathcal{P}'(I^+)$ we have a non-trivial involution $p^+ \mapsto \overline{p^+}$ where $\overline{p^+}$ has the same underlying partition as p^+ but has different marked singleton, and this involution satisfies $\epsilon(p^+) = -\epsilon(\overline{p^+}), c_{\mathsf{B}}(p^+,\mu) = c_{\mathsf{B}}(\overline{p^+},\mu)$.

It remains to treat the case where $i \in I^+$. Let $p^+ \in \mathcal{P}'(I^+)$. If one of the singletons in p^+ contains i, then for arbitrary $p^- \in \mathcal{P}^0_{\leq 2}(I^-)$, $L_{i,(1)}(p^+,p^-)$ is equal to a certain number times

$$\sum_{U \subset [k-1], W \subset [l]} (-1)^{|U|+|W|},$$

which is zero since either $k-1 \geq 1$ or $l \geq 1$. Thus the contribution of such p^+ to (8.15.6.2) is zero. If one of the cardinality-2 members of p^+ contains i, then so does one of the cardinality-2 members of $\overline{p^+}$. For such a pair $\left\{p^+, \overline{p^+}\right\}$, the contribution of p^+ to (8.15.6.2) is equal to the negative of the contribution of $\overline{p^+}$, since for any fixed $p^- \in \mathcal{P}^0_{\leq 2}(I^-)$ we have $L_{i,(1)}(p^+,p^-) = L_{i,(1)}(\overline{p^+},p^-)$, and as before we have $\epsilon(p^+) = -\epsilon(\overline{p^+}), c_{\mathsf{B}}(p^+,\mu) = c_{\mathsf{B}}(\overline{p^+},\mu)$. We have completed the proof that $M_{i,(1)} = 0$.

We now show that $M_{i,(2)} = 0$. By (8.15.6.3), it suffices to show that $L_{i,(2)}(p^+,p^-) = 0$ for all $p^+ \in \mathcal{P}^0_{\leq 2}(I^+), p^- \in \mathcal{P}^0_{\leq 2}(I^-)$. To show this, by symmetry we may assume without loss of generality that $s \in I^-$. Enumerate the elements of p^- as $I_1, \cdots, I_l$ such that $i \in I_1$. Using this enumeration we identify the sets p^- and $[l]$. (Here $l \geq 1$.) Then $\nabla_i(p^+,p^-,U,W) = \nabla_1(W)$ for all $W \subset p^- = [l]$. Hence

$$L_{i,(2)}(p^+,p^-) = \sum_{U \subset p^+, W \subset [l]} (-1)^{|U|+|W|} \nabla_1(W) = \sum_{U \subset [k], W \subset [l]} (-1)^{|U|+|W|} \nabla_1(W).$$

If $k > 0$, then $L_{i,(2)}(p^+, p^-) = 0$ because $\sum_{U\subset[k]}(-1)^{|U|} = 0$. If $k = 0$, then $l \geq 2$, and we have $\sum_{W\subset[l]}(-1)^{|W|}\nabla_1(W) = 0$ as in (8.15.3.2), from which $L_{i,(2)}(p^+, p^-) = 0$. The proof of the proposition for even $r \geq 3$ is complete. □

8.16. A vanishing result, even case

8.16.1. — Assume we are in the even case. We are to state and prove the analogue of Theorem 8.15.2. We only point out some new features in the even case, without repeating most of the identical steps.

As in §8.15.1, we consider a Levi subgroup M^* of G^* of the form $\mathbb{G}_m^r \times \mathrm{GL}_2^t \times \mathrm{SO}(\underline{W})$. Without loss of generality, we may and shall assume that $\mathrm{SO}(\underline{W})$ is not the split SO_2 over $\mathbb{Q}$, since in that case we can "absorb" it into the factor $\mathbb{G}_m^r$ (or more precisely, we can replace $\underline{W}^\perp$ by the whole $\underline{V}$, and extend the hyperbolic basis $\mathbb{B}_{\underline{W}^\perp}$ to a hyperbolic basis of $\underline{V}$, after which we obtain the same Levi subgroup M^* but presented in the form $M^* = M^{*,\mathrm{GL}} = \mathbb{G}_m^{r+1} \times \mathrm{GL}_2^t$). In the current even case we impose the assumption that M^* is cuspidal. This is equivalent to $\mathrm{SO}(\underline{W})_\mathbb{R}$ having anisotropic maximal tori (since $\mathrm{SO}(\underline{W})$ is not the split SO_2 over $\mathbb{Q}$), and equivalent to r being even.

Define $\dot{\mathscr{E}}_{G^*}(M^*)$ in the same way as in §8.15.1. As in §8.15.1, for each $\mathfrak{e}_{A,B,\mathfrak{p}} = \mathfrak{e}_{A,B,d^+,\delta^+,d^-,\delta^-} = (M', {}^LM', s_{M^*}, \eta_{M^*}) \in \dot{\mathscr{E}}_{G^*}(M^*)$, we let

$$(H, {}^LH, s, \eta) := \mathfrak{e}_{d^++2|A|+4|B|,\delta^+,d^-+2|A^c|+4|B^c|,\delta^-}$$

(viewed as an elliptic endoscopic datum for G), fix an embedding $M' \hookrightarrow H$ as in §5.5.9, and define $ST^H_{M'}(f^H)$ as in Definition 8.3.3. Then as in (8.15.1.1), we define

(8.16.1.1)
$$\mathrm{Tr}'_{M^*} \quad := \quad (n^{G^*}_{M^*})^{-1} \sum_{\substack{\mathfrak{e}=(M',{}^LM',s_{M^*},\eta_{M^*})\\ \in\dot{\mathscr{E}}_{G^*}(M^*)}} |\mathrm{Out}_{G^*}(\mathfrak{e})|^{-1}\, \tau(G)\tau(H)^{-1} ST^H_{M'}(f^H).$$

In the odd case, since $G_{\mathbb{Q}_p}$ is unramified, it is split, and this already implies that the quadratic space (V, q) is (quasi-)split over $\mathbb{Q}_p$ (see Proposition 1.2.8). In the even case, it no longer follows from the unramifiedness of $G_{\mathbb{Q}_p}$ that (V, q) is quasi-split over $\mathbb{Q}_p$. However, we shall impose this as a hypothesis[(14)] in the following theorem. By Proposition 1.2.8, given the unramifiedness of $G_{\mathbb{Q}_p}$, in order for (V, q) to be quasi-split over $\mathbb{Q}_p$ it is sufficient and necessary that the Hasse invariant of (V, q) at p is trivial.

Theorem 8.16.2. — *Keep the assumptions on M^* in §8.16.1, and assume that M^* does not transfer to G. Assume that the quadratic space (V, q) is quasi-split over $\mathbb{Q}_p$. Then* $\mathrm{Tr}'_{M^*} = 0$.

(14) This is equivalent to asking that $G_{\mathbb{Q}_p}$ as a pure inner form of $G^*_{\mathbb{Q}_p}$ is trivial.

Proof. — The proof is similar to the proof of Theorem 8.15.2. We follow most of the notations introduced in the proofs of Theorem 8.15.2 and Propositions 8.15.3, 8.15.4.

Recall that r is even. By hypothesis at least one of the following conditions holds:

$$rt > 0 \qquad \text{or} \qquad r \geq 4 \qquad \text{or} \qquad t \geq 2.$$

As in the proof of Theorem 8.15.2 we reduce the current proof to showing the vanishing of

$$R_i := \sum_{A,B} \nabla_i(A)\epsilon^{p,\infty}(A,B) \sum_{\varphi_H \in \Phi_H(\varphi_{\mathbb{V}^*})} \det(\omega_*(\varphi_H))\Phi^H_{M'}(\gamma'^{-1}, \Theta_{\varphi_H}), \qquad 1 \leq i \leq r,$$

$$T_j := \sum_{A,B} \nabla_j(B)\epsilon^{p,\infty}(A,B) \sum_{\varphi_H \in \Phi_H(\varphi_{\mathbb{V}^*})} \det(\omega_*(\varphi_H))\Phi^H_{M'}(\gamma'^{-1}, \Theta_{\varphi_H}), \qquad 1 \leq j \leq t,$$

$$S := \sum_{A,B} \epsilon^{p,\infty}(A,B) \sum_{\varphi_H \in \Phi_H(\varphi_{\mathbb{V}^*})} \det(\omega_*(\varphi_H))\Phi^H_{M'}(\gamma'^{-1}, \Theta_{\varphi_H}),$$

for an arbitrary element $\mathfrak{e} = (M', {}^L M', s'_{M^*}, \eta_{M^*}) \in \dot{\mathscr{E}}(M^*)^{c,\mathrm{ur}}$. Here $\dot{\mathscr{E}}(M^*)^{c,\mathrm{ur}}$ is defined in the beginning of the proof of Theorem 8.15.2, and in its definition we do impose that its elements $(M', {}^L M', s'_{M^*}, \eta_{M^*})$ should be such that M' is cuspidal (which was automatic in the odd case). In all the above summations, B runs through all subsets of $[t]$, while A only runs through *even-cardinality* subsets of $[r]$, because otherwise the resulting group H will not be cuspidal. On the other hand, indeed all choices of (A,B) with A having even cardinality will contribute, in the sense that if we write $\mathfrak{e} = \mathfrak{e}_{d^+,\delta^+,d^-,\delta^-}(M^*)$, then the usual formula $\mathfrak{e}_{d^+ + 2|A|+4|B|,\delta^+,d^- + 2|A^c|+4|B^c|,\delta^-}$ as in §8.16.1 defines an elliptic endoscopic datum $(H, {}^L H, s, \eta)$ for G. In other words, neither of $(d^+ + 2\,|A| + 4\,|B|\,, \delta^+)$ and $(d^- + 2\,|A^c| + 4\,|B^c|\,, \delta^-)$ is equal to $(2,1) \in \mathbb{Z}_{\geq 0} \times (\mathbb{Q}^\times/\mathbb{Q}^{\times,2})$. To see this, we recall that M^{SO} is assumed not to be the split SO_2 over $\mathbb{Q}$, so neither of $(d^\pm, \delta^\pm)$ is $(2,1)$. Then since $|A|$ and $|A^c|$ are even it is clear that neither of $(d^+ + 2\,|A| + 4\,|B|\,, \delta^+)$ and $(d^- + 2\,|A^c| + 4\,|B^c|\,, \delta^-)$ is $(2,1)$.

Since we are in the even case, when choosing B_0 as in the proof of Theorem 8.15.2, by making a different choice we can only replace an *even number* of the τ's by their inverses. This means that in condition **C**, we may not be able to arrange (8.15.2.4). Nevertheless, it is easy to see that we can always arrange either of the following two conditions:

– The original condition **C**.

– The modification of condition **C** where (8.15.2.2) and (8.15.2.3) are still in force, and (8.15.2.4) is replaced by the following condition:

$$\left|\epsilon_i(\gamma'^{-1})\right| < 1 \text{ for all } i < r, \text{ and } 1 < \left|\epsilon_r(\gamma'^{-1})\right| < \min_{r' < i < r} \left|\epsilon_i(\gamma'^{-1})\right|^{-1}.$$

In either case, it is still true that $\epsilon_{R_H}(\gamma'^{-1})$ is independent of (A,B). Moreover, (8.15.2.10) still holds, and it reads $\mathrm{sgn}(\sigma) = \omega_0(A)$ since $|A|$ is even. Instead of

(8.15.2.11) we have $q(H_{\mathbb{R}}) \equiv 0 \mod 2$ by the cuspidality of H. Hence

$$R_i = \text{Const.} \sum_{A,B} \nabla_i(A)\epsilon^{p,\infty}(A,B)\omega_0(A)\Phi(\gamma', A),$$

$$T_j = \text{Const.} \sum_{A,B} \nabla_j(B)\epsilon^{p,\infty}(A,B)\omega_0(A)\Phi(\gamma', A),$$

$$S = \text{Const.} \sum_{A,B} \epsilon^{p,\infty}(A,B)\omega_0(A)\Phi(\gamma', A).$$

To compute $\epsilon^{p,\infty}(A,B)$, in the proof of Theorem 8.15.2 we used the fact that the quadratic space $V_{\mathbb{Q}_p}$ is quasi-split. This is now an assumption in the current theorem. When we showed that the Whittaker normalization between M' and M^* at ∞ is inherited from the Whittaker normalization between H and G^* at ∞ in the even case in the proof of Proposition 8.9.5, we used that $m^- \equiv n^- \mod 2$. This is indeed true here since $m^- = n^- + |A^c| + 2\,|B^c|$ and we know that $|A^c| = r - |A|$ is even. Thus by the same argument as in the proof of Theorem 8.15.2, $\epsilon^{p,\infty}(A,B)$ is up to a multiplicative constant equal to the ratio of the Whittaker normalization between H and G at ∞ to the normalization $\Delta_{j,B_{G,H}}$. This ratio is equal to

$$(-1)^{\lfloor m^-/2 \rfloor} = (-1)^{\lfloor \frac{n^- + |A^c| + 2|B^c|}{2} \rfloor}$$

as shown in the proof of Proposition 8.9.5. Hence

$$\epsilon^{p,\infty}(A,B) = \text{Const.}(-1)^{|B| + |A|/2},$$

and we have

$$R_i = \text{Const.} \sum_{A,B} \nabla_i(A)(-1)^{|B|+|A|/2}\omega_0(A)\Phi(\gamma', A),$$

$$T_j = \text{Const.} \sum_{A,B} \nabla_j(B)(-1)^{|B|+|A|/2}\omega_0(A)\Phi(\gamma', A),$$

$$S = \text{Const.} \sum_{A,B} (-1)^{|B|+|A|/2}\omega_0(A)\Phi(\gamma', A).$$

Since $|A|$ is even, we have $\omega_0(A) = \omega_0(A^c)$. In particular,

$$\omega_0(A)(-1)^{|A|/2}\omega_0(A^c)(-1)^{|A^c|/2} = (-1)^{r/2}. \tag{8.16.2.1}$$

We now start to show the vanishing of R_i, T_j, S. As in the proof of Proposition 8.15.3, the case where $t \geq 2$ is the easiest. In this case we have

$$\sum_B (-1)^{|B|} = \sum_B \nabla_j(B)(-1)^{|B|} = 0,$$

so $R_i = T_j = S = 0$. Now consider the case where $t = 1$ and $r = 2$. Then $R_i = S = 0$ because $\sum_B (-1)^{|B|} = 0$. To show $T_j = 0$, we use the fact that (8.16.2.1) is equal to -1 and $\Phi(\gamma', A) = \Phi(\gamma', A^c)$.

Finally we treat the case where $r \geq 4$. The corresponding discussion in §8.15 for $r \geq 3$ needs almost no change to be carried over here. The only differences are:

– All the sets $I^+, I^-, A^+, A^{c,+}, A^-, A^{c,-}$ have to have even cardinality in the present case.

– The root system $R_{A,\gamma'}$ in the present case is of type $\mathsf{D}_{|A^+|} \times \mathsf{D}_{|A^{c,+}|} \times \mathsf{D}_{|A^-|} \times \mathsf{D}_{|A^{c,-}|}$.

– Herb's formula reads

$$\bar{c}_{R_{A,\gamma'}}(x,\chi) = \text{Const.} \sum_{p_1^+ \in \mathcal{P}^0_{\leq 2}(A^+)} \sum_{p_1^- \in \mathcal{P}^0_{\leq 2}(A^-)} \sum_{p_2^+ \in \mathcal{P}^0_{\leq 2}(A^{c,+})} \sum_{p_2^- \in \mathcal{P}^0_{\leq 2}(A^{c,-})} \epsilon(p_1^+)\epsilon(p_1^-)\epsilon(p_2^+)\epsilon(p_2^-) c_{\mathsf{D}}(p_1^+,\mu) c_{\mathsf{D}}(p_2^+,\mu) c_{\mathsf{D}}(p_1^-,\mu) c_{\mathsf{D}}(p_2^-,\mu). \tag{8.16.2.2}$$

As in the proof of Proposition 8.15.3, define

$$M_i := \sum_A \nabla_i(A)\omega_0(A)(-1)^{|A|/2}\bar{c}_{R_{A,\gamma'}}(x,\chi),$$
$$N := \sum_A \omega_0(A)(-1)^{|A|/2}\bar{c}_{R_{A,\gamma'}}(x,\chi),$$

where A runs through subsets of $[r]$ such that $|A^{\pm}|$ and $|A^{c,\pm}|$ are all even. Then the desired vanishing of R_i, T_j, S reduces to the vanishing of M_i and N, which we now show.

Write $k = |I^+|/2, l = |I^-|/2$. (They are both integers.) For $i \in I^+$, $p^+ \in \mathcal{P}^0_{\leq 2}(I^+)$, and $U \subset p^+$, define

$$\nabla_i(p^+,U) := \begin{cases} 1, & \text{if } i \in I \text{ for some } I \in U, \\ -1, & \text{otherwise.} \end{cases}.$$

Similarly, for $i \in I^-$, $p^- \in \mathcal{P}^0_{\leq 2}(I^-)$, and $W \subset p^-$, we define $\nabla_i(p^-,W)$.

Herb's formula (8.16.2.2) together with a similar argument as in the proof of Proposition 8.15.4 implies that

$$N = \sum_{p^+ \in \mathcal{P}^0_{\leq 2}(I^+)} \sum_{p^- \in \mathcal{P}^0_{\leq 2}(I^-)} \epsilon(p^+)\epsilon(p^-)c_{\mathsf{D}}(p^+,\mu)c_{\mathsf{D}}(p^-,\mu) \sum_{U \subset p^+, W \subset p^-} (-1)^{|U|+|W|}$$
$$= \text{Const.} \sum_{U \subset [k], W \subset [l]} (-1)^{|U|+|W|},$$

and for $i \in [r]$

$$M_i = \sum_{p^+ \in \mathcal{P}^0_{\leq 2}(I^+)} \sum_{p^- \in \mathcal{P}^0_{\leq 2}(I^-)} \epsilon(p^+)\epsilon(p^-)c_{\mathsf{D}}(p^+,\mu)c_{\mathsf{D}}(p^-,\mu)$$
$$\cdot \sum_{U \subset p^+, W \subset p^-} (-1)^{|U|+|W|}\nabla_i(p^+,p^-,U,W),$$

where

$$\nabla_i(p^+,p^-,U,W) := \begin{cases} \nabla_i(p^+,U), & \text{if } i \in I^+, \\ \nabla_i(p^-,W), & \text{if } i \in I^-. \end{cases}$$

Since $lk \neq 0$, we have $N = 0$. We now show $M_i = 0$. Fix $p^+ \in \mathcal{P}^0_{\leq 2}(I^+), p^- \in \mathcal{P}^0_{\leq 2}(I^-)$. It suffices to show that

$$L := \sum_{U \subset p^+, W \subset p^-} \nabla_i(p^+,p^-,U,W)(-1)^{|U|+|W|}$$

is zero. By symmetry we may assume that $i \in I^-$. After fixing an enumeration of the elements of p^- such that the first element contains i, we get

$$L = \sum_{U \subset [k], W \subset [l]} \nabla_1(W)(-1)^{|U|+|W|}.$$

If $k \geq 1$, then $L = 0$ because $\sum_{U \subset [k]} (-1)^{|U|} = 0$. If $k = 0$, then $l = r/2 \geq 2$, and $L = 0$ because $\sum_{W \subset [l]} \nabla_1(W)(-1)^{|W|} = 0$ as in (8.15.3.2). This concludes the proof. □

8.17. The main identity

8.17.1. — Keep the notation and setting in §1.8.3 and Theorem 1.8.4. Fix a prime $p \notin \Sigma(\mathbf{O}(V), \mathbb{V}, \lambda, K, f^\infty)$. In the even case, assume that the quadratic space (V,q) is quasi-split over $\mathbb{Q}_p$, or equivalently, that its Hasse invariant at p is trivial (cf. §8.16.1). Let $f^{p,\infty}$ and $dg^{p,\infty}$ be as in §1.8.3. Fix a set $\dot{\mathscr{E}}(G)$ of representatives of the isomorphism classes in $\mathscr{E}(G)$ such that each element of $\dot{\mathscr{E}}(G)$ is of the form $\mathfrak{e}_\mathfrak{p}$ for some $\mathfrak{p} = (d^+, \delta^+, d^-, \delta^-) \in \mathscr{P}_V$ with $d^+ \geq 2$ (cf. §8.4.1). As in §8.4.1, assume that $\mathbb{V}$ is absolutely irreducible. Then for each $\mathfrak{e}_\mathfrak{p} = (H, {}^L H, s, \eta) \in \dot{\mathscr{E}}(G)$, we have a test function $f^H \in C_c^\infty(H(\mathbb{A}))$ fixed in §8.4.

Corollary 8.17.2. — *For $a \in \mathbb{Z}_{\geq 1}$ large enough, we have*

$$\mathrm{Tr}_{M_1}(f^{p,\infty}dg^{p,\infty}, K, a) + \mathrm{Tr}_{M_2}(f^{p,\infty}dg^{p,\infty}, K, a) + \mathrm{Tr}_{M_{12}}(f^{p,\infty}dg^{p,\infty}, K, a) = \sum_{(H, {}^L H, s, \eta) \in \dot{\mathscr{E}}(G)} \iota(G,H)[ST^H(f^H) - ST^H_e(f^H)].$$

Here $\iota(G,H) := \tau(G)\tau(H)^{-1} \left|\mathrm{Out}(H, {}^L H, s, \eta)\right|^{-1}$, and $ST^H_e(f^H) := ST^H_H(f^H)$ as defined in §8.3.

Proof. — The right hand side of the desired identity is by definition

$$\sum_{(H, {}^L H, s, \eta) \in \dot{\mathscr{E}}(G)} \left|\mathrm{Out}(H, {}^L H, s, \eta)\right|^{-1} \sum_L (n^H_L)^{-1} \tau(G)\tau(H)^{-1} ST^H_L(f^H),$$

where L runs through a set of representatives of the $H(\mathbb{Q})$-conjugacy classes of proper Levi subgroups of H (cf. §8.3). By an observation of Kottwitz which can be verified directly in our case (see also [**Mor10b**, Lem. 2.4.2]), the above is equal to

$$\sum_{M\in\{M_1,M_2,M_{12}\}} \mathrm{Tr}'_M + \sum_{M^*} \mathrm{Tr}'_{M^*},$$

where

– For each $M \in \{M_1, M_2, M_{12}\}$, the term Tr'_M is as in §8.5.1.

– The second sum is over cuspidal Levi subgroups M^* of G^* of the form considered in §8.15.1 and §8.16.1 in such a way that each conjugacy class of cuspidal Levi subgroups of G^* that does not transfer to G is represented exactly once, and that no other conjugacy classes show up.[15]

– For each M^*, the term Tr'_{M^*} is as in (8.15.1.1) and (8.16.1.1).

The corollary then follows from Theorems 8.5.2, 8.15.2, 8.16.2. □

Remark 8.17.3. — In Corollary 8.17.2 we *defined* $ST^H_e(f^H)$ to be $ST^H_H(f^H)$, where ST^H_H is defined only when the test function at the archimedean place is stable cuspidal (see §8.3). On the other hand, ST^H_e has a more general definition, namely it is the elliptic part of the stable trace formula for H as in [**Kot86**]. Of course it is expected (and proved in Kottwitz's unpublished notes) that these two definitions agree when the test function at the archimedean place is stable cuspidal. For our particular f^H_∞, this compatibility is essentially proved in [**Kot90**, §7]. In fact, if we substitute the archimedean stable orbital integrals in the general definition of $ST^H_e(f^H)$ by the formula [**Kot90**, (7.4)], then we obtain precisely $ST^H_H(f^H)$.

The following is a special case of the main result of [**KSZ**].

Theorem 8.17.4. — *Keep the setting of §8.17.1. For $a \in \mathbb{Z}_{\geq 1}$ large enough, we have*

$$\mathrm{Tr}(\mathrm{Frob}^a_p \times f^\infty dg^\infty \mid \mathbf{H}^*_c(\mathrm{Sh}_K, \mathbb{V})) = \sum_{(H,{}^L H,s,\eta)\in\mathring{\mathscr{E}}(G)} \iota(G,H) ST^H_e(f^H).$$

□

Corollary 8.17.5. — *For $a \in \mathbb{Z}_{\geq 1}$ large enough, we have*

$$\mathrm{Tr}(\mathrm{Frob}^a_p \times f^\infty dg^\infty \mid \mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})) = \sum_{(H,{}^L H,s,\eta)\in\mathring{\mathscr{E}}(G)} \iota(G,H) ST^H(f^H). \tag{8.17.5.1}$$

Proof. — This follows from Theorem 1.8.4, Corollary 8.17.2, and Theorem 8.17.4. □

[15] Note that in general G^* has Levi subgroups which have direct factors GL_j with $j \geq 3$. These Levi subgroups are not conjugate to the ones considered in §8.15.1 and §8.16.1, but none of them are cuspidal. On the other hand, every cuspidal Levi subgroup of G^* is conjugate to the ones considered in §8.15.1 and §8.16.1.

Remark 8.17.6. — The right hand side of (8.17.5.1) is *a priori* a number in $\mathbb{C}$. However, as we have seen in Theorem 1.8.4, the left hand side is in fact a number in $\mathbb{E}$, the number field over which $\mathbb{V}$ is defined.

CHAPTER 9

APPLICATION: SPECTRAL EXPANSION AND HASSE–WEIL ZETA FUNCTIONS

9.1. Introductory remarks

9.1.1. — In [**Kot90**, Part II], Kottwitz explained how the formula in Corollary 8.17.5 would imply a description of $\sum_i (-1)^i \mathbf{IH}^i(\overline{\mathrm{Sh}_K}, \mathbb{V})$ in the Grothendieck group of $\mathcal{H}(G(\mathbb{A}_f) /\!\!/ K)_{\mathbb{Q}} \times \Gamma_{\mathbb{Q}}$-modules over $\overline{\mathbb{Q}}_\ell$. More precisely, the Grothendieck group is taken with respect to the category of $\mathcal{H}(G(\mathbb{A}_f) /\!\!/ K)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell$-modules which are finite-dimensional over $\overline{\mathbb{Q}}_\ell$ and are equipped with a continuous (with respect to the ℓ-adic topology) $\Gamma_{\mathbb{Q}}$-action that commutes with $\mathcal{H}(G(\mathbb{A}_f) /\!\!/ K)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell$. This description is in terms of the conjectural parametrization of automorphic representations by Arthur parameters. The main hypotheses assumed by Kottwitz are the following (see [**Kot90**, §8]):

(1) Arthur's conjectural parametrization and multiplicity formula for automorphic representations.

(2) The closely related conjectural spectral expansion of the stable trace formula in terms of Arthur parameters.

Recent developments have seen the proof of variations of these hypotheses in specific instances. For the groups that are relevant to this paper, Arthur [**Art13**] has established (1) and (2) for quasi-split special orthogonal groups over number fields, and Taïbi [**Taï19**] has generalized (1) to some inner forms of these groups (and under a regular algebraic assumption). Among the inputs to Taïbi's work are the theory of rigid inner forms established by Kaletha [**Kal16, Kal18**] and results of Arancibia–Moeglin–Renard [**AMR18**] on archimedean Arthur packets. (For the special orthogonal groups of interest to us, only the special case of Kaletha's theory, namely that of pure inner forms, is needed.) We mention that Arthur's work [**Art13**] depends on the stabilization of the twisted trace formula as a hypothesis, and the latter has been established by Moeglin–Waldspurger [**MW17**].[(1)] It is thus possible to combine

[(1)] However, see footnote 3 on p. 3.

Corollary 8.17.5 with the results from [**Art13**] and [**Taï19**] to obtain an unconditional description of $\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})$ in certain special cases. In the following we carry this out, for the special cases described in Lemma 9.4.2.

In the sequel, we shall assume the following hypothesis.

Hypothesis 9.1.2. — *Let H be a quasi-split reductive group over $\mathbb{Q}$. For test functions f on $H(\mathbb{A})$ which are stable cuspidal at infinity, we have $ST^H(f) = S^H(f)$. Here $ST^H(f)$ denotes Kottwitz's simplified geometric side of the stable trace formula (see §8.3), and $S^H(f)$ denotes Arthur's stable trace formula* [**Art02, Art01, Art03**].

This hypothesis essentially follows from Kottwitz's stabilization of the trace formula with stable cuspidal test functions at infinity in his unpublished notes. Recently an alternative proof has been given by Z. Peng [**Pen19**]. Let us make some comments on the former. Firstly we state and prove two lemmas that are well known and independent of Hypothesis 9.1.2.

Lemma 9.1.3. — *Let H be a semi-simple (for simplicity) reductive group over $\mathbb{R}$. Assume that H is cuspidal (Definition 1.1.6). Let $f : H(\mathbb{R}) \to \mathbb{C}$ be a stable cuspidal function (see* [**Art89**, §4], [**Mor10b**, 5.4]*). The following statements hold.*

(1) *The function f is equal to a finite linear combination $\sum_\varphi c_\varphi f_\varphi, c_\varphi \in \mathbb{C}$, where φ runs through the discrete Langlands parameters for H and each f_φ is a stable pseudo-coefficient for the L-packet of φ as in (8.4.3.1).*

(2) *Let $(H', \mathcal{H}', s, \eta : \mathcal{H}' \to {}^L H)$ be an elliptic endoscopic datum for H. For simplicity assume $\mathcal{H}' = {}^L H'$. Then a Langlands–Shelstad transfer of f_φ as in (1) to H' can be taken to be a stable cuspidal function on $H'(\mathbb{R})$ that is supported on those discrete Langlands parameters φ' for H' such that $\eta \circ \varphi'$ is equivalent to φ.*

Proof. — (1) is a formal consequence of the definitions. In fact, by the definition of being stable cuspidal, we know there exists a function f' of the desired form $\sum_{i=1}^k c_i f_{\varphi_i}, c_i \in \mathbb{C}^\times$ such that $\delta := f - f'$ has zero trace on all tempered representations of $H(\mathbb{R})$. By definition we have $f_{\varphi_1} = \sum_\pi f_\pi$, where π runs through the L-packet of φ_1 and each f_π is a pseudo-coefficient of π. Then for one such π we may replace f_π by $f_\pi + \delta/c_1$, which is still a pseudo-coefficient of π. After making this replacement f is precisely equal to $\sum_{i=1}^k c_i f_{\varphi_i}$, with the new definition of f_{φ_1}.

(2) follows from the fact, due to Shelstad (see for instance [**She10b, She08**]), that the spectral transfer factor between a tempered Langlands parameter φ' for H' and a tempered representation π for H vanishes unless π lies in the L-packet of $\eta \circ \varphi'$. For a summary of Shelstad's theory of spectral transfer factors see [**Kal16**, p. 621]. □

Lemma 9.1.4. — *Let H be a semi-simple (for simplicity) reductive group over $\mathbb{Q}$. Assume that H is cuspidal (Definition 1.1.6). Let $f_\infty \in C_c^\infty(H(\mathbb{R}))$ be a stable cuspidal function, and let $f^\infty \in C_c^\infty(H(\mathbb{A}_f))$. Let I_H denote the invariant trace formula* *for H and let $I_{H,\mathrm{disc}} = \sum_{t\geq 0} I_{\mathrm{disc},t}$ denote its discrete part; see* [**Art88**] *and* [**Art89**, §3]. *Then*

$$I_H(f_\infty f^\infty) = I_{H,\mathrm{disc}}(f_\infty f^\infty),$$

and they are also equal to

$$\mathrm{Tr}(f_\infty f^\infty \mid L^2_{\mathrm{disc}}(H(\mathbb{Q})\backslash H(\mathbb{A}))).$$

Proof. — By Lemma 9.1.3 we may assume that $f_\infty = f_\varphi$ for a discrete Langlands parameter φ. Then the lemma follows from [**Art89**, §3] (where our f_φ is equal to the function denoted by f_μ up to a multiplicative constant). □

9.1.5. — We now explain how Kottwitz's stabilization in the aforementioned unpublished notes is related to Hypothesis 9.1.2. For $f_\infty = f_\varphi$ as in the above proof, Arthur [**Art89**] shows that the value $I_H(f_\varphi f^\infty)$ has the interpretation as the L^2 Lefschetz number of a Hecke operator on a locally symmetric space, with coefficients in a sheaf determined by φ. This Lefschetz number is evaluated by Arthur [**Art89**] and independently by Goresky–Kottwitz–MacPherson [**GKM97**]. Hence the general $I_H(f_\infty f^\infty)$ with stable cuspidal f_∞ as in the above lemma is just a linear combination of these Lefschetz number formulas. Based on this, Kottwitz proves in his unpublished notes a stabilization

$$I_H(f_\infty f^\infty) = \sum_{H'\in\mathscr{E}(H)} \iota(H,H') ST^{H'}(f^{H'}), \tag{9.1.5.1}$$

where the terms are explained below:

– The left hand side is as in Lemma 9.1.4.

– In the sum H' runs through the elliptic endoscopic data for H up to isomorphism.

– For each $H' \in \mathscr{E}(H)$, the function $f^{H'}$ is of the form $f^{H'}_\infty f^{H',\infty}$, where $f^{H'}_\infty$ (resp. $f^{H',\infty}$) is a Langlands–Shelstad transfer of f_∞ (resp. of f^∞). Here by Lemma 9.1.3 we may and do take $f^{H'}_\infty$ to be stable cuspidal.

– For each $H' \in \mathscr{E}(H)$, the term $ST^{H'}(f^{H'})$ is the simplified geometric side of the stable trace formula, as recalled in §8.3.

– For each $H' \in \mathscr{E}(H)$, the term $\iota(H,H') \in \mathbb{Q}$ is the usual constant in the stabilization of trace formulas; cf. Corollary 8.17.2.

On the other hand, according to Arthur's stabilization [**Art02**] [**Art01**] [**Art03**], we have

$$I_H(f_\infty f^\infty) = \sum_{H'\in\mathscr{E}(H)} \iota(H,H') S^{H'}(f^{H'}), \tag{9.1.5.2}$$

$$I_{H,\mathrm{disc}}(f_\infty f^\infty) = \sum_{H'\in\mathscr{E}(H)} \iota(H,H') S^{H'}_{\mathrm{disc}}(f^{H'}) \tag{9.1.5.3}$$

where $S^{H'}$ (resp. $S^{H'}_{\mathrm{disc}}$) is Arthur's stable trace formula for H' (resp. the discrete part thereof[(2)]; see [**Art13**, §§3.1, 3.2]), and the rest of the notations are the same as in (9.1.5.1). Comparing (9.1.5.1) and (9.1.5.2) for H quasi-split (so that $H \in \mathscr{E}(H)$) and by induction on the dimension of the group in Hypothesis 9.1.2, we conclude that

$$ST^H(f_\infty f^\infty) = S^H(f_\infty f^\infty).$$

Thus Hypothesis 9.1.2 is proved. Moreover, comparing Lemma 9.1.4 and (9.1.5.2), (9.1.5.3) for H quasi-split and by induction, we also draw the following conclusion independently of Hypothesis 9.1.2:

Proposition 9.1.6. — *Keep the setting of Lemma 9.1.4 and assume in addition that H is quasi-split. Then*

$$S^H(f_\infty f^\infty) = S^H_{\mathrm{disc}}(f_\infty f^\infty).$$

□

Corollary 9.1.7. — *We may replace each ST^H in Corollary 8.17.5 by S^H_{disc}.*

Proof. — This follows from Hypothesis 9.1.2 and Proposition 9.1.6. □

9.2. Review of Arthur's results

We loosely follow [**Taï19**, §2] to recall some of the main constructions and results in [**Art13**]. We fix a quasi-split quadratic space $(\underline{V}, \underline{q})$ over $\mathbb{Q}$, of dimension d and discriminant $\delta \in \mathbb{Q}^\times/\mathbb{Q}^{\times,2}$. (See §1.2 for what we mean by a quasi-split quadratic space.) Let $G^* := \mathrm{SO}(\underline{V}, \underline{q})$. As usual we explicitly fix the L-group ${}^L G^*$, and fix explicit representatives $(H, \mathcal{H} = {}^L H, s, \eta : {}^L H \to {}^L G^*)$ for the isomorphism classes of elliptic endoscopic data for G^*, as discussed in §5.

Self-dual cuspidal automorphic representations of GL_N

9.2.1. — Let $N \in \mathbb{Z}_{\geq 1}$. Let π be a self-dual cuspidal automorphic representation of GL_N over $\mathbb{Q}$. Arthur [**Art13**, Thm. 1.4.1] associates to π a quasi-split orthogonal or symplectic group G_π over $\mathbb{Q}$, such that $\widehat{G_\pi}$ is isomorphic to $\mathrm{Sp}_N(\mathbb{C})$ or $\mathrm{SO}_N(\mathbb{C})$. We view $\mathrm{Sp}_N(\mathbb{C})$ and $\mathrm{SO}_N(\mathbb{C})$ as standard subgroups of $\mathrm{GL}_N(\mathbb{C})$ as in §5.2. There is a standard representation

$$\mathrm{Std}_\pi : {}^L G_\pi \longrightarrow {}^L\,\mathrm{GL}_N = \mathrm{GL}_N(\mathbb{C})$$

(2) More precisely, each of $I_{H,\mathrm{disc}}$ and $S^{H'}_{\mathrm{disc}}$ is formally a sum over a parameter $t \in \mathbb{R}_{\geq 0}$ of respective contributions $I_{H,\mathrm{disc},t}$ and $S^{H'}_{\mathrm{disc},t}$, and (9.1.5.3) could be stated parameter-wise for each t.

extending the inclusion $\widehat{G_\pi} \hookrightarrow \mathrm{GL}_N(\mathbb{C})$ determined as follows. The central character ω_π of π determines a character $\eta_\pi : \Gamma_{\mathbb{Q}} \to \{\pm 1\}$. Let $E/\mathbb{Q}$ be the degree one or two extension given by η_π. When $E = \mathbb{Q}$, the group G_π is split. In this case we may take ${}^L G_\pi = \widehat{G_\pi}$ and there is nothing to do. When $E \neq \mathbb{Q}$, the group G_π is either symplectic, or the non-split quasi-split even special orthogonal group over $\mathbb{Q}$ which is split over E. Thus when $E \neq \mathbb{Q}$ we have $\widehat{G_\pi} = \mathrm{SO}_N(\mathbb{C})$, and we may take ${}^L G_\pi$ to be $\widehat{G_\pi} \rtimes \mathrm{Gal}(E/\mathbb{Q})$ (which is a direct product when G_π is symplectic). When G_π is symplectic, we define Std_π to send the non-trivial element of $\mathrm{Gal}(E/\mathbb{Q})$ to $-1 \in \mathrm{GL}_N(\mathbb{C})$. When G_π is the non-split quasi-split even special orthogonal group, we define Std_π to send the non-trivial element of $\mathrm{Gal}(E/\mathbb{Q})$ to the permutation matrix switching $\hat{e}_{N/2}$ and $\hat{e}_{1+N/2}$ in the notation of §5.2. Thus in the last case Std_π maps ${}^L G_\pi$ isomorphically onto the subgroup $\mathrm{O}_N(\mathbb{C})$ of $\mathrm{GL}_N(\mathbb{C})$ as in §5.2.

Let v be a place of $\mathbb{Q}$. Under the local Langlands correspondence for GL_N, established by Langlands [**Lan89**] in the archimedean case and by Harris–Taylor [**HT01**], Henniart [**Hen00**], and Scholze [**Sch13**] in the non-archimedean case, the local component π_v of π corresponds to a Langlands parameter $\varphi_{\pi_v} : \mathrm{WD}_v \to \mathrm{GL}_N(\mathbb{C})$. Here WD_v denotes the Weil–Deligne group of $\mathbb{Q}_v$ (denoted by $L_{\mathbb{Q}_v}$ in [**Art13**]), which is by definition the Weil group when $\mathbb{Q}_v = \mathbb{R}$, and the direct product of the Weil group with $\mathrm{SU}_2(\mathbb{R})$ when $\mathbb{Q}_v$ is non-archimedean. Arthur shows [**Art13**, Thm. 1.4.1, Thm. 1.4.2] that φ_{π_v} is conjugate to $\mathrm{Std}_\pi \circ \varphi_v$ for some Langlands parameter

$$\varphi_v : \mathrm{WD}_v \longrightarrow {}^L G_\pi. \tag{9.2.1.1}$$

The $\mathrm{Aut}({}^L G_\pi)$-orbit of φ_v is uniquely determined by φ_{π_v}. (See [**Taï19**, §2.1] for $\mathrm{Aut}({}^L G_\pi)$, also cf. Remark 9.2.6 below.) Define

$$\mathrm{sgn}(\pi) := \begin{cases} 1, & \text{if } \widehat{G_\pi} \text{ is orthogonal,} \\ -1, & \text{if } \widehat{G_\pi} \text{ is symplectic.} \end{cases}$$

Substitutes for global Arthur parameters

9.2.2. — Similar to the definition of Std_π above, we have a standard representation

$$\mathrm{Std}_{G^*} : {}^L G^* \longrightarrow \mathrm{GL}_N(\mathbb{C}) \tag{9.2.2.1}$$

where $N = d - 1$ (resp. $N = d$) when d is odd (resp. even).

Let $\Psi(N)$ denote the set of formal unordered sums

$$\psi = \boxplus_{k \in K_\psi} \pi_k[d_k],$$

where K_ψ is a finite indexing set, each π_k is a unitary cuspidal automorphic representation of GL_{N_k} over $\mathbb{Q}$ for some $N_k \in \mathbb{Z}_{\geq 1}$, and each d_k is a positive integer, satisfying

$\sum_k N_k d_k = N$. Let $\widetilde{\Psi}(N)$ denote the set of

$$\psi = \boxplus_{k\in K_\psi} \pi_k[d_k] \in \Psi(N)$$

satisfying the condition that there is an involution $k \mapsto k^\vee$ on the indexing set K_ψ such that $(\pi_k)^\vee \cong \pi_{k^\vee}$ and $d_k = d_{k^\vee}$ for all $k \in K_\psi$. Let $\widetilde{\Psi}_{\mathrm{ell}}(N)$ be the subset of $\widetilde{\Psi}(N)$ defined by the conditions that each π_k should be self-dual and that the pairs (π_k, d_k) should be distinct (i.e., for $k \neq k'$, either π_k is not isomorphic to $\pi_{k'}$ or $d_k \neq d_{k'}$).

For any $\psi \in \widetilde{\Psi}(N)$, we write

$$\psi = \boxplus_{i\in I} \pi_i[d_i] \boxplus_{j\in J} (\pi_j[d_j] \boxplus \pi_j^\vee[d_j]),$$

where π_i is self-dual for each $i \in I$ and π_j is not self-dual for each $j \in J$. Let $\mathcal{L}_\psi$ be the fiber product over $\Gamma_{\mathbb{Q}}$ of ${}^L G_{\pi_i}$ and $\mathrm{GL}_{N_j}(\mathbb{C})$ for all $i \in I, j \in J$. For $j \in J$, we define

$$\begin{aligned} \mathrm{Std}_{N_j} \oplus \mathrm{Std}_{N_j}^\vee : \mathrm{GL}_{N_j}(\mathbb{C}) &\longrightarrow \mathrm{GL}_{2N_j}(\mathbb{C}) \\ g &\longmapsto g \oplus (g^{\mathsf{T}})^{-1}. \end{aligned}$$

Define

$$\tilde{\psi} := (\bigoplus_{i\in I} \mathrm{Std}_{\pi_i} \otimes \nu_{d_i}) \oplus \bigoplus_{j\in J} (\mathrm{Std}_{N_j} \oplus \mathrm{Std}_{N_j}^\vee) \otimes \nu_{d_j} : \mathcal{L}_\psi \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C}),$$

where ν_k denotes the irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension k for any positive integer k. Let $\widetilde{\Psi}(G^*)$ be the set of $\psi \in \widetilde{\Psi}(N)$ for which there exists

$$\dot{\psi} : \mathcal{L}_\psi \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L G^*$$

such that $\mathrm{Std}_{G^*} \circ \dot{\psi}$ is conjugate under $\mathrm{GL}_N(\mathbb{C})$ to $\tilde{\psi}$. Let $\Psi(G^*)$ be the set of pairs $(\psi, \dot{\psi})$ where $\psi \in \widetilde{\Psi}(G^*)$ and $\dot{\psi}$ is a choice as above. For $\psi \in \widetilde{\Psi}(G^*)$, we define

(9.2.2.2)

$$m_\psi := \text{the number of } \dot{\psi} \text{ modulo } \widehat{G^*}\text{-conjugation such that } (\psi, \dot{\psi}) \in \Psi(G^*).$$

We define[(3)]

$$\widetilde{\Psi}_2(G^*) := \widetilde{\Psi}_{\mathrm{ell}}(N) \cap \widetilde{\Psi}(G^*),$$

and define $\Psi_2(G^*)$ to be the preimage of $\widetilde{\Psi}_2(G^*)$ in $\Psi(G^*)$ along the forgetful map $\Psi(G^*) \to \widetilde{\Psi}(G^*)$. Recall that d and δ denote the dimension and discriminant of the quadratic space $\underline{V}$. For $\psi = \boxplus_k \pi_k[d_k] \in \widetilde{\Psi}_{\mathrm{ell}}(N)$, the following condition is equivalent to the condition that $\psi \in \widetilde{\Psi}_2(G^*)$:

– The character $\Gamma_{\mathbb{Q}} \to \{\pm 1\}$ given by $\prod_k \eta_{\pi_k}^{d_k}$ is trivial if G^* is split, and corresponds to the quadratic extension $\mathbb{Q}(\sqrt{\delta})/\mathbb{Q}$ if G^* is non-split, i.e., if d is even and

(3) In [**Taï19**], our $\widetilde{\Psi}_2(G^*)$ and $\Psi_2(G^*)$ are denoted by $\widetilde{\Psi}_{\mathrm{disc}}(G^*)$ and $\Psi_{\mathrm{disc}}(G^*)$ respectively. However, in [**Art13**], the usage of the subscript "disc" is different; see p. 172. We follow [**Art13**] to use the subscript "2" here.

$\delta \notin \mathbb{Q}^{\times,2}$. Moreover

(9.2.2.3) $$\operatorname{sgn}(\pi_k)(-1)^{d_k-1} = (-1)^d$$

for all k.

For $\psi \in \widetilde{\Psi}_2(G^*)$, we know that $m_\psi \leq 2$, and $m_\psi = 2$ if and only if d and all $N_k d_k$ are even; see [**Art13**, p. 47]. In the latter case the two $\widehat{G^*}$-conjugacy classes of $\dot{\psi}$ are interchanged by the non-trivial outer automorphism of $\widehat{G^*} = \mathrm{SO}_d(\mathbb{C})$.

For $(\psi, \dot{\psi}) \in \Psi(G^*)$, we define

$$S_{\dot{\psi}} := \operatorname{Cent}(\dot{\psi}, \widehat{G^*}),$$
$$\mathcal{S}_{\dot{\psi}} := S_{\dot{\psi}}/S^0_{\dot{\psi}} Z(\widehat{G^*})^{\Gamma_{\mathbb{Q}}}.$$

In fact $\mathcal{S}_{\dot{\psi}}$ is isomorphic to a finite power of $\mathbb{Z}/2\mathbb{Z}$. Moreover, $S_{\dot{\psi}}$ is finite if and only if $(\psi, \dot{\psi}) \in \Psi_2(G^*)$, in which case $S_{\dot{\psi}}$ is a finite power of $\mathbb{Z}/2\mathbb{Z}$. These statements follow easily from the description [**Art13**, (1.4.8)] of $S_{\dot{\psi}}$. By abuse of notation we shall write S_ψ and $\mathcal{S}_\psi$ for $S_{\dot{\psi}}$ and $\mathcal{S}_{\dot{\psi}}$ respectively.(4) In the case where $(\psi, \dot{\psi}) \in \Psi_2(G^*)$ (which is the only case relevant to us in practice), our abuse of notation is essentially harmless for the following reason. Since $S_{\dot{\psi}}$ is abelian, it depends on $\dot{\psi}$ only via its $\widehat{G^*}$-conjugacy class, up to canonical isomorphism. Moreover, in the even case with $m_\psi = 2$, it follows from the description [**Art13**, (1.4.9)] of $S_{\dot{\psi}}$ that there is an element of $\mathrm{O}_N(\mathbb{C}) - \mathrm{SO}_N(\mathbb{C}) = \mathrm{O}_N(\mathbb{C}) - \widehat{G^*}$ centralizing $S_{\dot{\psi}}$. Hence in both the odd and even cases, for $(\psi, \dot{\psi}) \in \Psi_2(G^*)$, the group $S_{\dot{\psi}}$ depends only on ψ up to canonical isomorphism. The similar remark applies to $\mathcal{S}_{\dot{\psi}}$. Moreover, it also follows from the above discussion that the $\widehat{G^*}$-conjugacy class of the subgroup $S_{\dot{\psi}} \subset \widehat{G^*}$ depends only on ψ.

For $\psi \in \widetilde{\Psi}(G^*)$, we define $s_\psi \in S_\psi$ by

(9.2.2.4) $$s_\psi := \dot{\psi}(-1), \text{ where } -1 \in \mathrm{SL}_2(\mathbb{C}).$$

(Here we implicitly fix a lift $(\psi, \dot{\psi}) \in \Psi(G^*)$.) We will also need the canonical character

(9.2.2.5) $$\epsilon_\psi : \mathcal{S}_\psi \longrightarrow \{\pm 1\}$$

defined on p. 48 of [**Art13**] using symplectic root numbers. We do not recall its definition here.

Let $(H, {}^L H, s, \eta : {}^L H \to {}^L G^*)$ be an elliptic endoscopic datum for G^*, presented in the explicit form as in §5.4. Recall that H is a direct product $H^+ \times H^-$ of two quasi-split special orthogonal groups over $\mathbb{Q}$. The above discussion for G^* applies

(4) Here we follow the notation of [**Art13**], which differs slightly from that in [**Kot84b**] and [**Taï19**]. In the latter two papers the notation S_ψ refers to a larger group, which in the present case is equal to $S_\psi Z(\widehat{G^*})$ in our notation. More specifically, in our notation we have $S_\psi \supset Z(\widehat{G^*}) = Z(\widehat{G^*})^{\Gamma_{\mathbb{Q}}}$ unless G^* is a non-split SO_2, in which case $S_\psi = Z(\widehat{G^*})^{\Gamma_{\mathbb{Q}}}$ and $Z(\widehat{G^*}) = \widehat{G^*}$. In particular, we see that the formula $S_\psi/S^0_\psi Z(\widehat{G^*})^{\Gamma_{\mathbb{Q}}}$ defines the same group $\mathcal{S}_\psi$ with both interpretations of the notation S_ψ.

equally to H^+ and H^-. We define

$$\widetilde{\Psi}(H) := \widetilde{\Psi}(H^+) \times \widetilde{\Psi}(H^-),$$
$$\Psi(H) := \Psi(H^+) \times \Psi(H^-).$$

Similarly we define $\widetilde{\Psi}_2(H)$ and $\Psi_2(H)$. For $\psi' = (\psi^+, \psi^-) \in \widetilde{\Psi}(H)$, we define

$$\begin{aligned} S_{\psi'} &:= S_{\psi^+} \times S_{\psi^-}, \\ \mathcal{S}_{\psi'} &:= \mathcal{S}_{\psi^+} \times \mathcal{S}_{\psi^-}, \\ s_{\psi'} &:= (s_{\psi^+}, s_{\psi^-}) \in S_{\psi'}, \\ m_{\psi'} &:= m_{\psi^+} m_{\psi^-}, \\ \epsilon_{\psi'} &:= \epsilon_{\psi^+} \otimes \epsilon_{\psi^-} : \mathcal{S}_{\psi'} \longrightarrow \{\pm 1\}. \end{aligned}$$

We have a natural map

$$\begin{aligned} \widetilde{\Psi}(H) &\longrightarrow \widetilde{\Psi}(G^*) \\ (\psi^+, \psi^-) &\longmapsto \psi^+ \boxplus \psi^-, \end{aligned}$$

which we shall denote by

$$\psi' \longmapsto \eta \circ \psi'.$$

Local Arthur packets

9.2.3. — Let v be a place of $\mathbb{Q}$. We abbreviate $G_v^* := G_{\mathbb{Q}_v}^*$. Let $\Psi^+(G_v^*)$ be the set of all *Arthur–Langlands parameters over* $\mathbb{Q}_v$

$$\psi : \mathrm{WD}_v \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L G_v^*$$

satisfying the usual axioms (without the requirement that $\psi(\mathrm{WD}_v)$ is bounded); see [**Taï19**, §2.5]. Let $\Psi(G_v^*)$ be the set of $\psi \in \Psi^+(G_v^*)$ such that $\psi(\mathrm{WD}_v)$ is bounded.

Following [**Art13**, §1.5] we define a subset $\Psi^+_{\mathrm{unit}}(G_v^*)$ of $\psi \in \Psi^+(G_v^*)$ as follows. For any $\psi \in \Psi^+(G_v^*)$, the parameter

$$\mathrm{Std}_{G^*} \circ \psi : \mathrm{WD}_v \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_N(\mathbb{C})$$

gives rise to an irreducible representation $\pi_1 \boxtimes \cdots \boxtimes \pi_r$ of a standard Levi subgroup $\prod_{i=1}^r \mathrm{GL}_{N_i}(\mathbb{Q}_v)$ of $\mathrm{GL}_N(\mathbb{Q}_v)$; see [**Art13**, p. 45] and [**KMSW14**, §1.2.2] for this construction (using the local Langlands correspondence for general linear groups). By definition, ψ is an element of $\Psi^+_{\mathrm{unit}}(G_v^*)$ if and only if the normalized parabolic induction $\pi_1 \times \cdots \pi_r$ of $\pi_1 \boxtimes \cdots \boxtimes \pi_r$ to $\mathrm{GL}_N(\mathbb{Q}_v)$ is irreducible and unitary. As on p. 45 of [**Art13**], we have a chain of subsets

$$\Psi(G_v^*) \subset \Psi^+_{\mathrm{unit}}(G_v^*) \subset \Psi^+(G_v^*).$$

For $\psi \in \Psi^+_{\mathrm{unit}}(G^*_v)$, we define

$$S_\psi := \mathrm{Cent}(\psi, \widehat{G^*}),$$
$$\mathcal{S}_\psi := S_\psi / S^0_\psi Z(\widehat{G^*})^{\Gamma_v}.$$

As in the global case, the group $\mathcal{S}_\psi$ is a finite abelian 2-group. We write $\mathcal{S}^D_\psi$ for its Pontryagin dual group. Denote by $s_\psi \in S_\psi$ the image of $-1 \in \mathrm{SL}_2(\mathbb{C})$ under ψ.

We fix a $\mathbb{Q}_v$-splitting $\mathbf{spl}_v$ for G^*_v. When d is even, let θ_v be the unique non-trivial automorphism of G^*_v fixing $\mathbf{spl}_v$ (which is of order 2). When d is odd we take $\theta_v = \mathrm{id}_{G^*_v}$. For both parities of d, we fix a Whittaker datum $\mathfrak{w}_v$ for G^*_v that is fixed by θ_v. (For instance, in the even case we can construct $\mathfrak{w}_v$ from $\mathbf{spl}_v$ and the choice of a non-trivial character $\mathbb{Q}_v \to \mathbb{C}^\times$ in the usual manner.)

In the even case, if we let $\mathbf{spl}_v$ vary over all $\mathbb{Q}_v$-splittings of G^*_v, then the resulting θ_v's are all of the form $\mathrm{Int}(g)|_{G^*_v}$ for certain $g \in \mathrm{O}(\underline{V})(\mathbb{Q}_v) - G^*(\mathbb{Q}_v)$. In fact, by explicit construction it is easy to see that there is one choice of θ_v that is of the asserted form. To see that *all* choices of θ_v are of the asserted form, use that all $\mathbb{Q}_v$-splittings of G^*_v are conjugate under $G^{*,\mathrm{ad}}(\mathbb{Q}_v)$, and that $G^{*,\mathrm{ad}}(\mathbb{Q}_v)$ naturally acts on $\mathrm{O}(\underline{V})(\mathbb{Q}_v)$ by conjugation since the center of G^* is central in $\mathrm{O}(\underline{V})$. As a consequence of this observation, if we have two choices θ_v and θ'_v, then $\theta_v = \theta'_v \circ \mathrm{Int}(g_0)$ for some $g_0 \in G^*_v(\mathbb{Q}_v)$. In particular, the way in which θ_v permutes isomorphism classes of representations of $G^*_v(\mathbb{Q}_v)$ (resp. conjugacy classes in $G^*_v(\mathbb{Q}_v)$) is the same as the way in which θ'_v permutes these objects.

Let $\psi \in \Psi^+_{\mathrm{unit}}(G^*_v)$. Then Arthur [**Art13**, §1.5] associates to ψ a finite multi-set[(5)] $\widetilde{\Pi}_\psi(G^*_v)$. Here each element of $\widetilde{\Pi}_\psi(G^*_v)$ is a $\{1, \theta_v\}$-orbit of isomorphism classes of finite-length smooth representations[(6)] of $G^*(\mathbb{Q}_v)$, and such an element is allowed to repeat itself for finitely many times in $\widetilde{\Pi}_\psi(G^*_v)$ (thus "multi-set"). If $\psi \in \Psi(G^*_v)$ then these representations are all irreducible and unitary. Moreover, for general $\psi \in \Psi^+_{\mathrm{unit}}(G^*_v)$, there is a canonical map (depending on the choice of $\mathfrak{w}_v$)

$$(9.2.3.1) \qquad \begin{aligned} \widetilde{\Pi}_\psi(G^*_v) &\longrightarrow \mathcal{S}^D_\psi \\ \pi &\longmapsto \langle \cdot, \pi \rangle. \end{aligned}$$

Definition 9.2.4. — We define the *Hecke algebra* $\mathcal{H}(G^*_v)$ as follows. When v is finite, we define $\mathcal{H}(G^*_v)$ to be $C^\infty_c(G^*(\mathbb{Q}_v))$. When $v = \infty$, we fix a maximal compact subgroup $K_\infty \subset G^*(\mathbb{R})$, and define $\mathcal{H}(G^*_v)$ to consist of smooth compactly supported functions on $G^*(\mathbb{R})$ that are bi-finite under K_∞. Moreover for each place v we define

$$\widetilde{\mathcal{H}}(G^*_v) := \mathcal{H}(G^*_v)^{\theta_v = 1},$$

(5) In [**Taï19**], this set is simply denoted by Π_ψ.

(6) By construction these representations are obtained as parabolic inductions of irreducible representations, and are hence finite-length smooth representations.

and define

$$\widetilde{\mathcal{H}}^{\mathrm{st}}(G_v^*) \subset \mathcal{H}(G_v^*)$$

to be the subspace consisting of $f \in \mathcal{H}(G_v^*)$ such that $f - \theta_v^* f$ has all stable orbital integrals equal to 0.

9.2.5. — Let $\psi \in \Psi^+_{\mathrm{unit}}(G_v^*)$. In [**Art13**, Thm. 2.2.1], Arthur gives a characterization of $\widetilde{\Pi}_\psi(G_v^*)$ and the map $\pi \mapsto \langle \cdot, \pi \rangle$, and proves that the linear form

$$\begin{aligned} \Lambda_\psi : \widetilde{\mathcal{H}}(G_v^*) &\longrightarrow \mathbb{C} \\ f &\longmapsto \sum_{\pi \in \widetilde{\Pi}_\psi(G_v^*)} \langle s_\psi, \pi \rangle \operatorname{Tr}(\pi(f dg)) \end{aligned} \tag{9.2.5.1}$$

is *stable*, in the sense that $\Lambda_\psi(f) = 0$ if all stable orbital integrals of f vanish. (In *loc. cit.* these results are explicitly stated only for $\psi \in \Psi(G_v^*)$, but see Remark 9.2.10 below.) We explain the notations. Here dg is a fixed Haar measure on $G^*(\mathbb{Q}_v)$. The summation takes into account the multiplicities of the elements π in the multi-set $\widetilde{\Pi}_\psi(G_v^*)$. For each such element π, which is a $\{1, \theta_v\}$-orbit of representations of $G^*(\mathbb{Q}_v)$, we let $\dot\pi$ be any element of this orbit, and define $\operatorname{Tr}(\pi(f dg)) := \operatorname{Tr}(\dot\pi(f dg))$. Since $f \in \widetilde{\mathcal{H}}(G_v^*)$ is by definition fixed by θ_v and since dg is obviously fixed by θ_v (as θ_v has order at most 2), this definition is independent of the choice of $\dot\pi$.

It is clear from the characterization in [**Art13**, Thm. 2.2.1] that Λ_ψ is independent of the choice of $\mathfrak{w}_v$, although the definition of the map $\pi \mapsto \langle \cdot, \pi \rangle$ depends on $\mathfrak{w}_v$. Moreover, since Λ_ψ is stable, we can naturally extend its domain of definition to $\widetilde{\mathcal{H}}^{\mathrm{st}}(G_v^*)$, and still obtain a stable distribution

$$\begin{aligned} \Lambda_\psi : \widetilde{\mathcal{H}}^{\mathrm{st}}(G_v^*) &\longrightarrow \mathbb{C} \\ f &\longmapsto \Lambda_\psi\left(\frac{f + \theta_v^* f}{2}\right). \end{aligned}$$

In §9.2.3, we observed that different choices of θ_v permute conjugacy classes in $G_v^*(\mathbb{Q}_v)$ in the same way. In particular, $\widetilde{\mathcal{H}}^{\mathrm{st}}(G_v^*)$ is independent of the choice of θ_v. If we view Λ_ψ as being defined over $\widetilde{\mathcal{H}}^{\mathrm{st}}(G_v^*)$, then it is also independent of the choice of θ_v, as follows from the characterization in [**Art13**, Thm. 2.2.1].

Remark 9.2.6. — We note that $\widetilde{\Pi}_\psi(G_v^*)$ depends on $\psi \in \Psi^+_{\mathrm{unit}}(G_v^*)$ only via its orbit under $\operatorname{Aut}({}^L G_v^*)$. In the odd case such an orbit is the same as a $\widehat{G^*}$-conjugacy class, since $\operatorname{Aut}({}^L G_v^*) = (\widehat{G^*})^{\mathrm{ad}}$. In the even case, by contrast, such an orbit could contain up to two $\widehat{G^*}$-conjugacy classes. This is because $\operatorname{Aut}({}^L G_v^*)$ is identified with $\mathrm{O}_N(\mathbb{C})^{\mathrm{ad}}$, whose action on ${}^L G_v^*$ is determined by the following two conditions:

(1) The projection map from ${}^L G_v^*$ to the Galois factor is preserved.

(2) The map ${}^L G_v^* \to {}^L G^* \xrightarrow{\mathrm{Std}_{G^*}} \mathrm{O}_N(\mathbb{C}) \subset \mathrm{GL}_N(\mathbb{C})$ is $\mathrm{O}_N(\mathbb{C})^{\mathrm{ad}}$-equivariant, where $\mathrm{O}_N(\mathbb{C})^{\mathrm{ad}}$ acts on $\mathrm{O}_N(\mathbb{C})$ by conjugation.

In particular, $(\widehat{G^*})^{\mathrm{ad}}$ is of index 2 in $\mathrm{Aut}({}^LG_v^*)$. When the $\mathrm{Aut}({}^LG_v^*)$-orbit of ψ contains two $\widehat{G^*}$-conjugacy classes, one should regard $\widetilde{\Pi}_\psi(G_v^*)$ as the concoction of two conjectural Arthur packets.

Remark 9.2.7. — As remarked in [**Art13**, §1.5], it follows from the work of Moeglin [**Mœg11**] that the multi-set $\widetilde{\Pi}_\psi(G_v^*)$ for $\psi \in \Psi(G_v^*)$ (and therefore also for $\psi \in \Psi^+_{\mathrm{unit}}(G_v^*)$ by construction) is in fact multiplicity free in the non-archimedean case.

9.2.8. — Let $(H, {}^LH, s, \eta)$ be an endoscopic datum for G_v^*, and assume that it is the localization of an elliptic endoscopic datum for G^* over $\mathbb{Q}$. Thus $H = H^+ \times H^-$ is the direct product of two quasi-split special orthogonal groups over $\mathbb{Q}_v$. (Under our assumption, the endoscopic datum $(H, {}^LH, s, \eta)$ over $\mathbb{Q}_v$ itself may still be non-elliptic. More precisely, in the odd case it is always elliptic, while in the even case it is elliptic if and only if either G_v^* is the split SO_2 over $\mathbb{Q}_v$ or neither of $H^\pm$ is the split SO_2 over $\mathbb{Q}_v$; cf. the discussion at the beginning of §7.3.2.)

As in §9.2.3, let $\Psi^+(H), \Psi^+(H^+), \Psi^+(H^-)$ be the sets of all Arthur–Langlands parameters for H, H^+, H^- over $\mathbb{Q}_v$ respectively. We have a natural identification $\Psi^+(H) \cong \Psi^+(H^+) \times \Psi^+(H^-)$, to be viewed as the identity. We define $\Psi^+_{\mathrm{unit}}(H)$ to be the preimage of $\Psi^+_{\mathrm{unit}}(G_v^*)$, defined in §9.2.3, under the map $\Psi^+(H) \to \Psi^+(G_v^*), \psi \mapsto \eta \circ \psi$. Also, we define $\Psi^+_{\mathrm{unit}}(H^\pm)$ in a similar way as in §9.2.3, with G_v^* replaced by the quasi-split special orthogonal group $H^\pm$. We have

$$\Psi^+_{\mathrm{unit}}(H^+) \times \Psi^+_{\mathrm{unit}}(H^-) \subset \Psi^+_{\mathrm{unit}}(H).$$

Indeed, this containment boils down to the fact that every representation of $\mathrm{GL}_N(\mathbb{Q}_v)$ that is the normalized parabolic induction of an irreducible unitary representation of a Levi subgroup is irreducible unitary. In the non-archimedean case this fact is Bernstein's theorem [**Ber84**]. In the archimedean case this fact is implicit in the work of Vogan [**Vog86**] and also follows from Kirillov's conjecture proved by Baruch [**Bar03**] plus the work of Sahi [**Sah89**]. We note, however, that in general

$$\Psi^+_{\mathrm{unit}}(H^+) \times \Psi^+_{\mathrm{unit}}(H^-) \subsetneqq \Psi^+_{\mathrm{unit}}(H).$$

Now let $\psi \in \Psi^+_{\mathrm{unit}}(H)$, and write $\psi^\pm$ for the components of ψ in $\Psi^+(H^\pm)$. Similarly as in §9.2.5, we have stable distributions

$$\Lambda_{\psi^+} : \widetilde{\mathcal{H}}^{\mathrm{st}}(H^+) \longrightarrow \mathbb{C},$$
$$\Lambda_{\psi^-} : \widetilde{\mathcal{H}}^{\mathrm{st}}(H^-) \longrightarrow \mathbb{C},$$

(after fixing Haar measures). We define

$$\widetilde{\mathcal{H}}^{\mathrm{st}}(H) := \widetilde{\mathcal{H}}^{\mathrm{st}}(H^+) \otimes_{\mathbb{C}} \widetilde{\mathcal{H}}^{\mathrm{st}}(H^-).$$

Taking the product of Λ_{ψ^+} and Λ_{ψ^-}, we obtain a stable distribution

$$\Lambda_\psi : \widetilde{\mathcal{H}}^{\mathrm{st}}(H) \longrightarrow \mathbb{C}.$$

We have an expansion of Λ_ψ similar to (9.2.5.1). To make this precise, similarly as in §9.2.3, we fix a $\mathbb{Q}_v$-splitting $\mathbf{spl}_{H^\pm}$ of $H^\pm$, and let $\theta_{H^\pm}$ be the unique non-trivial automorphism of $H^\pm$ fixing $\mathbf{spl}_{H^\pm}$ in the even case, and the identity on $H^\pm$ in the odd case. Fix a Whittaker datum $\mathfrak{w}_{H^\pm}$ for $H^\pm$ that is fixed by $\theta_{H^\pm}$. Then similarly as in §9.2.3, we have the local packet $\widetilde{\Pi}_{\psi^+}(H^+)$, which is a multi-set whose elements are $\langle\theta_{H^+}\rangle$-orbits of isomorphism classes of representations of $H^+(\mathbb{Q}_v)$. Similarly we have $\widetilde{\Pi}_{\psi^-}(H^-)$. Define the packet $\widetilde{\Pi}_\psi(H)$ as the product of $\widetilde{\Pi}_{\psi^\pm}(H^\pm)$, and we regard its elements as $\langle\theta_{H^+}\rangle \times \langle\theta_{H^-}\rangle$-orbits of isomorphism classes of representations of $H(\mathbb{Q}_v) = H^+(\mathbb{Q}_v) \times H^-(\mathbb{Q}_v)$. We have maps $\widetilde{\Pi}_{\psi^\pm}(H^\pm) \to \mathcal{S}^D_{\psi^\pm}$ as in (9.2.3.1), and taking the product we obtain a map $\widetilde{\Pi}_\psi(H) \to \mathcal{S}^D_\psi$, which we still denote by $\pi \mapsto \langle\cdot,\pi\rangle$. Define

$$\widetilde{\mathcal{H}}(H^\pm) := \mathcal{H}(H^\pm)^{\theta_{H^\pm}=1},$$

and

$$\widetilde{\mathcal{H}}(H) := \widetilde{\mathcal{H}}(H^+) \otimes \widetilde{\mathcal{H}}(H^-).$$

We then have the expansion

$$\Lambda_\psi(h) = \sum_{\pi\in\widetilde{\Pi}_\psi(H)} \langle s_\psi,\pi\rangle \operatorname{Tr}(\pi(h)), \qquad \forall h \in \widetilde{\mathcal{H}}(H). \tag{9.2.8.1}$$

Here, as in (9.2.5.1), the summation takes into account the multiplicities, and for each π we define $\operatorname{Tr}(\pi(h))$ to be $\operatorname{Tr}(\dot\pi(h))$ for any $\dot\pi \in \pi$, the Haar measure on $H(\mathbb{Q}_v)$ being implicit.

We comment that the constructions of the packets $\widetilde{\Pi}_{\psi^\pm}(H^\pm)$, the maps from them to $\mathcal{S}^D_{\psi^\pm}$, and the stable distributions $\Lambda_{\psi^\pm}$, are of a slightly more general nature than the previous constructions for G^*_v in §§9.2.3 and 9.2.5, since $\psi^\pm$ may not lie in $\Psi^+_{\mathrm{unit}}(H^\pm)$. Nevertheless, the assumption that $\psi = (\psi^+,\psi^-)$ lies in $\Psi^+_{\mathrm{unit}}(H)$ implies that $\psi^\pm$ can be constructed from a Levi subgroup $M \subset H^\pm$, a parameter in $\Psi(M)$, and a point $\lambda \in \mathfrak{a}^*_M$ as on p. 45 of [**Art13**], in exactly the same way as any element of $\Psi^+_{\mathrm{unit}}(H^\pm)$ can be constructed from such data. The proof of this fact, which is implicitly used in [**Art13**], is an elementary exercise using [**Tad86**, Thm. D] in the non-archimedean case and [**Tad09**] in the archimedean case. Thus the construction using parabolic induction on the representation side and analytic continuation on the character side as indicated on pp. 45–46 of [**Art13**] works for the current $\psi^\pm$ in the same way as it works for elements of $\Psi^+_{\mathrm{unit}}(H^\pm)$.

9.2.9. — Fix $\psi \in \Psi^+_{\mathrm{unit}}(G^*_v)$ and fix a semi-simple element $s \in S_\psi$. Then there is an induced endoscopic datum $(H,\mathcal{H},s,\eta : \mathcal{H} \to {}^LG^*_v)$ over $\mathbb{Q}_v$. Arthur has proved an endoscopic character relation for such ψ and s. For our applications, we only need the case where the endoscopic datum $(H,\mathcal{H},s,\eta)$ is the localization over $\mathbb{Q}_v$ of an elliptic endoscopic datum for G^* over $\mathbb{Q}$, so we assume this for simplicity. Thus as in §9.2.8, $H = H^+ \times H^-$ is the direct product of two quasi-split special orthogonal groups over $\mathbb{Q}_v$, and as usual we choose an identification $\mathcal{H} \cong {}^LH$. We have $\psi = \eta\circ\psi'$ for a unique

$\psi' \in \Psi^+_{\mathrm{unit}}(H)$. As in §9.2.8, we have the stable distribution $\Lambda_{\psi'} : \widetilde{\mathcal{H}}^{\mathrm{st}}(H) \longrightarrow \mathbb{C}$ after fixing a Haar measure dh on $H(\mathbb{Q}_v)$.

The Whittaker datum $\mathfrak{w}_v$ for G^*_v determines a normalization of the transfer factors between H and G^*_v; cf. §6.2.1. For any $f \in \widetilde{\mathcal{H}}(G^*_v)$, let f' be a Langlands–Shelstad transfer in $\mathcal{H}(H)$, with respect to the normalization of transfer factors just mentioned and the Haar measures dg on $G^*_v(\mathbb{Q}_v)$, dh on $H(\mathbb{Q}_v)$. Then $f' \in \widetilde{\mathcal{H}}^{\mathrm{st}}(H)$; see [**Art13**, §2.1] or [**Taï19**, Prop. 3.3.1]. We have the following *endoscopic character relation* ([**Art13**, Thm. 2.2.1 (b)]):

$$\sum_{\pi \in \widetilde{\Pi}_\psi(G^*_v)} \langle s_\psi s, \pi \rangle \operatorname{Tr}(\pi(f dg)) = \Lambda_{\psi'}(f'). \tag{9.2.9.1}$$

Remark 9.2.10. — In [**Art13**, Thm. 2.2.1], the stability of $\Lambda_{\psi'}$ and the relation (9.2.9.1) are explicitly stated only in the case where $\psi \in \Psi(G^*_v)$ and $\psi' \in \Psi(H)$. The generalization to the case where $\psi \in \Psi^+_{\mathrm{unit}}(G^*_v)$ and $\psi' \in \Psi^+_{\mathrm{unit}}(H)$ can be easily obtained by analytic continuation, as explained on p. 46 of [**Art13**].

Unramified parameters and representations

9.2.11. — We complement our exposition with a discussion on how unramified representations appear in local Arthur packets. Keep the setting and notation of §9.2.3, and assume that the place v is finite. We say that a parameter $\psi : \mathrm{WD}_v \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G^*_v$ in $\Psi^+(G^*_v)$ is *unramified*, if the reductive group G^*_v over $\mathbb{Q}_v$ is unramified, and the restriction of ψ to $\mathrm{WD}_v = W_{\mathbb{Q}_v} \times \mathrm{SU}_2(\mathbb{R})$ is trivial on $\mathrm{SU}_2(\mathbb{R})$ and sends every element τ of the inertia subgroup of $W_{\mathbb{Q}_v}$ to $1 \rtimes \tau \in {}^L G^*_v$.

The existence of an unramified $\psi \in \Psi^+(G^*_v)$ by definition presupposes that G^*_v is unramified. We assume that this is the case. Then inside $G^*(\mathbb{Q}_v)$, there is a unique $G^*(\mathbb{Q}_v)$-conjugacy class of hyperspecial subgroups which are compatible with the fixed Whittaker datum $\mathfrak{w}_v$, in the sense of [**CS80**]. Let K^*_v be such a hyperspecial subgroup. Since θ_v fixes $\mathfrak{w}_v$, we know that θ_v stabilizes the $G^*(\mathbb{Q}_v)$-conjugacy class of K^*_v. In particular, θ_v permutes isomorphism classes of K^*_v-unramified representations of $G^*(\mathbb{Q}_v)$.

Lemma 9.2.12. — *Assume that G^*_v is unramified, and let K^*_v be a hyperspecial subgroup of $G^*(\mathbb{Q}_v)$ as in §9.2.11. Let $\psi \in \Psi^+_{\mathrm{unit}}(G^*_v)$. The following statements hold.*

(1) *The packet $\widetilde{\Pi}_\psi(G^*_v)$ contains at most one element that is a $\{1, \theta_v\}$-orbit of K^*_v-unramified representations of $G^*(\mathbb{Q}_v)$. It contains one if and only if ψ is unramified.*

(2) *Assume that ψ is unramified, and let $\pi \in \widetilde{\Pi}_\psi(G^*_v)$ be the unique element that is a $\{1, \theta_v\}$-orbit of K^*_v-unramified representations, as in (1). Then for any $\dot{\pi} \in \pi$, we have $\dim(\dot{\pi}^{K^*_v}) = 1$, or equivalently, $\dot{\pi}$ has a unique K^*_v-unramified Jordan–Hölder constituent. Moreover, the unramified Langlands parameter $\mathrm{WD}_v \to {}^L G^*_v$ of that*

Jordan–Hölder constituent (with respect to the unramified local Langlands correspondence) is in the same $\mathrm{Aut}({}^L G_v^*)$*-orbit (see Remark 9.2.6) as the Langlands parameter* φ_ψ *associated to* ψ*. Here* $\varphi_\psi(w) := \psi(w, \mathrm{diag}(\|w\|^{1/2}, \|w\|^{-1/2}))$ *for* $w \in \mathrm{WD}_v$*.*

(3) *Let* ψ *and* π *be as in (2). We have* $\langle \cdot, \pi \rangle = 1 \in \mathcal{S}_\psi^D$*.*

Proof. — If $\psi \in \Psi(G_v^*)$, then parts (1) and (3) are proved in [**Taï17**, Lem. 4.1.1], and part (2) follows from the characterization in [**Art13**, Thm. 2.2.1]. (In this case, all elements of $\widetilde{\Pi}_\psi(G_v^*)$ are $\{1, \theta_v\}$-orbits of smooth *irreducible* representations of $G^*(\mathbb{Q}_v)$.) For general $\psi \in \Psi^+_{\mathrm{unit}}(G_v^*)$, we know that ψ arises from a standard Levi subgroup $M \subset G^*$, an element $\psi_M \in \Psi(M)$ (i.e., a local Arthur–Langlands parameter for M which is bounded on WD_v), and an element $\lambda \in \mathfrak{a}_M^*$, as on p. 45 of [**Art13**]. The packet $\widetilde{\Pi}_\psi(G_v^*)$ is constructed from the packet $\widetilde{\Pi}_{\psi_M}(M)$ of $M(\mathbb{Q}_v)$-representations associated to ψ_M via a certain parabolic induction process which involves λ; see *loc. cit.* for more details. It is easy to see that ψ is unramified if and only if ψ_M is unramified. Moreover, the obvious analogue of the current lemma holds for (M, ψ_M) in place of (G_v^*, ψ). (More precisely, M is a direct product of several general linear groups and one unramified special orthogonal group. The special case of the lemma for parameters bounded on WD_v, which we have already proved, takes care of the special orthogonal factor of M. The general linear factors are taken care of by the local Langlands correspondence.) The lemma for (G_v^*, ψ) then follows from the lemma for (M, ψ_M), by basic properties of the parabolic induction process used in the definition of $\widetilde{\Pi}_\psi(G_v^*)$. (More specifically, we may assume that the standard parabolic subgroup $P \subset G_v^*$ containing M as the Levi component is compatible with K_v^* in the sense that $G^*(\mathbb{Q}_v) = P(\mathbb{Q}_v)K_v^*$. Let K_M be the hyperspecial subgroup of $M(\mathbb{Q}_v)$ given by the image of $P(\mathbb{Q}_v) \cap K_v^*$ under the projection $P(\mathbb{Q}_v) \to M(\mathbb{Q}_v)$. Then for any irreducible smooth representation τ of $M(\mathbb{Q}_v)$, the parabolic induction $\mathcal{I}_P(\tau)$ of τ to $G^*(\mathbb{Q}_v)$ satisfies $\dim \mathcal{I}_P(\tau)^{K_v^*} = \dim \tau^{K_M} \in \{0, 1\}$. Moreover, when this number is 1, we have compatibility between the unramified Langlands parameter of the unique K_v^*-unramified constituent of $\mathcal{I}_P(\tau)$ and that of τ.) □

9.2.13. — We have an obvious analogue of Lemma 9.2.12 with G_v^* replaced by the group $H = H^+ \times H^-$ over $\mathbb{Q}_v$ as in §9.2.8. To set up the notation, we assume that H is unramified, and let $K_{H^\pm}$ be a hyperspecial subgroup of $H^\pm(\mathbb{Q}_v)$ that is compatible with the Whittaker datum $\mathfrak{w}_{H^\pm}$ for $H^\pm$ (so $K_{H^\pm}$ is unique up to $H^\pm(\mathbb{Q}_v)$-conjugacy). Let $K_H := K_{H^+} \times K_{H^-} \subset H(\mathbb{Q}_v)$. Since $\mathfrak{w}_{H^\pm}$ is fixed by $\theta_{H^\pm}$, we know that elements of the group $\langle \theta_{H^+} \rangle \times \langle \theta_{H^-} \rangle \subset \mathrm{Aut}(H)$ stabilize the $H(\mathbb{Q}_v)$-conjugacy class of K_H. In particular, $\langle \theta_{H^+} \rangle \times \langle \theta_{H^-} \rangle$ permutes isomorphism classes of K_H-unramified representations of $H(\mathbb{Q}_v)$.

Lemma 9.2.14. — *Keep the setting of §9.2.13. Let* $\psi \in \Psi^+_{\mathrm{unit}}(H)$*. The following statements hold.*

(1) *The packet* $\widetilde{\Pi}_\psi(H)$ *contains at most one element that is a* $\langle\theta_{H^+}\rangle \times \langle\theta_{H^-}\rangle$*-orbit of* K_H*-unramified representations of* $H(\mathbb{Q}_v)$*. It contains one if and only if* ψ *is unramified.*

(2) *Assume that* ψ *is unramified. Let* $\pi \in \widetilde{\Pi}_\psi(H)$ *be the unique element that is a* $\langle\theta_{H^+}\rangle\times\langle\theta_{H^-}\rangle$*-orbit of* K_H*-unramified representations, as in (1). Then for any* $\dot\pi \in \pi$*,* $\dot\pi$ *has a unique* K_H*-unramified Jordan–Hölder constituent. Moreover, the unramified Langlands parameter* $\mathrm{WD}_v \to {}^L H$ *of that Jordan–Hölder constituent is in the same* $\mathrm{Aut}({}^L H)$*-orbit as the Langlands parameter* φ_ψ *associated to* ψ*.*

(3) *Let* ψ *and* π *be as in (2). We have* $\langle\cdot,\pi\rangle = 1 \in \mathcal{S}_\psi^D$*.*

Proof. — This follows from Lemma 9.2.12 applied to H^+ and H^- separately. More precisely, write $\psi = (\psi^+,\psi^-)$ with $\psi^\pm \in \Psi^+(H^\pm)$. Although $\psi^\pm$ may not lie in $\Psi^+_{\mathrm{unit}}(H^\pm)$, the proof of Lemma 9.2.12 still applies to $(H^\pm,\psi^\pm)$ in place of (G_v^*,ψ), in view of the comment at the end of §9.2.8. □

The spectral expansion of the discrete part of the stable trace formula

9.2.15. — Consider an elliptic endoscopic datum $(H = H^+ \times H^-, {}^L H, s, \eta)$ for G^* over $\mathbb{Q}$, presented in the explicit form as in §5.4. Let $\psi \in \widetilde{\Psi}(H)$. For each place v of $\mathbb{Q}$, there is a natural *localization*

$$\psi_v = (\psi_v^+,\psi_v^-) \in \Psi^+_{\mathrm{unit}}(H^+_{\mathbb{Q}_v}) \times \Psi^+_{\mathrm{unit}}(H^-_{\mathbb{Q}_v}) \subset \Psi^+_{\mathrm{unit}}(H_{\mathbb{Q}_v})$$

of ψ that is well defined up to the action of $\mathrm{Aut}({}^L H_{\mathbb{Q}_v}) = \mathrm{Aut}({}^L H^+_{\mathbb{Q}_v}) \times \mathrm{Aut}({}^L H^-_{\mathbb{Q}_v})$, and there are natural homomorphisms $S_\psi \to S_{\psi_v}$ and $\mathcal{S}_\psi \to \mathcal{S}_{\psi_v}$; see [**Art13**, §1.4 and pp. 46–47]. Note that the image of $s_\psi \in S_\psi$ under $S_\psi \to S_{\psi_v}$ is precisely s_{ψ_v}.

Let $\widetilde{\mathcal{H}}^{\mathrm{st}}(H)$ be the restricted tensor product of $\widetilde{\mathcal{H}}^{\mathrm{st}}(H_{\mathbb{Q}_v})$ over all places v. More precisely, consider a large enough finite set of prime numbers Σ such that H extends to a reductive group scheme H' over $\mathbb{Z}[1/\Sigma]$, and such that the image of a fixed admissible splitting $\mathrm{Out}(H) \to \mathrm{Aut}(H)$ is contained in $\mathrm{Aut}(H') \subset \mathrm{Aut}(H)$. Then for all primes $p \notin \Sigma$, the function $1_{H'(\mathbb{Z}_p)}$ is in $\widetilde{\mathcal{H}}^{\mathrm{st}}(H_{\mathbb{Q}_p})$. We form the restricted tensor product with respect to these distinguished elements for almost all p. As usual, the result is independent of the choices of Σ and H'.

The discrete part of Arthur's stable trace formula for H is a formal sum

$$S^H_{\mathrm{disc}} = \sum_{t\geq 0} S^H_{\mathrm{disc},t}$$

of stable distributions over all real numbers $t \geq 0$; see [**Art13**, §§3.1, 3.2], and cf. §9.1. For each $t \geq 0$ and any $f \in \widetilde{\mathcal{H}}^{\mathrm{st}}(H)$, we have the following spectral expansion by [**Art13**, Lem. 3.3.1, Prop. 3.4.1, Thm. 4.1.2]:

$$S^H_{\mathrm{disc},t}(f) = \sum_{\psi\in\widetilde{\Psi}(H), t(\psi)=t} m_\psi \left|\mathcal{S}_\psi\right|^{-1} \sigma(\bar S^0_\psi)\epsilon_\psi(s_\psi)\Lambda_\psi(f), \qquad (9.2.15.1)$$

where Λ_ψ is the product[(7)] of the local stable distributions $\Lambda_{\psi_v} : \widetilde{\mathcal{H}}^{\mathrm{st}}(H_{\mathbb{Q}_v}) \to \mathbb{C}$ as in §9.2.8, and $\sigma(\bar{S}_\psi^0)$ is an invariant associated to the following connected complex reductive group (see [**Art13**, Prop. 4.1.1]):

$$\bar{S}_\psi^0 := (S_\psi / Z(\widehat{H})^{\Gamma_{\mathbb{Q}}})^0.$$

Thus formally we have

$$S_{\mathrm{disc}}^H(f) = \sum_{\psi \in \widetilde{\Psi}(H)} m_\psi \left|\mathcal{S}_\psi\right|^{-1} \sigma(\bar{S}_\psi^0) \epsilon_\psi(s_\psi) \Lambda_\psi(f). \tag{9.2.15.2}$$

9.3. Taïbi's parametrization of local packets for certain pure inner forms

9.3.1. — We keep the setting of §9.2. In particular, we fix $G^* = \mathrm{SO}(\underline{V}, \underline{q})$. For each place v of $\mathbb{Q}$, we shall consider a *pure inner form* (G_v, Ξ_v, z_v) of $G_v^* = G_{\mathbb{Q}_v}^*$, by which we mean the following data:

– a reductive group G_v over $\mathbb{Q}_v$;

– an isomorphism $\Xi_v : G^*_{\overline{\mathbb{Q}}_v} \xrightarrow{\sim} (G_v)_{\overline{\mathbb{Q}}_v}$ defined over $\overline{\mathbb{Q}}_v$;

– a (continuous) cocycle $z_v \in Z^1(\Gamma_v, G_v^*)$ such that ${}^\rho\Xi_v^{-1}\Xi_v = \mathrm{Int}(z_v(\rho))^{-1}$ for all $\rho \in \Gamma_v$.

We recall Taïbi's parametrization in [**Taï19**] of the Arthur packets for G_v under special hypotheses. For each place v, note the equivalence of the following conditions:

(1) The image of z_v in $\mathbf{H}^1(\mathbb{Q}_v, G^{*,\mathrm{ad}})$ is trivial.

(2) The reductive group G_v over $\mathbb{Q}_v$ is quasi-split.

Indeed, that (1) implies (2) is clear, and the converse amounts to the assertion that only the trivial element of $\mathbf{H}^1(\mathbb{Q}_v, G^{*,\mathrm{ad}})$ goes to the trivial element of $\mathbf{H}^1(\Gamma_v, \mathrm{Aut}(G^*_{\overline{\mathbb{Q}}_v}))$. This is clear in the odd case since all automorphisms of $G^*_{\overline{\mathbb{Q}}_v}$ are inner. In the even case, this is true because the inner automorphisms form an index 2 subgroup of $\mathrm{Aut}(G^*_{\overline{\mathbb{Q}}_v})$, and in the complement there is an element invariant under Γ_v, for instance the conjugation action on G_v^* by any element of $\mathrm{O}(\underline{V})(\mathbb{Q}_v)$ of determinant -1.

Finite places

9.3.2. — Let v be a finite place of $\mathbb{Q}$. We assume that the image of z_v in $\mathbf{H}^1(\mathbb{Q}_v, G^{*,\mathrm{ad}})$ is trivial, or equivalently (see §9.3.1), that G_v is quasi-split as an abstract reductive group over $\mathbb{Q}_v$. We caution the reader that under our assumption it could still happen that z_v has non-trivial image in $\mathbf{H}^1(\mathbb{Q}_v, G^*)$ (when d is even).

(7) Here it is implicit that if we fix a finite set Σ of primes and fix a reductive model H' of H over $\mathbb{Z}[1/\Sigma]$, then for almost all primes $p \notin \Sigma$ we have $\Lambda_{\psi_p}(1_{H'(\mathbb{Z}_p)}) = 1$. It follows that Λ_ψ is well defined on $\widetilde{\mathcal{H}}^{\mathrm{st}}(H)$, i.e., there is no issue with infinite products.

In the odd case, let θ_{G_v} be the identity automorphism of G_v. In the even case, fix a $\mathbb{Q}_v$-splitting of G_v and let θ_{G_v} be the unique non-trivial automorphism of G_v fixing that splitting (which is of order 2). As we have observed in §9.2.3, the way in which θ_{G_v} permutes isomorphism classes of representations of $G_v(\mathbb{Q}_v)$ or conjugacy classes in $G_v(\mathbb{Q}_v)$ is canonical.

Fix a Whittaker datum $\mathfrak{w}_v$ for G_v^*. As explained in [**Kal11**, §2.2] (cf. Remark 5.1.4), the datum $(\mathfrak{w}_v, \Xi_v, z_v)$ determines a normalization of transfer factors between any endoscopic datum H for G_v and G_v. We denote this normalization by $\Delta_H^{G_v}(\mathfrak{w}_v, \Xi_v, z_v)$. We summarize in the next proposition the construction in [**Taï19**, §3.3].

Proposition 9.3.3. — *For each $\psi \in \Psi^+_{\mathrm{unit}}(G_v^*)$, there is a finite multi-set*[(8)] *$\widetilde{\Pi}_\psi(G_v)$ of $\{1, \theta_{G_v}\}$-orbits of isomorphism classes of finite-length smooth representations of $G_v(\mathbb{Q}_v)$, and a canonical map (depending on $(\mathfrak{w}_v, \Xi_v, z_v)$)*

$$\begin{aligned} \widetilde{\Pi}_\psi(G_v) &\longrightarrow \pi_0(S_\psi)^D \\ \pi &\longmapsto \langle \cdot, \pi \rangle. \end{aligned}$$

Moreover, if all the representations in $\widetilde{\Pi}_\psi(G_v^*)$ are irreducible, then so are those in $\widetilde{\Pi}_\psi(G_v)$. For each semi-simple $s \in S_\psi$ inducing an endoscopic datum $(H, \mathcal{H}, s, \eta)$ over $\mathbb{Q}_v$, we have an endoscopic character relation. For simplicity, we describe it only under the same assumption on $(H, \mathcal{H}, s, \eta)$ as in §9.2.9. As usual fix an identification ${}^L H \cong \mathcal{H}$. Let $\psi' \in \Psi^+_{\mathrm{unit}}(H)$ be such that $\psi = \eta \circ \psi'$. Fix Haar measures on $G_v(\mathbb{Q}_v)$ and $H(\mathbb{Q}_v)$. Let $f \in \mathcal{H}(G_v)$, and assume that the orbital integrals of f are invariant under θ_{G_v}. Let $f' \in \mathcal{H}(H)$ be a Langlands–Shelstad transfer of f with respect to the normalization $\Delta_H^{G_v}(\mathfrak{w}_v, \Xi_v, z_v)$ of transfer factors. Then we have $f' \in \widetilde{\mathcal{H}}^{\mathrm{st}}(H)$, and

$$\sum_{\pi \in \widetilde{\Pi}_\psi(G_v)} \langle s_\psi s, \pi \rangle \operatorname{Tr}(\pi(f)) = \Lambda_{\psi'}(f').$$

Here we understand that $s, s_\psi \in S_\psi$ are naturally mapped into $\pi_0(S_\psi)$ in writing $\langle s_\psi s, \pi \rangle$.

Proof. — In [**Taï19**, §3.3], it is assumed that $\psi \in \Psi(G_v^*)$, and $\langle \cdot, \pi \rangle$ is constructed as a character on $\mathcal{S}_\psi^+$ rather than a character on $\pi_0(S_\psi)$. Here $\mathcal{S}_\psi^+$ is a certain finite extension of $\mathcal{S}_\psi$ sitting in a chain of surjective group homomorphisms

$$\mathcal{S}_\psi^+ \longrightarrow \pi_0(S_\psi) \longrightarrow \mathcal{S}_\psi.$$

We indicate why the reformulation as in the present proposition is valid.

We first note that the construction in [**Taï19**, §3.3] generalizes verbatim from $\psi \in \Psi(G_v^*)$ to $\psi \in \Psi^+_{\mathrm{unit}}(G_v^*)$, based on the "$\Psi^+_{\mathrm{unit}}$-version" of Arthur's results recalled in §§9.2.3–9.2.9 and Remark 9.2.10. Moreover the finite-length and irreducible properties stated in the proposition follow from the corresponding properties of $\widetilde{\Pi}_\psi(G_v^*)$, since

[(8)] This is denoted by $\Pi_\psi(G_v)$ in [**Taï19**, §3.3]. By its construction and by Remark 9.2.7, this multi-set is actually multiplicity free.

by construction $\widetilde{\Pi}_\psi(G_v)$ contains the same representations as $\widetilde{\Pi}_\psi(G_v^*)$, with respect to a certain $\mathbb{Q}_v$-isomorphism $G_v^* \xrightarrow{\sim} G_v$ which we do not explain.

It remains to explain why it is valid to replace $\mathcal{S}_\psi^+$ by $\pi_0(S_\psi)$ (which is denoted by $\pi_0(C_\psi)$ in [**Taï19**]). The reason that one needs to consider $\mathcal{S}_\psi^+$ in general is due to the fact that when G_v is fixed as a rigid inner form of G_v^*, in order to normalize transfer factors between an endoscopic datum and G_v one needs to upgrade the former to a "refined endoscopic datum", which roughly means picking a lift in $\mathcal{S}_\psi^+$ of the image of $s \in S_\psi$ in $\pi_0(S_\psi)$. In our present case, this is not necessary thanks to the fact that (G_v, Ξ_v, z_v) is a pure inner form of G^*: Each semi-simple element $s \in S_\psi$ determines an endoscopic datum $(H, \mathcal{H}, s, \eta)$, and the datum $(\mathfrak{w}_v, \Xi_v, z_v)$ already determines canonically a normalization of transfer factors between H and G_v. Moreover, as noted in [**Taï19**, Rmk. 3.3.2], the pairing $\langle \cdot, \pi\rangle$ for $\pi \in \widetilde{\Pi}(G_v)$ descends to a character on $\pi_0(S_\psi)$ in our case. In conclusion it is valid to replace the group $\mathcal{S}_\psi^+$ in [**Taï19**, §3.3] by $\pi_0(S_\psi)$ in our case. ☐

The archimedean place

9.3.4. — Let $v = \infty$. Assume that G_v^* contains anisotropic maximal tori. Let (G_v, Ξ_v, z_v) be an arbitrary pure inner form of G_v^* as in §9.3.1. Thus G_v also contains anisotropic maximal tori. As in the non-archimedean case, we fix a Whittaker datum $\mathfrak{w}_v$ for G_v^*, and then the datum $(\mathfrak{w}_v, \Xi_v, z_v)$ determines a normalization of transfer factors between any endoscopic datum H for G_v and G_v, which we denote by $\Delta_H^{G_v}(\mathfrak{w}_v, \Xi_v, z_v)$.

Recall that any Arthur–Langlands parameter $\psi \in \Psi^+(G_v^*)$ (through its associated Langlands parameter φ_ψ) has a well-defined *infinitesimal character*, which is an $\Omega_{\mathbb{C}}(G,T)$-orbit in $X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}$. Here T is any maximal torus in $G_{\mathbb{C}}^*$, and $\Omega_{\mathbb{C}}(G,T)$ is the complex Weyl group. For an account see for instance [**Taï17**, §4.1.2], where the infinitesimal character is denoted by μ_1. Following the terminology of Buzzard–Gee [**BG14**], we say that the infinitesimal character is *C-algebraic* (resp. *regular C-algebraic*) if it is the $\Omega_{\mathbb{C}}(G,T)$-orbit of an element of $\rho + X^*(T)$ (resp. a regular element of $\rho + X^*(T)$), where $\rho \in \frac{1}{2}X^*(T)$ is the half sum of a system of positive roots.

For $\psi \in \Psi^+(G_v^*)$, we say that it is *Adams–Johnson* if it is bounded on $W_{\mathbb{R}}$ (i.e., $\psi \in \Psi(G_v^*)$) and has regular C-algebraic infinitesimal character. For more details see [**Taï17**, §4.2.2] and [**AMR18**, §8.1]. We denote by $\Psi^{\mathrm{AJ}}(G_v^*)$ the set of Adams–Johnson parameters for G_v^*. We know that all $\psi \in \Psi^{\mathrm{AJ}}(G_v^*)$ are *discrete*, in the sense that $S_\psi = \pi_0(S_\psi)$.

For each $\psi \in \Psi^{\mathrm{AJ}}(G_v^*)$, Adams–Johnson [**AJ87**] have explicitly constructed a packet $\Pi_\psi^{\mathrm{AJ}}(G_v)$ of representations of $G_v(\mathbb{R})$. Using the rigidifying datum $(\mathfrak{w}_v, \Xi_v, z_v)$, Taïbi [**Taï19**, §§3.2.2–3.2.3] associates to each $\pi \in \Pi_\psi^{\mathrm{AJ}}(G_v)$ a character $\langle \cdot, \pi\rangle$ of $\mathcal{S}_\psi^+$.

Here as in the proof of Proposition 9.3.3 the finite group $\mathcal{S}_\psi^+$ sits in a chain of surjective group homomorphisms

$$\mathcal{S}_\psi^+ \longrightarrow \pi_0(S_\psi) \longrightarrow \mathcal{S}_\psi,$$

and its introduction is in fact unnecessary in our situation thanks to the fact that we have fixed G_v as a pure inner form of G_v^* (as opposed to a more general rigid inner form). Namely, for each $\psi \in \Psi^{\mathrm{AJ}}(G_v^*)$ and $\pi \in \Pi_\psi^{\mathrm{AJ}}(G_v)$, the pairing $\langle \cdot, \pi\rangle$ descends to a character on $\pi_0(S_\psi) = S_\psi$. This assertion could either be directly checked by going through Taïbi's construction, or be proved as follows: By the well-definedness of the normalization $\Delta_H^{G_v}(\mathfrak{w}_v, \Xi_v, z_v)$ of transfer factors between an endoscopic datum H and the pure inner form G_v, we know that the right hand side of the endoscopic character relation in [**Taï19**, Prop. 3.2.5] depends on $\dot{s} \in \mathcal{S}_\psi^+$ only via its image in $\pi_0(S_\psi) = S_\psi$. It follows that so does the left hand side, which means that $\langle \cdot, \pi\rangle$ descends to S_ψ as desired.

With the above modification, we summarize the results in [**Taï19**, §§3.2.2–3.2.3] together with a comparison result in [**AMR18**] as follows.

Proposition 9.3.5. — *For any $\psi \in \Psi^{\mathrm{AJ}}(G_v^*)$, let $\Pi_\psi^{\mathrm{AJ}}(G_v)$ be the associated (finite) Adams–Johnson packet. There is a canonical map (depending on $(\mathfrak{w}_v, \Xi_v, z_v)$)*[(9)]

$$\Pi_\psi^{\mathrm{AJ}}(G_v) \longrightarrow \pi_0(S_\psi)^D = S_\psi^D$$
$$\pi \longmapsto \langle \cdot, \pi\rangle_{\mathrm{AJT}}.$$

Fix $s \in S_\psi$, and let $(H, \mathcal{H}, s, \eta)$ be the induced endoscopic datum over $\mathbb{R}$, which is necessarily an elliptic endoscopic datum because S_ψ is discrete. We have an endoscopic character relation described as follows. As usual fix an identification ${}^L H \cong \mathcal{H}$, and let $\psi' \in \Psi^+(H)$ be such that $\psi = \eta \circ \psi'$. Then $\psi' \in \Psi^{\mathrm{AJ}}(H)$. Fix Haar measures on $G_v(\mathbb{R})$ and $H(\mathbb{R})$. The following statements hold.

(1) *Fix a Haar measure dh on $H(\mathbb{R})$. The distribution*

$$\Lambda_{\psi'}^{\mathrm{AJ}} : C_c^\infty(H(\mathbb{R})) \longrightarrow \mathbb{C}$$
$$f' \longmapsto \sum_{\pi' \in \Pi_{\psi'}^{\mathrm{AJ}}(H)} \langle s_{\psi'}, \pi'\rangle_{\mathrm{AJT}} \ \operatorname{Tr}(\pi'(f' dh))$$

is stable.

(2) *For $f' \in \widetilde{\mathcal{H}}^{\mathrm{st}}(H)$, we have*

$$\Lambda_{\psi'}^{\mathrm{AJ}}(f') = \Lambda_{\psi'}(f').$$

Here $\Lambda_{\psi'}$ is as in §9.2.8.

(3) *Fix a Haar measure dg on $G_v(\mathbb{R})$. Let $f \in C_c^\infty(G_v(\mathbb{R}))$, and let f' be a Langlands–Shelstad transfer of f in $C_c^\infty(H(\mathbb{R}))$, with respect to the normalization*

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(9) Here the subscript "AJT" stands for Adams–Johnson–Taïbi.

$\Delta_H^{G_v}(\mathfrak{w}_v, \Xi_v, z_v)$ *of transfer factors and the Haar measures* dg, dh. *We have*

$$e(G_v) \sum_{\pi \in \Pi_\psi^{\mathrm{AJ}}(G_v)} \langle s_\psi s, \pi \rangle_{\mathrm{AJT}} \ \mathrm{Tr}(\pi(f)) = \Lambda_{\psi'}^{\mathrm{AJ}}(f'). \tag{9.3.5.1}$$

Here $e(G_v)$ *is the Kottwitz sign of* G_v. □

Remark 9.3.6. — By the formula $\langle s_\psi, \pi_{\psi,\mathbf{Q},\mathbf{L}} \rangle = e(\mathbf{L})$ in the proof of [**Taï19**, Prop. 3.2.5], the distribution $\Lambda_{\psi'}^{\mathrm{AJ}}$ in part (1) of Proposition 9.3.5 is none other than the distribution that appears in [**AJ87**, Thm. 2.13]. With this understanding, part (1) is the same as [**AJ87**, Thm. 2.13], and part (2) is proved in [**AMR18**].

9.4. The global group G

9.4.1. — Fix $d = 2m+1$ or $2m$ where $m \in \mathbb{Z}_{\geq 1}$, and fix $\delta \in \mathbb{Q}^\times/\mathbb{Q}^{\times,2}$. Assume that $(-1)^m\delta > 0$. Let $(\underline{V}, \underline{q})$ be the quasi-split quadratic space (in the sense of Definition 1.2.3) over $\mathbb{Q}$ of dimension d and discriminant δ, which is unique up to isomorphism. Let $G^* = \mathrm{SO}(\underline{V}, \underline{q})$. We note that by our assumption on δ, there exist inner twistings between the $\mathbb{R}$-groups $\mathrm{SO}(d-2,2)$ and $G^*_{\mathbb{R}}$. We fix a $G^*(\mathbb{C})$-conjugacy class[(10)] of such inner twistings, and thereby view $\mathrm{SO}(d-2,2)$ as an inner form of $G^*_{\mathbb{R}}$.

Lemma 9.4.2. — *The following statements hold.*

(1) *There exists at most one isomorphism class* G *of inner forms of* G^* *such that* G *is isomorphic to* $\mathrm{SO}(d-2,2)$ *as inner forms of* G^* *over* $\mathbb{R}$ *and* G *is quasi-split over* $\mathbb{Q}_v$ *as a reductive group (or equivalently,* $G_{\mathbb{Q}_v}$ *is isomorphic to* $G^*_{\mathbb{Q}_v}$ *as inner forms of* $G^*_{\mathbb{Q}_v}$*; see §9.3.1) for all finite places* v.

(2) *Assume either of the following two conditions:*

- $d \equiv 2,3,4,5,6 \mod 8$.
- $d \equiv 0 \mod 8$ *and* $\delta \neq 1 \in \mathbb{Q}^\times/\mathbb{Q}^{\times,2}$.

Then there is a quadratic space (V,q) *over* $\mathbb{Q}$*, of dimension* d*, discriminant* δ*, and signature* $(d-2,2)$ *at* ∞*, such that* $G := \mathrm{SO}(V,q)$ *is quasi-slit at all finite places.*

Proof. — Let F be $\mathbb{Q}$ or $\mathbb{Q}_v$. The set $\mathbf{H}^1(F, G^*)$ classifies isomorphism classes of pure inner forms of G^* over F, and it also classifies isomorphism classes of quadratic spaces (V,q) over F whose dimension is d and discriminant is δ. Thus the lemma is just a reformulation of parts 1,2 of [**Taï19**, Prop. 3.1.2], in the special case where the base number field is $\mathbb{Q}$. In fact, the condition in part 1 of that proposition reads $d \equiv 3,5 \mod 8$. The condition in part 2 (a) reads $d \equiv 2,6 \mod 8$. The condition in part 3 reads $d \equiv 4 \mod 8$, or ($d \equiv 0 \mod 4$ and $\delta \neq 1$). □

(10) In the even case there are two such conjugacy classes to choose from. Nevertheless, the resulting two ways of viewing $\mathrm{SO}(d-2,2)$ as an inner form of $G^*_{\mathbb{R}}$ give rise to isomorphic inner forms of $G^*_{\mathbb{R}}$. This is because the two $G^*(\mathbb{C})$-conjugacy classes of inner twistings are interchanged under any non-inner automorphism of $\mathrm{SO}(d-2,2)_{\mathbb{C}}$, and there exists one such automorphism defined over $\mathbb{R}$.

Remark 9.4.3. — In part (2) of the above lemma, the isomorphism class of (V,q) may not be unique in the even case. The quadratic space $(V,q)\otimes_{\mathbb{Q}}\mathbb{Q}_v$ may not be quasi-split (in the sense of Definition 1.2.3) for all finite places v.

9.4.4. — In the rest of the paper we fix $d\geq 5,\delta,(\underline{V},\underline{q}),G^*$ as in §9.4.1, and fix $(V,q),G$ as in part (2) of Lemma 9.4.2. We shall apply the preceding parts of this paper, in particular Corollary 8.17.5, to (V,q) and G. As in §5.1, we fix an isometry $\phi_V:(V,q)\otimes\overline{\mathbb{Q}}\xrightarrow{\sim}(\underline{V},\underline{q})\otimes\overline{\mathbb{Q}}$, and use it to define the inner twisting $\psi_V:G_{\overline{\mathbb{Q}}}\xrightarrow{\sim}G^*_{\overline{\mathbb{Q}}},g\mapsto\phi_V g\phi_V^{-1}$ as well as the function $u_V:\Gamma_{\mathbb{Q}}\to G^*(\overline{\mathbb{Q}}),\rho\mapsto{}^{\rho}\phi_V\phi_V^{-1}$. To conform with the convention of [**Taï19**], we let Ξ be ψ_V^{-1} and let z be the function $\Gamma_{\mathbb{Q}}\to G^*(\overline{\mathbb{Q}}),\rho\mapsto u_V(\rho)^{-1}$. Then according to that convention it is z rather than u_V that is a cocycle, and (G,Ξ,z) is a global pure inner form of G^* over $\mathbb{Q}$.

At each place v of $\mathbb{Q}$, by localization we obtain a pure inner form (G_v,Ξ_v,z_v) of G^*_v, where $G_v:=G_{\mathbb{Q}_v}$. By construction this pure inner form satisfies the hypothesis in §9.3.2 when v is finite.

We fix once and for all a global Whittaker datum $\mathfrak{w}$ for G^*.

We also fix an automorphism θ_G of G once and for all, as follows. In the odd case let $\theta_G=\mathrm{id}_G$. In the even case, we fix an element $\mathbf{r}$ of $\mathrm{O}(V)(\mathbb{Q})-G(\mathbb{Q})$ of order 2 (for instance, the reflection on V associated to an anisotropic vector), and let $\theta_G=\mathrm{Int}(\mathbf{r})|_G$. Thus in this case θ_G is of order 2.

We know that there exists a large enough finite set Σ of prime numbers such that G^* (resp. G) admits a reductive model $\mathcal{G}$ (resp. $\mathcal{G}^*$) over $\mathbb{Z}[1/\Sigma]$. In particular, for any prime $p\notin\Sigma$, the group G^* (resp. G) is unramified over $\mathbb{Q}_p$, and $\mathcal{G}^*(\mathbb{Z}_p)$ (resp. $\mathcal{G}(\mathbb{Z}_p)$) is a hyperspecial subgroup of $G^*(\mathbb{Q}_p)$ (resp. $G(\mathbb{Q}_p)$). Moreover, we may and shall assume that θ_G stabilizes $\mathcal{G}(\mathbb{Z}_p)$ for all $p\notin\Sigma$, up to enlarging Σ. In fact, the $\mathbb{Q}$-automorphism θ_G of G extends to a $\mathbb{Z}[1/\Sigma]$-automorphism of the model $\mathcal{G}$ after suitably enlarging Σ.

As argued in [**Taï19**, §3.4], we may further enlarge Σ to a finite set of prime numbers, denoted by $\Sigma(\mathcal{G}^*,\mathcal{G},\Xi,z,\mathfrak{w},\theta_G)$, such that the following conditions hold for all primes p outside the set:

(1) As we have already assumed, θ_G stabilizes $\mathcal{G}(\mathbb{Z}_p)$.

(2) The localization $\mathfrak{w}_p$ of $\mathfrak{w}$, which is a Whittaker datum for G^*_p, is compatible with the hyperspecial subgroup $\mathcal{G}^*(\mathbb{Z}_p)\subset G^*(\mathbb{Q}_p)$ in the sense of [**CS80**].

(3) The pure inner form (G_p,Ξ_p,z_p) of G^*_p over $\mathbb{Q}_p$ is trivial. Equivalently, the quadratic spaces $(V,q)\otimes\mathbb{Q}_p$ and $(\underline{V},\underline{q})\otimes\mathbb{Q}_p$ are abstractly isomorphic over $\mathbb{Q}_p$ (but ϕ_V itself may not be defined over $\mathbb{Q}_p$). In particular, we have a canonical $G(\mathbb{Q}_p)$-conjugacy class of $\mathbb{Q}_p$-isomorphisms $G^*_p\xrightarrow{\sim}G_p$, consisting of isomorphisms induced by isometries $V\otimes\mathbb{Q}_p\xrightarrow{\sim}\underline{V}\otimes\mathbb{Q}_p$ that differ from ϕ_V by elements of $G^*(\overline{\mathbb{Q}}_v)$ (as opposed to $\mathrm{O}(\underline{V})(\overline{\mathbb{Q}}_v)$).

(4) Inside the canonical $G(\mathbb{Q}_p)$-conjugacy class of $\mathbb{Q}_p$-isomorphisms $G_p^* \xrightarrow{\sim} G_p$ as in (3), there is one that extends to a $\mathbb{Z}_p$-isomorphism $\mathcal{G}_{\mathbb{Z}_p}^* \xrightarrow{\sim} \mathcal{G}_{\mathbb{Z}_p}$.

Definition 9.4.5. — Let S be a finite set of places of $\mathbb{Q}$. Let ϑ^S be the infinite direct product group $\prod_{v\notin S} \mathbb{Z}/2\mathbb{Z}$, where the product is over all places of $\mathbb{Q}$ outside S. Let ϑ^S act on $G(\mathbb{A}^S)$ by

$$(\epsilon_v)_v \cdot (g_v)_v := (\theta_G^{\epsilon_v}(g_v))_v, \qquad \forall (\epsilon_v)_v \in \vartheta^S, (g_v)_v \in G(\mathbb{A}^S).$$

Since θ_G fixes $\mathcal{G}(\mathbb{Z}_p)$ for almost all primes p, this action is well defined, and each element of ϑ^S acts via a topological group automorphism of $G(\mathbb{A}^S)$. Similarly, we define $\vartheta_S := \prod_{v\in S} \mathbb{Z}/2\mathbb{Z}$ and let ϑ_S act on $\prod_{v\in S} G(\mathbb{Q}_v)$ by the same formula.

9.4.6. — Let v be a finite place of $\mathbb{Q}$ and let $\psi_v \in \Psi^+_{\mathrm{unit}}(G_v^*)$. As in Proposition 9.3.3 the local packet $\widetilde{\Pi}_{\psi_v}(G_v)$ is a set of $\{1, \theta_{G_v}\}$-orbits of isomorphism classes of representations of $G(\mathbb{Q}_v)$, where $\theta_{G_v} \in \mathrm{Aut}(G_v)$ is chosen as in §9.3.2. Since θ_{G_v} is of the form $\mathrm{Int}(g_v)|_{G_v}$ for some $g_v \in \mathrm{O}(V)(\mathbb{Q}_v) - G(\mathbb{Q}_v)$, we have $\theta_G = \theta_{G_v} \circ \mathrm{Int}(h_v)$ for some $h_v \in G(\mathbb{Q}_v)$. Therefore we can view each element of $\widetilde{\Pi}_{\psi_v}(G_v)$ as a $\{1, \theta_G\}$-orbit, or equivalently, a ϑ_v-orbit, of isomorphism classes of representations of $G(\mathbb{Q}_v)$. We normalize the map $\widetilde{\Pi}_{\psi_v}(G_v) \to \pi_0(S_{\psi_v})^D, \pi_v \mapsto \langle \cdot, \pi_v\rangle$ as in Proposition 9.3.3 with respect to the localization $(\mathfrak{w}_v, \Xi_v, z_v)$ of $(\mathfrak{w}, \Xi, z)$ at v, where $(\mathfrak{w}, \Xi, z)$ is fixed in §9.4.4. Similarly, for any $\psi_\infty \in \Psi^{\mathrm{AJ}}(G^*_\infty)$, we have the local packet $\Pi_\psi^{\mathrm{AJ}}(G_\infty)$ as in Proposition 9.3.5, and we normalize the map $\pi \mapsto \langle \cdot, \pi\rangle_{\mathrm{AJT}}$ in that proposition with respect to the localization $(\mathfrak{w}_\infty, \Xi_\infty, z_\infty)$ of $(\mathfrak{w}, \Xi, z)$ at ∞. In the sequel we always keep these normalizations, without explicitly mentioning them.

Now let $\psi \in \widetilde{\Psi}(G^*)$. For each place v of $\mathbb{Q}$, we fix a localization $\psi_v \in \Psi^+_{\mathrm{unit}}(G_v^*)$ of ψ; see §9.2.15. Let S be a finite set of places of $\mathbb{Q}$ containing ∞. We define the global (away from S) Arthur packet $\widetilde{\Pi}^S_\psi(G)$ to be the set of $(\pi_v)_{v\notin S} \in \prod_{v\notin S} \widetilde{\Pi}_{\psi_v}(G_v)$ such that π_v is a ϑ_v-orbit of isomorphism classes of $\mathcal{G}(\mathbb{Z}_v)$-unramified representations for almost all v. (Here note that for almost all v, ϑ_v permutes isomorphism classes of $\mathcal{G}(\mathbb{Z}_v)$-unramified representations.) Now for all primes v not in $\Sigma(\mathcal{G}^*, \mathcal{G}, \Xi, z, \mathfrak{w}, \theta_G)$, the packet $\widetilde{\Pi}_{\psi_v}(G_v)$ together with the map from it to $\pi_0(S_{\psi_v})^D$ is constructed from $\widetilde{\Pi}_{\psi_v}(G_v^*)$ via an isomorphism $G_v \xrightarrow{\sim} G_v^*$ as in §9.4.4 (4); see [**Taï19**, §3.3]. Moreover, ψ_v is unramified for almost all v. Thus for almost all v, by Lemma 9.2.12 applied to $(G_v^*, \psi_v, \mathcal{G}^*(\mathbb{Z}_v))$, there is a unique $\pi_v \in \widetilde{\Pi}_{\psi_v}(G_v)$ which is a ϑ_v-orbit of $\mathcal{G}(\mathbb{Z}_v)$-representations, and moreover for this π_v we have $\langle \cdot, \pi_v\rangle = 1 \in \pi_0(S_{\psi_v})^D$ and $\dim(\dot{\pi}_v)^{\mathcal{G}(\mathbb{Z}_v)} = 1$ for any $\dot{\pi}_v \in \pi_v$. We conclude that for $\pi^S = (\pi_v)_{v\notin S} \in \widetilde{\Pi}^S_\psi(G)$, we have $\langle \cdot, \pi_v\rangle = 1 \in \pi_0(S_{\psi_v})^D$ for almost all v.

For $\pi^S = (\pi_v)_{v\notin S} \in \widetilde{\Pi}^S_\psi(G)$, we choose a member $\dot{\pi}_v \in \pi_v$ for each v, and form the restricted tensor product $\dot{\pi}^S := \bigotimes'_{v\notin S} \dot{\pi}_v$, which makes sense as a smooth admissible representation of $G(\mathbb{A}^S)$ since almost all $\dot{\pi}_v$ satisfy $\dim(\dot{\pi}_v)^{\mathcal{G}(\mathbb{Z}_v)} = 1$. The isomorphism class of the $G(\mathbb{A}^S)$-representation $\dot{\pi}^S$ is well defined up to the ϑ^S-action.

9.5. Spectral evaluation

9.5.1. — In the following we keep the setting and notation of §1.8.3, Theorem 1.8.4, and Corollary 8.17.5, for the quadratic space (V, q) fixed in §9.4.4. In particular we fix a neat compact open subgroup $K \subset G(\mathbb{A}_f)$, and fix $f^\infty dg^\infty \in \mathcal{H}(G(\mathbb{A}_f) /\!\!/ K)_{\mathbb{Q}}$.

We need a modified version of Corollary 8.17.5 as follows. In §8.4.1, we assumed that $\mathbb{V}$ is absolutely irreducible. In the odd case we keep that assumption, but in the even case we assume either one of the following two conditions:

(1) The algebraic $G_{\mathbb{E}}$-representation $\mathbb{V}$ is absolutely irreducible, and the isomorphism class of the $G_{\overline{\mathbb{Q}}}$-representation $\mathbb{V} \otimes_{\mathbb{E}} \overline{\mathbb{Q}}$ is preserved by outer automorphisms of $G_{\overline{\mathbb{Q}}}$.

(2) We have $\mathbb{V} \cong \mathbb{V}_0 \oplus \mathbb{V}_1$, where $\mathbb{V}_0$ and $\mathbb{V}_1$ are absolutely irreducible algebraic $G_{\mathbb{E}}$-representations such that the isomorphism classes of the $G_{\overline{\mathbb{Q}}}$-representations $\mathbb{V}_0 \otimes_{\mathbb{E}} \overline{\mathbb{Q}}$ and $\mathbb{V}_1 \otimes_{\mathbb{E}} \overline{\mathbb{Q}}$ are unequal and interchanged with each other under an outer automorphism of $G_{\overline{\mathbb{Q}}}$.

We shall call case (1) the *even symmetric case*, and case (2) the *even composite case*. In the odd case and the even symmetric case, Corollary 8.17.5 directly applies. In the even composite case, as in Theorem 1.8.4, for each fixed $f^\infty dg^\infty$ we obtain two finite sets of prime numbers $\Sigma(\mathbf{O}(V), \mathbb{V}_i, \lambda, K, f^\infty)$ for $i = 0, 1$. We define $\Sigma(\mathbf{O}(V), \mathbb{V}, \lambda, K, f^\infty)$ to be the union of these two sets. Clearly (8.17.5.1) still holds in this case, for any prime p outside $\Sigma(\mathbf{O}(V), \mathbb{V}, \lambda, K, f^\infty)$ and satisfying the assumption in §8.17.1, if on the right hand side we define f_∞^H to be the *sum* of the two test functions corresponding to $\mathbb{V}_0$ and $\mathbb{V}_1$. Indeed one obtains this by simply summing the two cases of (8.17.5.1) corresponding to $\mathbb{V}_0$ and $\mathbb{V}_1$.

In all of the odd case, the even symmetric case, and the even composite case, we define the finite sets of primes

$$\Sigma'_{\mathrm{bad}}(K, f^\infty) := \Sigma(\mathbf{O}(V), \mathbb{V}, \lambda, K, f^\infty) \cup \Sigma(\mathcal{G}^*, \mathcal{G}, \Xi, z, \mathfrak{w}, \theta_G),$$

and

$$\Sigma_{\mathrm{bad}}(K, f^\infty) := \Sigma'_{\mathrm{bad}}(f^\infty) \cup \{p \notin \Sigma'_{\mathrm{bad}}(K, f^\infty) \mid K_p \neq \mathcal{G}(\mathbb{Z}_p)\}. \tag{9.5.1.1}$$

We now fix a prime $p \notin \Sigma_{\mathrm{bad}}(K, f^\infty)$, and apply (the modified) (8.17.5.1) to p. Note that the extra assumption on p in the even case in §8.17.1 is satisfied here by condition (2) in §9.4.4. We thus obtain

$$\mathrm{Tr}(\mathrm{Frob}_p^a \times f^\infty dg^\infty \mid \mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})) = \sum_{(H, {}^L H, s, \eta) \in \dot{\mathscr{E}}(G)} \iota(G, H) ST^H(f^H), \tag{9.5.1.2}$$

for every sufficiently large a. On the right hand side, as we have already indicated, the archimedean test function f_∞^H is defined to be the sum of the test functions constructed in §8.4 corresponding to $\mathbb{V}_0$ and $\mathbb{V}_1$ in the even composite case. Here we view the two sides of (9.5.1.2) as numbers in $\mathbb{C}$, but recall from Theorem 1.8.4 and Remark 8.17.6 that the left hand side is actually in $\mathbb{E}$.

Remark 9.5.2. — In the even composite case, $\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})$ is the direct sum of $\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V}_0)$ and $\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V}_1)$ as $\mathcal{H}(G(\mathbb{A}_f) /\!\!/ K)_{\mathbb{Q}} \times \Gamma_{\mathbb{Q}}$-modules. We explain how the latter two are related to each other. Let $K' = K \cap \theta_G(K)$. Then K' is a compact open subgroup of K, and the $\mathcal{H}(G(\mathbb{A}_f) /\!\!/ K)_{\mathbb{Q}} \times \Gamma_{\mathbb{Q}}$-module $\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V}_i)$ is obtained from the $\mathcal{H}(G(\mathbb{A}_f) /\!\!/ K')_{\mathbb{Q}} \times \Gamma_{\mathbb{Q}}$-module $\mathbf{IH}^*(\overline{\mathrm{Sh}_{K'}}, \mathbb{V}_i)$ by taking K-invariants. It is easier to describe the relation between $\mathbf{IH}^*(\overline{\mathrm{Sh}_{K'}}, \mathbb{V}_i)$ and $\mathbf{IH}^*(\overline{\mathrm{Sh}_{K'}}, \mathbb{V}_i)$, so we replace K by K'. Write $\mathcal{H}$ for $\mathcal{H}(G(\mathbb{A}_f) /\!\!/ K)_{\mathbb{Q}}$. Then θ_G induces a ring automorphism of $\mathcal{H}$. Now observe[11] that the automorphism $\theta_G = \mathrm{Int}(\mathbf{r})|_G$ of G induces an automorphism of the Shimura datum $\mathbf{O}(V) = (G, \mathcal{X}, h)$, since $\mathbf{r} \in \mathrm{O}(V)(\mathbb{Q})$ induces an automorphism of the space $\mathcal{X}$ of oriented negative definite planes in $V_{\mathbb{R}}$, and h intertwines this automorphism with the automorphism $f \mapsto \theta_G \circ f$ of $\mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$. Moreover θ_G interchanges the isomorphism classes of the $G_{\overline{\mathbb{Q}}}$-representations $\mathbb{V}_{0,\overline{\mathbb{Q}}}$ and $\mathbb{V}_{1,\overline{\mathbb{Q}}}$. Therefore by transport of structure we have an $\mathcal{H} \times \Gamma_{\mathbb{Q}}$-module isomorphism

$$\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V}_1) \cong \mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V}_0) \otimes_{\mathcal{H}, \theta_G} \mathcal{H}$$

Lemma 9.5.3. — *Suppose that f^∞, as a function on $G(\mathbb{A}_f)$, is fixed by the group ϑ^∞ (see Definition 9.4.5). Then for each $(H, {}^L H, s, \eta) \in \dot{\mathscr{E}}(G)$ we have $f^H \in \widetilde{\mathcal{H}}^{\mathrm{st}}(H)$, where $\widetilde{\mathcal{H}}^{\mathrm{st}}(H)$ is defined in §9.2.15.*

Proof. — If $(H, {}^L H, s, \eta)$ does not satisfy the conditions (†) and (‡) in §8.4.1, then by definition $f^H = 0$. In the following we assume that these conditions are satisfied. We can factorize $f^{\infty,p}$ as $f_S f^{\infty,p,S}$, where S is a finite set of primes not containing p, $f_S \in C_c^\infty(\prod_{v \in S} G(\mathbb{Q}_v))$, and $f^{\infty,p,S} = 1_{\mathcal{G}(\widehat{\mathbb{Z}}^{p,S})} \in C_c^\infty(G(\mathbb{A}^{\infty,p,S}))$. Moreover, up to enlarging S, we may assume that $1_{\mathcal{G}(\widehat{\mathbb{Z}}^{p,S})}$ is fixed by $\vartheta^{\infty,p,S}$. Since p is not in $\Sigma_{\mathrm{bad}}(f^\infty)$, we also know that $1_{K_p} = 1_{\mathcal{G}(\mathbb{Z}_p)}$ is fixed by ϑ_p. Hence our assumption that f^∞ is invariant under ϑ^∞ implies that f_S is invariant under ϑ_S. By induction on $|S|$, it is an elementary exercise to show that f_S can be written as a sum of functions in $C_c^\infty(\prod_{v \in S} G(\mathbb{Q}_v))$ each of which is completely factorizable (i.e., a product over $v \in S$ of functions in $C_c^\infty(G(\mathbb{Q}_v))$) and invariant under ϑ_S. Hence $f^{\infty,p}$ is a sum of functions in $C_c^\infty(G(\mathbb{A}_f^p))$ each of which is completely factorizable and invariant under ϑ_S. We have thus reduced to the case where $f^{\infty,p} = \prod_{v \neq \infty,p} f_v$, with each $f_v \in C_c^\infty(G(\mathbb{Q}_v))$ invariant under θ_G, and $f_v = 1_{\mathcal{G}(\mathbb{Z}_v)}$ for almost all v.

For each finite place $v \neq p$, we can choose an automorphism θ_{G_v} of G_v as in §9.3.2. As we have observed in §9.4.6, $\theta_G = \theta_{G_v} \circ \mathrm{Int}(h_v)$ for some $h_v \in G(\mathbb{Q}_v)$. Therefore the fact that f_v is invariant under θ_G implies that f_v has θ_{G_v}-invariant orbital integrals. By Proposition 9.3.3 we know that f_v^H, which is a Langlands–Shelstad transfer of f_v, lies in $\widetilde{\mathcal{H}}^{\mathrm{st}}(H_{\mathbb{Q}_v})$. It remains to check that $f_v^H \in \widetilde{\mathcal{H}}^{\mathrm{st}}(H_{\mathbb{Q}_v})$ for $v = \infty, p$.

The fact that $f_\infty^H \in \widetilde{\mathcal{H}}^{\mathrm{st}}(H_{\mathbb{R}})$ follows from the following ingredients:

[11] We thank the anonymous referee for bringing this observation to our attention.

– The implicit fact that we may (and do) take f^H_∞ inside $\mathcal{H}(H_{\mathbb{R}}) \subset C_c^\infty(H(\mathbb{R}))$. (By the construction in §8.4, this reduces to the fact [**Art89**, Lem. 3.1] that for any discrete series representation of $H(\mathbb{R})$, a pseudo-coefficient of it may be taken to be bi-finite under a prescribed maximal compact subgroup of $H(\mathbb{R})$.)

– The formula [**Kot90**, (7.4)] for the stable orbital integrals of f^H_∞.

– The invariance properties of the transfer factors shown in the proof of [**Taï19**, Prop. 3.2.6].

– The fact that for any semi-simple elliptic element $\gamma_0 \in G(\mathbb{R})$ the term $e(I)\operatorname{vol}^{-1}\operatorname{Tr}\xi_{\mathbb{C}}(\gamma_0)$ (where $\xi_{\mathbb{C}} = \mathbb{V}_{\mathbb{C}}$) in [**Kot90**, (7.4)] is invariant under replacing γ_0 by its image under any automorphism of $G_{\mathbb{R}}$. (Note that in the even case this is false if we take $\xi_{\mathbb{C}}$ to be a general irreducible representation of $G_{\mathbb{C}}$.)

We now prove that $f^H_p \in \widetilde{\mathcal{H}}^{\mathrm{st}}(H_{\mathbb{Q}_p})$. By the discussion in §7.1.2, we have canonical actions of $\operatorname{Aut}(H_{\mathbb{Q}_p})$ on $\mathcal{H}^{\mathrm{ur}}(H_{\mathbb{Q}_p})$ and on $\mathscr{A}_{H_{\mathbb{Q}_p}}$, under which the subgroup $H^{\mathrm{ad}}(\mathbb{Q}_p) \subset \operatorname{Aut}(H_{\mathbb{Q}_p})$ (consisting of inner automorphisms) acts trivially. Thus the outer automorphism group $\operatorname{Out}(H_{\mathbb{Q}_p}) = \operatorname{Aut}(H_{\mathbb{Q}_p})/H^{\mathrm{ad}}(\mathbb{Q}_p)$ acts on $\mathcal{H}^{\mathrm{ur}}(H_{\mathbb{Q}_p})$ and $\mathscr{A}_{H_{\mathbb{Q}_p}}$. Moreover the canonical Satake isomorphism $\mathcal{H}^{\mathrm{ur}}(H_{\mathbb{Q}_p}) \xrightarrow{\sim} \mathscr{A}_{H_{\mathbb{Q}_p}}$ is $\operatorname{Out}(H_{\mathbb{Q}_p})$-equivariant. We need only show that the Satake transform of f^H_p in $\mathscr{A}_{H_{\mathbb{Q}_p}} = \mathscr{A}_{H^+_{\mathbb{Q}_p}} \otimes \mathscr{A}_{H^-_{\mathbb{Q}_p}}$, which is computed in (7.4.2.1), is invariant under $\operatorname{Out}(H_{\mathbb{Q}_p}) = \operatorname{Out}(H^+_{\mathbb{Q}_p}) \times \operatorname{Out}(H^+_{\mathbb{Q}_p})$. In all the five cases in (7.4.2.1), the image of $\operatorname{Out}(H_{\mathbb{Q}_p})$ in $\operatorname{Aut}(\mathscr{A}_{H_{\mathbb{Q}_p}})$ is generated by the automorphism $Z_1 \mapsto -Z_1$ of $\mathscr{A}_{H^+_{\mathbb{Q}_p}}$ (non-trivial in the second and fourth cases) and the automorphism $Y_1 \mapsto -Y_1$ of $\mathscr{A}_{H^-_{\mathbb{Q}_p}}$ (non-trivial in the second and third cases). By (7.4.2.1), the Satake transform of f^H_p is indeed invariant under $\operatorname{Out}(H_{\mathbb{Q}_p})$. □

9.5.4. — We keep the assumption in Lemma 9.5.3 that f^∞ is fixed by ϑ^∞. We assume Hypothesis 9.1.2. By Corollary 9.1.7, the expansion (9.2.15.2), and Lemma 9.5.3, we can rewrite (9.5.1.2) as

$$(9.5.4.1)\quad \operatorname{Tr}(\mathrm{Frob}_p^a \times f^\infty dg^\infty \mid \mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})) = \sum_{(H,{}^LH,s,\eta)\in\dot{\mathscr{E}}(G)} \iota(G,H) \sum_{\psi'\in\widetilde{\Psi}(H)} m_{\psi'}\, |\mathcal{S}_{\psi'}|^{-1}\, \sigma(\bar{S}^0_{\psi'})\epsilon_{\psi'}(s_{\psi'})\Lambda_{\psi'}(f^H).$$

Lemma 9.5.5. — *Assume that $\psi' \in \widetilde{\Psi}(H)$ contributes non-trivially to the RHS of (9.5.4.1). Then H is cuspidal, and $\eta \circ \psi'_\infty \in \Psi^{\mathrm{AJ}}(G^*_\infty)$. (In particular, $\psi'_\infty \in \Psi^{\mathrm{AJ}}(H_{\mathbb{R}})$.) Moreover, $\eta \circ \psi'_\infty$ has the same infinitesimal character as that of $\mathbb{V}^*_{\mathbb{C}}$ (resp. that of $\mathbb{V}^*_{0,\mathbb{C}}$ or $\mathbb{V}^*_{1,\mathbb{C}}$) in the odd case and the even symmetric case (resp. the even composite case).*

Proof. — We only treat the even composite case, the other two cases being similar. Recall that $f^H_\infty = f^H_{\infty,0} + f^H_{\infty,1}$ where $f^H_{\infty,i}$ is the analogue of f^H_∞ constructed in §8.4 with $\mathbb{V}$ replaced by $\mathbb{V}_i$. Thus $f^H_\infty = 0$ unless H is cuspidal; see §8.4.1. Assume that H is cuspidal. By [**Taï17**, Lem. 4.1.3] we know that for any $\psi''_\infty \in \Psi(H_{\mathbb{R}})$, all the representations in $\widetilde{\Pi}_{\psi''_\infty}(H_\mathbb{R})$ have the same infinitesimal character as that of ψ''_∞. By analytic continuation (see [**Art13**, p. 46] and cf. Remark 9.2.10), the same conclusion holds for all $\psi''_\infty \in \Psi^+_{\mathrm{unit}}(H_\mathbb{R})$. Hence in order that $\Lambda_{\psi'_\infty}(f^H_\infty) \neq 0$, the infinitesimal character of $\eta \circ \psi'_\infty$ must be the same as that of $\mathbb{V}^*_{0,\mathbb{C}}$ or $\mathbb{V}^*_{1,\mathbb{C}}$, which are regular C-algebraic. It remains to check that $\eta \circ \psi'_\infty$ is bounded on $W_\mathbb{R}$. But this follows from the fact that $\eta \circ \psi'_\infty$ is the localization of the global parameter $\eta \circ \psi'$, the fact that it has C-algebraic infinitesimal character, and Clozel's purity lemma [**Clo90b**, Lem. 4.9]. (For a similar argument cf. [**Taï17**, p. 309].) □

9.5.6. — Let $(H, {}^LH, s, \eta) \in \dot{\mathscr{E}}(G)$. For each place v of $\mathbb{Q}$, let $(\mathfrak{w}_v, \Xi_v, z_v)$ be the localization at v of $(\mathfrak{w}, \Xi, z)$ fixed in §9.4.4. In §9.3.2 and §9.3.4, we introduced a normalization $\Delta^{G_v}_{H_{\mathbb{Q}_v}}(\mathfrak{w}_v, \Xi_v, z_v)$ of transfer factors between $H_{\mathbb{Q}_v}$ and G_v for each place v. In §8.4.7 we also introduced a normalization $(\Delta^G_H)_v$. Thus we have

$$a^G_{H,v}\Delta^{G_v}_{H_{\mathbb{Q}_v}}(\mathfrak{w}_v, \Xi_v, z_v) = (\Delta^G_H)_v,$$

for a constant $a^G_{H,v} \in \mathbb{C}^\times$. By construction, the normalizations $(\Delta^G_H)_v$ are the canonical unramified normalizations at almost all places v (associated to hyperspecial subgroups determined by a reductive model of G over some Zariski open of $\operatorname{Spec}\mathbb{Z}$), and satisfy the global product formula. The same holds for the normalizations $\Delta^{G_v}_{H_{\mathbb{Q}_v}}(\mathfrak{w}_v, \Xi_v, z_v)$, since $\mathfrak{w}_v$ for various v are localizations of the global Whittaker datum $\mathfrak{w}$, and (Ξ_v, z_v) for various v are localizations of a global pure inner twist (Ξ, z); see [**Art13**, p. 137] or [**Kal18**, Prop. 4.4.1]. It follows that

$$\prod_v a^G_{H,v} = 1, \tag{9.5.6.1}$$

where almost all terms in the product are 1.

Let $\psi' \in \widetilde{\Psi}(H)$. In the following we compute the contribution of ψ' to the RHS of (9.5.4.1), based on Kottwitz's results in [**Kot90**, §9]. For each place v, let

$$\psi'_v \in \Psi^+_{\mathrm{unit}}(H_{\mathbb{Q}_v})$$

be a localization of ψ' as in §9.2.15. Let

$$\psi_v := \eta \circ \psi'_v \in \Psi^+_{\mathrm{unit}}(G^*_v),$$

and let

$$\psi := \eta \circ \psi \in \widetilde{\Psi}(G^*)$$

as in §9.2.2. For each place v, ψ_v is indeed a localization of ψ, so our notation is consistent. In Lemma 9.5.5, we have already seen that a necessary condition for ψ' to contribute non-trivially to the RHS of (9.5.4.1) is that ψ_∞ is Adams–Johnson with infinitesimal character determined by $\mathbb{V}$. In the following we assume this condition (but we do not assume that ψ' has a non-zero contribution *a priori*). In particular, ψ'_∞ is discrete, and so $\bar{S}^0_{\psi'} = \{1\}$. Thus by the definition of $\sigma(\bar{S}^0_{\psi'})$ in [**Art13**, Prop. 4.1.1], we have

$$\sigma(\bar{S}^0_{\psi'}) = 1. \tag{9.5.6.2}$$

We make several observations and definitions which will be understood in the statement of the next lemma. Recall from §9.4.6 that for each finite place v we view elements of $\widetilde{\Pi}_{\psi_v}(G_v)$ as ϑ_v-orbits of isomorphism classes of representations of $G(\mathbb{Q}_v)$. Since $p \notin \Sigma_{\mathrm{bad}}(K, f^\infty)$, we know that $K_p = \mathcal{G}(\mathbb{Z}_p)$ and that ϑ_p stabilizes $\mathcal{G}(\mathbb{Z}_p)$. Hence ϑ_p permutes the isomorphism classes of K_p-unramified representations of $G(\mathbb{Q}_p)$. Thus we can speak of whether an element of $\widetilde{\Pi}_{\psi_p}(G_p)$ is a ϑ_p-orbit of K_p-unramified representations. We write $\Lambda^{p,\infty}_{\psi'}$ for the product of the local stable distributions $\Lambda_{\psi'_v}$ over all places $v \notin \{\infty, p\}$, so we have

$$\Lambda_{\psi'}(f^H) = \Lambda_{\psi'_\infty}(f^H_\infty)\Lambda_{\psi'_p}(f^H_p)\Lambda^{p,\infty}_{\psi'}(f^{H,p,\infty}).$$

As in §9.4.6, we define the global packet $\widetilde{\Pi}^{p,\infty}_{\psi}(G)$, and for each $\pi^{p,\infty} \in \widetilde{\Pi}^{p,\infty}_{\psi}(G)$ we define the $G(\mathbb{A}^p_f)$-representation $\dot{\pi}^{p,\infty}$ (which depends on arbitrary choices).

Lemma 9.5.7. — *Let $(H, {}^LH, s, \eta), \psi', \psi, \psi'_v, \psi_v$ be as in §9.5.6, and keep assuming that ψ_∞ is Adams–Johnson with infinitesimal character determined by $\mathbb{V}$ as in Lemma 9.5.5. The following statements hold.*

(1) *We have*

$$\Lambda^{\mathrm{AJ}}_{\psi'_\infty}(f^H_\infty) = (-1)^{q(G_\infty)}\langle s_\psi s, \lambda_{\pi_\infty}\rangle\langle s_\psi s, \pi_\infty\rangle_{\mathrm{AJT}}\ a^G_{H,\infty}.$$

Here

- *π_∞ is any element of $\Pi^{\mathrm{AJ}}_{\psi_\infty}(G_{\mathbb{R}})$.*
- *$\langle\cdot, \lambda_{\pi_\infty}\rangle$ is a character on $S_{\psi_\infty} = \pi_0(S_{\psi_\infty})$ defined on p. 195 of* [**Kot90**].
- *The pairing $\langle s_\psi s, \pi_\infty\rangle_{\mathrm{AJT}}$ is as in Proposition 9.3.5, defined with respect to $(\mathfrak{w}_\infty, \Xi_\infty, z_\infty)$.*
- *The product $\langle s_\psi s, \lambda_{\pi_\infty}\rangle\langle s_\psi s, \pi_\infty\rangle_{\mathrm{AJT}}$ is independent of the choice of π_∞.*

(2) *We have*

$$\Lambda_{\psi'_\infty}(f^H_\infty) = (-1)^{q(G_\infty)}\langle s_\psi s, \lambda_{\pi_\infty}\rangle\langle s_\psi s, \pi_\infty\rangle_{\mathrm{AJT}}\ a^G_{H,\infty}$$

(3) *For each finite place v, ψ_v is unramified if and only if G^*_v is unramified and ψ'_v is unramified.*

(4) *If $\Lambda_{\psi'_p}(f^H_p) \neq 0$, then ψ_p is unramified. Conversely, assume that ψ_p is unramified. Then inside $\widetilde{\Pi}_{\psi_p}(G_p)$, there is a unique element π_p that is a ϑ_p-orbit of K_p-unramified representations of $G(\mathbb{Q}_p)$. Each $\dot{\pi}_p \in \pi_p$ satisfies $\dim(\dot{\pi}_p)^{K_p} = 1$. We have*

$$\Lambda_{\psi'_p}(f^H_p) = \langle s_\psi s, \pi_p\rangle p^{an/2}\,\mathrm{Tr}(s\varphi_{\psi_p}(\mathrm{Frob}^a_p) \mid \mathrm{Std}_G)\ a^G_{H,p}. \tag{9.5.7.1}$$

Here

- *$n = d - 2$ is the dimension of the Shimura variety; see §1.5.*

– $\mathrm{Std}_G = \mathrm{Std}_{G^*}$ *is the standard representation (9.2.2.1) of* ${}^LG = {}^LG^*$.

– Frob_p *denotes any choice of a lift of the geometric Frobenius in* $W_{\mathbb{Q}_p}$.

(5) *We have*

$$\Lambda^{p,\infty}_{\psi'}(f^{H,p,\infty}) = \sum_{\pi^{p,\infty}=(\pi_v)_v\in\widetilde{\Pi}^{p,\infty}_{\psi}(G)} \mathrm{Tr}\left(\dot{\pi}^{p,\infty}(f^{p,\infty}dg^{p,\infty})\right) \prod_{v\neq p,\infty} \langle s_\psi s, \pi_v\rangle a^G_{H,v},$$

where $f^{p,\infty}dg^{p,\infty}$ *is determined by* $f^\infty dg^\infty$ *in the same manner as in §1.8.3.*

Proof. — **(1)** This follows from [**Kot90**, Lem. 9.2]. More precisely, we know (Remark 9.3.6) that $\Lambda^{\mathrm{AJ}}_{\psi'_\infty}$ is the stable distribution considered by Adams–Johnson [**AJ87**] and Kottwitz [**Kot90**, §9], and the latter is Kottwitz's *definition* of [**Kot90**, (9.4)]. We know from Proposition 9.3.5 (3) that $\langle s_\psi s, \pi_\infty\rangle_{\mathrm{AJT}}$ serves as the spectral transfer factor that is denoted by $\Delta_\infty(\psi_H, \pi)$ (for $\psi_H = \psi'$, $\pi = \pi_\infty$) in [**Kot90**, Lem. 9.2], up to the correction factor $a^G_{H,\infty}$. Here $a^G_{H,\infty}$ arises because the spectral transfer factors used in *loc. cit.* are assumed to be compatible with the normalization $(\Delta^G_H)_\infty = a^G_{H,\infty}\Delta^{G_\infty}_{H_\mathbb{R}}(\mathfrak{w}_\infty, \Xi_\infty, z_\infty)$ of the geometric transfer factors, whereas the endoscopic character relation (9.3.5.1) is with respect to the normalization $\Delta^{G_\infty}_{H_\mathbb{R}}(\mathfrak{w}_\infty, \Xi_\infty, z_\infty)$. Note that in the even symmetric case, the fact that f^H_∞ is a sum $f^H_{\infty,0} + f^H_{\infty,1}$, where $f^H_{\infty,i}$ corresponds to $\mathbb{V}_i$, does not affect the validity of [**Kot90**, Lem. 9.2]. This is because the infinitesimal characters of $\mathbb{V}_{0,\mathbb{C}}$ and $\mathbb{V}_{1,\mathbb{C}}$ are unequal, and in the evaluation $\Lambda^{\mathrm{AJ}}_{\psi'_\infty}(f^H_\infty)$ only one of $f^H_{\infty,i}$ will contribute, according to whether the infinitesimal character of $\eta\circ\psi'_\infty$ is equal to that of $\mathbb{V}^*_{0,\mathbb{C}}$ or $\mathbb{V}^*_{1,\mathbb{C}}$.

(2) This follows from part (1) together with Proposition 9.3.5 (2) and the fact that $f^H_\infty \in \widetilde{\mathcal{H}}^{\mathrm{st}}(H_\mathbb{R})$ shown in the proof of Lemma 9.5.3.

(3) Write I_v for the inertia subgroup of $W_{\mathbb{Q}_v}$. For each $\tau \in I_v$, write $\psi'_v(\tau) = a_\tau \rtimes \tau$, with $a_\tau \in \widehat{H}$.

Assume that ψ_v is unramified. Then by definition G^*_v is unramified. It also immediately follows that ψ'_v is trivial on $\mathrm{SU}_2(\mathbb{R})$, and $\eta(\tau) = \eta(a_\tau^{-1}) \rtimes \tau$ for all $\tau \in I_v$. Since τ acts trivially on $\widehat{G^*}$, for all $x \in \widehat{H}$ we have $\eta({}^\tau x) = \eta(\tau)\eta(x)\eta(\tau)^{-1} = \eta(a_\tau^{-1}xa_\tau)$. Therefore I_v acts on $\widehat{H}$ via inner automorphisms, which implies that $H_{\mathbb{Q}_v}$ is unramified. Then by our explicit presentation we know that the endoscopic datum $(H, {}^LH, s, \eta)$ is unramified over $\mathbb{Q}_v$ (cf. §8.4.1), that is, $\eta(\tau) = 1 \rtimes \tau$ for all $\tau \in I_v$. This implies that $a_\tau = 1$ for all $\tau \in I_v$. Since $H_{\mathbb{Q}_v}$ is unramified and we have already seen that ψ'_v is trivial on $\mathrm{SU}_2(\mathbb{R})$, we know that ψ'_v is unramified.

Conversely, assume that ψ'_v is unramified and G^*_v is unramified. Then $H_{\mathbb{Q}_v}$ is unramified, and as before the endoscopic datum $(H, {}^LH, s, \eta)$ is unramified over $\mathbb{Q}_v$. Since $\psi_v = \eta\circ\psi'_v$, we know that ψ_v is trivial on $\mathrm{SU}_2(\mathbb{R})$ and sends every $\tau \in I_v$ to $1 \rtimes \tau$. Thus ψ_v is unramified since G^*_v is unramified.

(4) Suppose $\Lambda_{\psi'_p}(f^H_p) \neq 0$. Then $f^H_p \neq 0$, so by the definition of f^H_p we know that H is unramified over $\mathbb{Q}_p$. Fix a Whittaker datum $\mathfrak{w}_{H,p} = (\mathfrak{w}_{H^+_{\mathbb{Q}_p}}, \mathfrak{w}_{H^-_{\mathbb{Q}_p}})$ for $H_{\mathbb{Q}_p}$, and fix a hyperspecial subgroup $K_{H,p}$ of $H(\mathbb{Q}_p)$ that is compatible with $\mathfrak{w}_{H,p}$, as in

§§9.2.8 and 9.2.13. Recall from Definition 8.4.9 and Remark 8.4.10 that f_p^H is well defined as an element of the canonical unramified Hecke algebra $\mathcal{H}^{\mathrm{ur}}(H_{\mathbb{Q}_p})$, and its stable orbital integrals are independent of how we realize f_p^H in $C_c^\infty(H(\mathbb{Q}_p))$. Thus we may assume that $f_p^H \in \mathcal{H}(H(\mathbb{Q}_p) /\!\!/ K_{H,p})$ without loss of generality. Then by (9.2.8.1) and Lemma 9.2.14 (1), we know that ψ'_p is unramified. By part (3) above, this implies that ψ_p is unramified.

Conversely, assume that ψ_p is unramified. By part (3) above, ψ'_p is unramified (since we know that $G^*_{\mathbb{Q}_p}$ is unramified), so in particular $H_{\mathbb{Q}_p}$ is unramified. Fix $\mathfrak{w}_{H,p}$ and $K_{H,p}$ as in the preceding paragraph. Inside $G^*(\mathbb{Q}_p)$, we have the hyperspecial subgroup $\mathcal{G}^*(\mathbb{Z}_p)$, and it is compatible with the Whittaker datum $\mathfrak{w}_p$ for G^*_p since $p \notin \Sigma(\mathcal{G}^*, \mathcal{G}, \Xi, z, \mathfrak{w}, \theta_G)$; see §9.4.4. We normalize the Haar measures on $G^*(\mathbb{Q}_p)$ and $H(\mathbb{Q}_p)$ once and for all such that hyperspecial subgroups have volume 1. By Lemmas 9.2.12 and 9.2.14, we know that inside $\widetilde{\Pi}_{\psi_p}(G^*_p)$ (resp. $\widetilde{\Pi}_{\psi'_p}(H_{\mathbb{Q}_p})$) there is a unique element π_{p,G^*} (resp. $\pi_{p,H}$) whose members are $\mathcal{G}^*(\mathbb{Z}_p)$-unramified (resp. $K_{H,p}$-unramified), and moreover the members of π_{p,G^*} (resp. $\pi_{p,H}$) have 1-dimensional fixed spaces under $\mathcal{G}^*(\mathbb{Z}_p)$ (resp. $K_{H,p}$). As in the preceding paragraph we may assume that $f_p^H \in \mathcal{H}(H(\mathbb{Q}_p) /\!\!/ K_{H,p})$. Then by (9.2.8.1), we have

$$\Lambda_{\psi'_p}(f_p^H) = \langle s_{\psi'_p}, \pi_{p,H}\rangle \operatorname{Tr}(\pi_{p,H}(f_p^H)).$$

Here the pairing $\langle s_{\psi'_p}, \pi_{p,H}\rangle$ is defined with respect to $\mathfrak{w}_{H,p}$. In view of the compatibility between local unramified Arthur parameters and unramified Langlands parameters in Lemma 9.2.14 (2), the same argument as Kottwitz's proof that [**Kot90**, (9.3)] is equal to [**Kot90**, (9.7)] gives

$$\operatorname{Tr}(\pi_{p,H}(f_p^H)) = p^{an/2} \operatorname{Tr}(s\varphi_{\psi_p}(\mathrm{Frob}_p^a) \mid \mathrm{Std}_G).$$

Indeed, one easily checks that the irreducible representation of ${}^LG$ determined by the Shimura datum appearing in [**Kot90**, (9.7)] is Std_G, and that the ambiguity in φ_{ψ_p} up to the $\mathrm{Aut}({}^LG^*_p)$-action disappears when we consider the $\mathrm{GL}_N(\mathbb{C})$-conjugacy class of the composition of φ_{ψ_p} with $\mathrm{Std}_G : {}^LG = {}^LG^* \to \mathrm{GL}_N(\mathbb{C})$. In conclusion we have

$$\Lambda_{\psi'_p}(f_p^H) = \langle s_{\psi'_p}, \pi_{p,H}\rangle p^{an/2} \operatorname{Tr}(s\varphi_{\psi_p}(\mathrm{Frob}_p^a) \mid \mathrm{Std}_G).$$

As we have already mentioned in §9.4.6, since $p \notin \Sigma(\mathcal{G}^*, \mathcal{G}, \Xi, z, \mathfrak{w}, \theta_G)$, the packet $\widetilde{\Pi}_{\psi_p}(G_p)$ together with the map to $\mathcal{S}^D_{\psi_p}$ is constructed from $\widetilde{\Pi}_{\psi_p}(G^*_p)$ by identifying G_p with G^*_p via an isomorphism $G^*_p \xrightarrow{\sim} G_p$ as in condition (4) in §9.4.4. Hence the existence and uniqueness of π_p and the fact that members of π_p have 1-dimensional fixed spaces under $K_p = \mathcal{G}(\mathbb{Z}_p)$ follow from the existence and uniqueness of π_{p,G^*} and the fact that members of π_{p,G^*} have 1-dimensional fixed spaces under $\mathcal{G}^*(\mathbb{Z}_p)$. Also we have $\langle \cdot, \pi_p\rangle = \langle \cdot, \pi_{p,G^*}\rangle \in \mathcal{S}^D_{\psi_p}$. To finish the proof it suffices to show that[12]

$$\langle s_{\psi'_p}, \pi_{p,H}\rangle = \langle s_\psi s, \pi_{p,G^*}\rangle a^G_{H,p}. \tag{9.5.7.2}$$

Comparing the Fundamental Lemma (Theorem 8.1.4 (2)) with the endoscopic character relation (9.2.9.1) and the expansion (9.2.8.1), we get

(9.5.7.3)
$$\sum_{\xi\in\widetilde{\Pi}_{\psi_p}(G^*_p)} \langle s_\psi s, \xi\rangle \operatorname{Tr}(\xi(1_{\mathcal{G}^*(\mathbb{Z}_p)})) = (a^G_{H,p})^{-1} \sum_{\xi'\in\widetilde{\Pi}_{\psi'_p}(H_{\mathbb{Q}_p})} \langle s_{\psi'_p}, \xi'\rangle \operatorname{Tr}(\xi'(1_{K_{H,p}})).$$

Here, the pairing $\langle s_{\psi'_p}, \xi'\rangle$ is defined with respect to $\mathfrak{w}_{H,p}$, and the factor $(a^G_{H,p})^{-1}$ appears because it is $(a^G_{H,p})^{-1}1_{K_{H,p}}$, rather than $1_{K_{H,p}}$, that is a Langlands–Shelstad transfer of $1_{\mathcal{G}^*(\mathbb{Z}_p)}$ with respect to the Whittaker normalization of transfer factors between $H_{\mathbb{Q}_p}$ and G^*_p associated to $\mathfrak{w}_p$. By Lemmas 9.2.12 and 9.2.14, the two sides of (9.5.7.3) are equal to $\langle s_\psi s, \pi_{p,G^*}\rangle$ and $(a^G_{H,p})^{-1}\langle s_{\psi'_p}, \pi_{p,H}\rangle$ respectively. This proves (9.5.7.2).

(5) First observe that for each $\pi^{p,\infty} \in \widetilde{\Pi}^{p,\infty}_\psi(G)$, the ambiguity in the $G(\mathbb{A}^p_f)$-representation $\dot{\pi}^{p,\infty}$ up to the $\vartheta^{p,\infty}$-action does not affect the value of $\operatorname{Tr}\left(\dot{\pi}^{p,\infty}(f^{p,\infty}dg^{p,\infty})\right)$. Indeed, since θ_G is an automorphism of G of order at most 2, it is clear that $dg^{p,\infty}$ is fixed by $\vartheta^{p,\infty}$. In the proof of Lemma 9.5.3 we observed that $f^{p,\infty}$ is fixed by $\vartheta^{p,\infty}$ (under the overall assumption that f^∞ is fixed by ϑ^∞). Hence the trace of $f^{p,\infty}dg^{p,\infty}$ on a $G(\mathbb{A}^p_f)$-representation depends only on the $\vartheta^{p,\infty}$-orbit of the isomorphism class of that representation.

Now as in the proof of Lemma 9.5.3, we may assume that $f^{p,\infty} = \prod_{v\neq p,\infty} f_v$ with each $f_v \in \mathcal{H}(G_v)$ being fixed by ϑ_v. The desired statement then follows from the endoscopic character relation in Proposition 9.3.3 applied to each f_v. Here the term $a^G_{H,v}$ appears because it is $(a^G_{H,v})^{-1}f^H_v$, rather than f^H_v, that is a Langlands–Shelstad transfer of f_v with respect to the normalization $\Delta^{G_v}_{H_{\mathbb{Q}_v}}(\mathfrak{w}_v, \Xi_v, z_v)$ of transfer factors. □

We summarize the results we have obtained so far in the following proposition.

Proposition 9.5.8. — *Let $(H, {}^LH, s, \eta) \in \dot{\mathscr{E}}(G)$ and $\psi' \in \widetilde{\Psi}(H)$. For each place v of $\mathbb{Q}$, let $\psi'_v \in \Psi^+_{\mathrm{unit}}(H_{\mathbb{Q}_v})$ be a localization of ψ', and let $\psi_v := \eta\circ\psi'_v \in \Psi^+_{\mathrm{unit}}(G^*_v)$. Let $\psi = \eta\circ\psi' \in \widetilde{\Psi}(G^*)$. The following statements hold:*

(1) *For ψ' to contribute non-trivially to the RHS of (9.5.4.1), it is necessary that H is cuspidal, and that ψ_∞ is Adams–Johnson with infinitesimal character determined by $\mathbb{V}$ as in Lemma 9.5.5.*

[12] By Lemmas 9.2.12 and 9.2.14 we know that $\langle \cdot, \pi_{p,H}\rangle$ and $\langle \cdot, \pi_{p,G^*}\rangle$ are trivial, but in the current proof we do not need this.

(2) *Assume that the necessary conditions in (1) are satisfied. Then the contribution of* ψ' *to the RHS of (9.5.4.1), without the factor* $\iota(G,H)$*, is equal to*

$$(9.5.8.1)\quad m_{\psi'}\,|\mathcal{S}_{\psi'}|^{-1}\,\epsilon_{\psi'}(s_{\psi'})\langle s_\psi s,\lambda_{\pi_\infty}\rangle\langle s_\psi s,\pi_\infty\rangle_{\mathrm{AJT}}\ A(\psi,s,p,a) \sum_{\pi^\infty=(\pi_v)_v\in\widetilde{\Pi}_\psi^\infty(G)} \mathrm{Tr}\left(\dot{\pi}^\infty(f^\infty dg^\infty)\right)\prod_{v\neq\infty}\langle s_\psi s,\pi_v\rangle.$$

with notations explained below:

– *The product* $\langle s_\psi s,\lambda_{\pi_\infty}\rangle\langle s_\psi s,\pi_\infty\rangle_{\mathrm{AJT}}$ *is as in Lemma 9.5.7 (1).*
– *We define*

$$A(\psi,s,p,a) := (-1)^{q(G_\infty)}p^{an/2}\,\mathrm{Tr}(s\varphi_{\psi_p}(\mathrm{Frob}_p^a)\mid \mathrm{Std}_G).$$

The notations n*,* Frob_p*, and* Std_G *are as in Lemma 9.5.7 (3), and we have* $q(G_\infty)=n$*.*

Proof. — This follows from Lemma 9.5.5, Lemma 9.5.7, (9.5.6.1), (9.5.6.2), and the following simple observations:

(1) For any finite-length smooth representation τ_p of $G(\mathbb{Q}_p)$, we have

$$\mathrm{Tr}(\tau_p(1_{K_p}dg_p)) = \dim\tau_p^{K_p}.$$

Here, as in §1.8.3, dg_p is the Haar measure on $G(\mathbb{Q}_p)$ giving volume 1 to hyperspecial subgroups.

(2) If ψ_p is ramified, then no element of $\widetilde{\Pi}_{\psi_p}(G_p)$ is a ϑ_p-orbit of K_p-unramified representations. Indeed, as we have mentioned in the proof of Lemma 9.5.7 (4), the packet $\widetilde{\Pi}_{\psi_p}(G_p)$ is constructed from the packet $\widetilde{\Pi}_{\psi_p}(G_p^*)$ via an isomorphism $G_p^*\xrightarrow{\sim}G_p$ as in condition (4) in §9.4.4. Hence the current assertion follows from Lemma 9.2.12 applied to G_p^* and ψ_p.

□

9.6. Spectral expansion of the intersection cohomology

We keep the same setting and notation as in §9.5. In particular, $\mathbb{V}$ is as in §9.5.1, and we speak of the odd case, the even symmetric case, and the even composite case.

Definition 9.6.1. — We denote by $\widetilde{\Psi}(G^*)_{\mathbb{V}}$ the set of $\psi\in\widetilde{\Psi}(G^*)$ such that the localization ψ_∞ of ψ at ∞ lies in $\Psi^{\mathrm{AJ}}(G^*_\infty)$ and has the same infinitesimal character as that of $\mathbb{V}^*_{\mathbb{C}}$ in the odd case and the even symmetric case, and the same infinitesimal character as that of $\mathbb{V}^*_{0,\mathbb{C}}$ or $\mathbb{V}^*_{1,\mathbb{C}}$ in the even composite case. (This condition is insensitive to the ambiguity in ψ_∞ up to the $\mathrm{Aut}({}^LG^*_\infty)$-action.) In particular, for any $\psi\in\widetilde{\Psi}(G^*)_{\mathbb{V}}$, we have $\psi\in\widetilde{\Psi}_2(G^*)$, and $S_\psi=\pi_0(S_\psi)$ is a finite abelian group.

Definition 9.6.2. — We say that a compact open subgroup $K \subset G(\mathbb{A}_f)$ is *completely symmetric*, if $K = \prod_v K_v$ where v runs through all primes, with each K_v a compact open subgroup of $G(\mathbb{Q}_v)$ that is stable under θ_G.

Remark 9.6.3. — Completely symmetric compact open subgroups of $G(\mathbb{A}_f)$ form a cofinal system of compact open subgroups. Indeed, given any compact open subgroup W of $G(\mathbb{A}_f)$, we know that W contains a compact open subgroup of the form $\prod_{v \in S} U_v \times \prod_{v \notin S} \mathcal{G}(\mathbb{Z}_v)$, where S is a sufficiently large finite set of primes, $\mathcal{G}$ is as in §9.4.4, and U_v is a compact open subgroup of $G(\mathbb{Q}_v)$ for each $v \in S$. If S is sufficiently large, we know that $\mathcal{G}(\mathbb{Z}_v)$ is θ_G-stable for all $v \notin S$; see §9.4.4. Note that $U'_v := U_v \cap \theta_G(U_v)$ is a θ_G-stable compact open subgroup of $G(\mathbb{Q}_v)$ for each $v \in S$. Hence W contains the completely symmetric compact open subgroup $\prod_{v \in S} U'_v \times \prod_{v \notin S} \mathcal{G}(\mathbb{Z}_v)$.

Theorem 9.6.4. — *Assume Hypothesis 9.1.2. Fix a neat compact open subgroup K of $G(\mathbb{A}_f)$, and fix $f^\infty dg^\infty \in \mathcal{H}(G(\mathbb{A}_f) /\!\!/ K)_{\mathbb{Q}}$. Assume that K is completely symmetric, and that f^∞ is fixed by ϑ^∞. Let p be a prime not in the set $\Sigma_{\mathrm{bad}}(K, f^\infty)$ as in (9.5.1.1). Let $a \in \mathbb{Z}$ be arbitrary. We have*

$$\begin{aligned}(9.6.4.1)\quad & \operatorname{Tr}(\mathrm{Frob}_p^a \times f^\infty dg^\infty \mid \mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})) \\ &= \sum_{\psi \in \widetilde{\Psi}(G^*)_{\mathbb{V}}} m_\psi \sum_{\pi^\infty \in \widetilde{\Pi}_\psi^\infty(G)} \operatorname{Tr}\left(\dot{\pi}^\infty(f^\infty dg^\infty)\right) |S_\psi|^{-1} \sum_{s \in S_\psi} B(\psi, s, \pi^\infty, p, a),\end{aligned}$$

with notations explained below:

- *For each ψ, $m_\psi \in \{1, 2\}$ is as in (9.2.2.2).*
- *For each $\psi \in \widetilde{\Psi}(G^*)_{\mathbb{V}}$, $\pi^\infty = (\pi_v)_v \in \widetilde{\Pi}_\psi^\infty(G)$, and $s \in S_\psi$, we define*

$$B(\psi, s, \pi^\infty, p, a) := \epsilon_\psi(s) \langle s, \lambda_{\pi_\infty} \rangle \langle s, \pi_\infty \rangle_{\mathrm{AJT}}\ A(\psi, s_\psi s, p, a) \prod_{v \neq \infty} \langle s, \pi_v \rangle,$$

where the terms $\langle s, \lambda_{\pi_\infty} \rangle \langle s, \pi_\infty \rangle_{\mathrm{AJT}}$ and $A(\psi, s_\psi s, p, a)$ are as in Proposition 9.5.8, but with a change of variable from s to $s_\psi s$. (Recall that $s_\psi \in S_\psi$ and $s_\psi^2 = 1$.)

Moreover, the summands on the right hand side of (9.6.4.1) vanish outside a finite set of summation indices (ψ, π^∞) which depends only on $K, \mathbb{V}, \mathfrak{w}, \Xi, z$ and not on $f^\infty dg^\infty, p, a$.

Proof. — Throughout the proof, it will always be understood that the data $K, \mathbb{V}, \mathfrak{w}, \Xi, z$ are fixed. Also, since varying $f^\infty dg^\infty$ is equivalent to varying f^∞ while keeping dg^∞ fixed, we will omit dg^∞ in the notations throughout. We first prove that when f^∞ and p are fixed, (9.6.4.1) holds for all sufficiently large a (in a way depending on f^∞ and p). By (9.5.4.1) and Proposition 9.5.8, we know that when a is sufficiently large, the LHS of (9.6.4.1) is equal to

$$\sum_{\psi\in\widetilde{\Psi}(G^*)_{\mathbb{V}}}\sum_{s\in S_\psi} C(\psi,s)\sum_{\pi^\infty\in\widetilde{\Pi}^\infty_\psi(G)} \operatorname{Tr}\left(\dot{\pi}^\infty(f^\infty)\right)B(\psi,s_\psi s,\pi^\infty,p,a),$$

if we define

$$C(\psi,s):=\sum_{\substack{\mathfrak{e}=(H,{}^LH,s_1,\eta)\in\dot{\mathscr{E}}(G)\\ H\text{ is cuspidal}}}\ \sum_{\substack{\psi'\in\widetilde{\Psi}(H)\\ (\mathfrak{e},\psi')\mapsto(\psi,s)}}\iota(G,H)m_{\psi'}\left|\mathcal{S}_{\psi'}\right|^{-1}\epsilon_{\psi'}(s_{\psi'})\epsilon_\psi(s_\psi s)^{-1},$$

where the second summation is over $\psi'\in\widetilde{\Psi}(H)$ such that $(H,{}^LH,s_1,\eta,\psi')$ gives rise to (ψ,s) as on p. 36 of [**Art13**]. Now in the definition of $C(\psi,s)$ we can drop the condition that H is cuspidal in the first summation, for the following reason. If there exists $\psi'\in\widetilde{\Psi}(H)$ such that $(H,{}^LH,s_1,\eta,\psi')$ gives rise to (ψ,s), then by the argument in the last paragraph of [**Kot90**, p. 196], elliptic maximal tori in $G^*_{\mathbb{R}}$ (which are anisotropic) must come from $H_{\mathbb{R}}$ since ψ_∞ is Adams–Johnson. It follows that $H_{\mathbb{R}}$ contains anisotropic maximal tori, and hence H is cuspidal.

Thus the proof of (9.6.4.1) reduces to the proof of the identity

$$m_\psi\left|S_\psi\right|^{-1}=\sum_{\mathfrak{e}=(H,{}^LH,s_1,\eta)\in\dot{\mathscr{E}}(G)}\ \sum_{\substack{\psi'\in\widetilde{\Psi}(H)\\ (\mathfrak{e},\psi')\mapsto(\psi,s)}}\iota(G,H)m_{\psi'}\left|\mathcal{S}_{\psi'}\right|^{-1}\epsilon_{\psi'}(s_{\psi'})\epsilon_\psi(s_\psi s)^{-1}$$

for all $\psi\in\widetilde{\Psi}(G^*)_{\mathbb{V}}$ and $s\in S_\psi$. This step is identical to the corresponding step in the proof of [**Taï19**, Thm. 4.0.1]. Without the extra complication in the even case (i.e., the integers $m_{\psi'},m_\psi$ being possibly larger than 1), this step is also given in [**Kot90**, §10]. Both references rely on Arthur's identity $\epsilon_{\psi'}(s_{\psi'})=\epsilon_\psi(s_\psi s)$, which is known in our case by [**Art13**, Lem. 4.4.1].

Before showing that (9.6.4.1) holds for all $a\in\mathbb{Z}$, we show that the summands on the right hand side of it vanish outside a finite set of summation indices (ψ,π^∞) independently of f^∞,p,a. Here f^∞ is allowed to range over all ϑ^∞-fixed elements of $\mathcal{H}(G(\mathbb{A}_f)\,/\!\!/\,K)_{\mathbb{Q}}$, p is allowed to range over all primes that are hyperspecial for K and unramified for f^∞, and a is allowed to range over all positive integers, not necessarily "sufficiently large" with respect to f^∞ and p in the previous sense. (Afterwards we will show the stronger finiteness result when a is allowed to range over all integers.)

Since K is completely symmetric, we have $K=\prod_v K_v$ with each K_v a θ_G-stable compact open subgroup of $G(\mathbb{Q}_v)$. Let Σ_0 be a finite set of primes containing the set $\Sigma(\mathcal{G}^*,\mathcal{G},\Xi,z,\mathfrak{w},\theta_G)$ from §9.4.4 such that $K_v=\mathcal{G}(\mathbb{Z}_v)$ for all $v\notin\Sigma_0$. Now since f^∞ is bi-invariant under K, any $\pi^\infty=(\pi_v)_v$ appearing in (9.6.4.1) such that $\operatorname{Tr}\left(\dot{\pi}^\infty(f^\infty dg^\infty)\right)\neq 0$ must satisfy the condition that π_v is a ϑ_v-orbit of $\mathcal{G}(\mathbb{Z}_v)$-unramified representations for all primes $v\notin\Sigma_0$. By the discussion in §9.4.6, we know that for each $\psi\in\widetilde{\Psi}(G^*)_{\mathbb{V}}$, there are only finitely many elements $\pi^\infty\in\widetilde{\Pi}^\infty_\psi(G)$ satisfying the aforementioned condition, and these elements exist only when the localizations ψ_v of ψ are unramified for all primes $v\notin\Sigma_0$. By our above proof of (9.6.4.1), the desired finiteness of the summation range independently of f^∞, p, a follows from the following statements:

(1) If $(H, {}^LH, s, \eta) \in \dot{\mathscr{E}}(G)$ and $\psi' \in \widetilde{\Psi}(H)$ are such that $\eta \circ \psi'_v$ is unramified for a prime v, then ψ'_v is unramified, and in particular $H_{\mathbb{Q}_v}$ is unramified.

(2) There are only finitely many elements $(H, {}^LH, s, \eta) \in \dot{\mathscr{E}}(G)$ such that $H_{\mathbb{Q}_v}$ is unramified for all primes $v \notin \Sigma_0$.

(3) Fix $(H, {}^LH, s, \eta) \in \dot{\mathscr{E}}(G)$. For each choice of (f^∞, p, a) (with $a \in \mathbb{Z}_{\geq 1}$), define $f^H = f^H_{(f^\infty,p,a)} \in C_c^\infty(H(\mathbb{A}))$ as in §8.4 (cf. Lemma 9.5.3). Then in the expansion (9.2.15.2) with respect to the test function f^H, the summands vanish outside a finite subset of $\widetilde{\Psi}(H)$ which is independent of (f^∞, p, a).

Now statement (1) follows from Lemma 9.5.7 (3). By the explicit presentation of $(H, {}^LH, s, \eta)$ and by Proposition 1.2.8, statement (2) reduces to the fact that there are only finitely many elements in $\mathbb{Q}^\times/\mathbb{Q}^{\times,2}$ that have even valuations at all primes not in Σ_0. For (3), we may assume that H is cuspidal, as otherwise $f^H = 0$. Now note that for our given test function f^H_∞ on $H(\mathbb{R})$, there are only finitely many values of $t \geq 0$ (depending only on $\mathbb{V}$) that contribute non-trivially to the expansion $S^H_{\mathrm{disc}}(f^H) = \sum_{t\geq 0} S^H_{\mathrm{disc},t}(f^H)$ (see §9.2.15) by Lemma 9.5.5. Thus it suffices to show that for a fixed t the summands in (9.2.15.1) with respect to f^H vanish outside a finite subset of $\widetilde{\Psi}(H)$ independently of (f^∞, p, a). By [**Art13**, Thm. 1.3.2, Lem. 3.3.1], we need only check that f^H has a *Hecke type* (see [**Art13**, p. 129]) that is independent of (f^∞, p, a). Since f^H_∞ is independent of (f^∞, p, a), this amounts to the existence of a compact open subgroup $K_H \subset H(\mathbb{A}_f)$ such that $f^{H,\infty} = f^{H,p,\infty} f^H_p$ can be chosen to be bi-invariant under K_H independently of (f^∞, p, a).

We now construct K_H. Let S be the set of primes v such that either $G_{\mathbb{Q}_v}$ is ramified or $H_{\mathbb{Q}_v}$ is ramified. For each prime $v \notin S$, we pick a hyperspecial subgroup $U_v \subset H(\mathbb{Q}_v)$, in such a way that $\prod_{v\notin S} U_v$ is a compact open subgroup of $H(\mathbb{A}_f^S)$. By the two main theorems of [**Art96**, §6] (cf. the proof of [**Art13**, Lem. 3.3.1]), we know that for every prime v there is a compact open subgroup $V_v \subset H(\mathbb{Q}_v)$ with the property that every K_v-bi-invariant function in $C_c^\infty(G(\mathbb{Q}_v))$ has a Langlands–Shelstad transfer in $C_c^\infty(H(\mathbb{Q}_v))$ that is bi-invariant under V_v. By the Fundamental Lemma for the full unramified Hecke algebra proved by Hales [**Hal95**] (which is conditional on the Fundamental Lemma for the unit as recalled in Theorem 8.1.4), for every prime $v \notin S$ we may and shall take V_v to be U_v. We take K_H to be the product of V_v over all primes, which is a compact open subgroup of $H(\mathbb{A}_f)$. Now for every choice of (f^∞, p, a), the corresponding function $f^{H,\infty}$ is non-zero only when $p \notin S$, and in the latter case we can choose $f^{H,p,\infty}$ to be bi-invariant under $\prod_{v\neq p} V_v$, and choose f^H_p to be bi-invariant under $U_p = V_p$, as is clear from the construction in §8.4. It follows that $f^{H,\infty}$ is bi-invariant under K_H as desired.

We have proved that the summands on the RHS of (9.6.4.1) vanish outside a finite set of summation indices (ψ, π^∞) independently of $f^\infty, p, a \in \mathbb{Z}_{\geq 1}$. Note that the same holds even if a is allowed to range over all integers. This is because each summand, as a function in $a \in \mathbb{Z}$, is of the form $\sum_{i=1}^{k} c_i z_i^a$, where $c_i, z_i \in \mathbb{C}$ are independent of a. Thus if a summand is zero for all $a \in \mathbb{Z}_{\geq 1}$, then it is zero for all $a \in \mathbb{Z}$.

To finish the proof it remains to show that (9.6.4.1) holds for all $a \in \mathbb{Z}$. By what we have already shown, for each fixed f^∞ and p, the right hand side of (9.6.4.1) is of the form $\sum_{i=1}^{k} c_i z_i^a$, where $c_i, z_i \in \mathbb{C}$ are independent of a. It is easy to see that the left hand side is also of a similar form as a function in a. Hence the identity (9.6.4.1) holding for all sufficiently large a implies that it holds for all $a \in \mathbb{Z}$. □

Remark 9.6.5. — A form of Theorem 9.6.4 is conjectured in [**Kot90**, (10.2)].

9.7. The Hasse–Weil zeta function

We deduce an immediate consequence of Theorem 9.6.4 concerning the Hasse–Weil zeta function associated to $\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V}))$.

Definition 9.7.1. — Let p be a prime number. Let $\mathcal{M}$ be a finite-dimensional representation over $\mathbb{C}$ of WD_p.

(1) We view $\mathcal{M}$ as a Weil–Deligne representation of $W_{\mathbb{Q}_p}$, and define its local L-factor at p in the usual way as in [**Tat79**], denoted by $L_p(\mathcal{M}, s)$. In particular, when the representation is unramified (i.e. trivial on $\mathrm{SU}_2(\mathbb{R})$ and on the inertia subgroup), we have

$$L_p(\mathcal{M}, s) := \big(\exp(\sum_{a\geq 1} \mathrm{Tr}(\mathrm{Frob}_p^a \mid \mathcal{M}) p^{-as}/a)\big)^{-1} = \det(1 - \mathrm{Frob}_p\, p^{-s} \mid \mathcal{M})^{-1}$$

where Frob_p is any lift of geometric Frobenius in $W_{\mathbb{Q}_p}$.

(2) For any real number α, we define $\|\cdot\|^\alpha \mathcal{M}$ to be the twist of $\mathcal{M}$ by the quasi-character $\|\cdot\|^\alpha$ on $W_{\mathbb{Q}_p}$. Here the normalization is such that $\|\mathrm{Frob}_p\| = p^{-1}$.

(3) For any positive integer n, we define $\mathcal{M}^{(n)}$ to be

$$\|\cdot\|^{(n-1)/2} \mathcal{M} \oplus \|\cdot\|^{(n-3)/2} \mathcal{M} \oplus \cdots \oplus \|\cdot\|^{(1-n)/2} \mathcal{M}.$$

Remark 9.7.2. — We have

$$L_p(\|\cdot\|^\alpha \mathcal{M}, s) = L_p(\mathcal{M}, \alpha + s),$$

and

$$L_p(\mathcal{M}^{(n)}, s) = L_p(\mathcal{M}, s + \frac{n-1}{2}) L_p(\mathcal{M}, s + \frac{n-3}{2}) \cdots L_p(\mathcal{M}, s + \frac{1-n}{2}).$$

9.7.3. — Let $\psi \in \widetilde{\Psi}_2(G^*)$. Recall from §9.2.2 that S_ψ is a finite power of $\mathbb{Z}/2\mathbb{Z}$. Let $\nu : S_\psi \to \mathbb{C}^\times$ be a character. Let $\mathcal{V} = \mathbb{C}^N$ be the vector space used to define $\mathrm{GL}_N(\mathbb{C})$. The group S_ψ acts on $\mathcal{V}$ via

$$S_\psi \subset {}^L G^* \xrightarrow{\mathrm{Std}_{G^*}} \mathrm{GL}_N(\mathbb{C}).$$

Let $\mathcal{V}_\nu \subset \mathcal{V}$ be the ν-eigenspace for this action. For each prime number p, consider the action of WD_p on $\mathcal{V}$ defined by

$$\mathrm{WD}_p \xrightarrow{\varphi_{\psi_p}} {}^L G^* \xrightarrow{\mathrm{Std}_{G^*}} \mathrm{GL}_N(\mathbb{C}).$$

(Note that in the even case, although ψ_p is not always well defined up to $\widehat{G^*}$-conjugacy, the above composite map is always well defined up to $\mathrm{GL}_N(\mathbb{C})$-conjugacy.) This action commutes with the action of S_ψ, so we have an action of WD_p on $\mathcal{V}_\nu$. We denote this WD_p-representation on $\mathcal{V}_\nu$ by $\mathcal{V}_p(\psi, \nu)$. Define

$$\mathcal{M}_p(\psi, \nu) := \|\cdot\|^{-n/2} \mathcal{V}_p(\psi, \nu),$$

where $n = d-2$ is the dimension of the Shimura variety Sh_K. The motivation for this twist is to account for the factor $p^{an/2}$ in the definition of $A(\psi, s, p, a)$ in Proposition 9.5.8.

We can classify the WD_p-representations $\mathcal{V}_p(\psi, \nu)$ and $\mathcal{M}_p(\psi, \nu)$ more explicitly in terms of the local Langlands correspondence for general linear groups, as follows. Since $\psi \in \widetilde{\Pi}_2(G^*)$, it is of the form

$$\psi = \boxplus_{i \in I} \pi_i[d_i],$$

where each π_i is a self-dual cuspidal automorphic representation of GL_{N_i}, d_i are positive integers such that $\sum N_i d_i = N$, and the pairs (π_i, d_i) are distinct. For any irreducible admissible representation π_p of a general linear group over $\mathbb{Q}_p$, we write $\mathcal{V}(\pi_p)$ for the representation of WD_p corresponding to π_p under the local Langlands correspondence. By the explicit description of S_ψ in [**Art13**, (1.4.9)] (the notation N_i in *loc. cit.* corresponding to our $N_i d_i$), we have the following classification of $\mathcal{V}_p(\psi, \nu)$.

(1) The odd case. We have $S_\psi \cong \{\pm 1\}^I$. Set $I_\nu = \{i\}$ if ν is given by the i-th projection $\{\pm 1\}^I \to \{\pm 1\}$ for some $i \in I$. Otherwise, set $I_\nu = \emptyset$.

(2) The even case. Let I_{odd} be the set of $i \in I$ such that $\widehat{G_{\pi_i}}$ is odd orthogonal (or equivalently, $N_i d_i$ is odd), and let $I_{\mathrm{even}} = I - I_{\mathrm{odd}}$. We have $S_\psi \cong \{\pm 1\}^{I_{\mathrm{even}}} \times \{\pm 1\}^{I_{\mathrm{odd}},\prime}$, where as usual we write $\{\pm 1\}^{J,\prime}$ for the kernel of the map $\{\pm 1\}^J \to \{\pm 1\}, (z_j)_j \mapsto \prod_j z_j$ for any finite set J. Suppose ν is the restriction to $\{\pm 1\}^{I_{\mathrm{even}}} \times \{\pm 1\}^{I_{\mathrm{odd}},\prime}$ of the i-th projection $\{\pm 1\}^I \to \{\pm 1\}$ for some $i \in I$. Then we set $I_\nu = \{i\}$ unless $i \in I_{\mathrm{odd}}$ and $|I_{\mathrm{odd}}| = 2$, in which case we set $I_\nu = I_{\mathrm{odd}}$. In all the other cases, set $I_\nu = \emptyset$.

Then in both the odd and even cases we have

$$\mathcal{V}_p(\psi, \nu) = \bigoplus_{i \in I_\nu} \mathcal{V}(\pi_{i,p})^{(d_i)}$$

for all p.

For any finite set S of prime numbers, we define

$$L^S(\mathcal{M}(\psi, \nu), s) := \prod_{p \notin S} L_p(\mathcal{M}_p(\psi, \nu), s),$$

where $\mathcal{M}(\psi,\nu)$ is just a formal symbol, and the product is over all prime numbers $p \notin S$. By the previous classification, $L^S(\mathcal{M}(\psi,\nu),s)$ is nothing but a finite product of the S-partial standard L-functions associated to automorphic representations of general linear groups with some shifting in the variable s. Therefore the infinite product defining $L^S(\mathcal{M}(\psi,\nu),s)$ converges absolutely in some right half plane and continues to a meromorphic function in s over the whole $\mathbb{C}$. Specifically, letting $I_\nu \subset I$ be as above, we have

$$L^S(\mathcal{M}(\psi,\nu),s) = \prod_{i\in I_\nu}\prod_{j=0}^{d_i-1} L^S(\pi_i, s - \frac{n}{2} + \frac{d_i-1}{2} - j).$$

9.7.4. — Let $\mathbb{V}$ be as in §9.5.1, and fix a neat compact open subgroup K of $G(\mathbb{A}_f)$ assumed to be completely symmetric (see Definition 9.6.2). In the following we fix an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. For each prime p unequal to ℓ and unramified for the $\Gamma_{\mathbb{Q}}$-module $\mathbf{IH}^*(\overline{\mathrm{Sh}_K},\mathbb{V})$ over $\overline{\mathbb{Q}}_\ell$ (that is, unramified for each degree $*$), we define

$$\zeta_p(\mathbf{IH}^*(\overline{\mathrm{Sh}_K},\mathbb{V}),s) := \prod_j \det(1-\mathrm{Frob}_p\, p^{-s} \mid \mathbf{IH}^j(\overline{\mathrm{Sh}_K},\mathbb{V}))^{(-1)^{j+1}},$$

where on the right hand side $\mathbf{IH}^j(\overline{\mathrm{Sh}_K},\mathbb{V})$ is viewed as a vector space over $\mathbb{C}$. (The product is finite, since $\mathbf{IH}^j(\overline{\mathrm{Sh}_K},\mathbb{V})$ is non-zero only for $0 \le j \le 2\dim \mathrm{Sh}_K$.) This is the Euler factor at p of the Hasse–Weil zeta function of $\mathbf{IH}^*(\overline{\mathrm{Sh}_K},\mathbb{V})$, and it is a rational function in p^s. If S is a finite set of primes containing ℓ such that every prime p outside S is unramified for $\mathbf{IH}^*(\overline{\mathrm{Sh}_K},\mathbb{V})$, then we define the formal Dirichlet series

$$\zeta^S(\mathbf{IH}^*(\overline{\mathrm{Sh}_K},\mathbb{V}),s) := \prod_{p\notin S}\zeta_p(\mathbf{IH}^*(\overline{\mathrm{Sh}_K},\mathbb{V}),s).$$

This is the S-partial Hasse–Weil zeta function of $\mathbf{IH}^*(\overline{\mathrm{Sh}_K},\mathbb{V})$.

Theorem 9.7.5. — *Assume Hypothesis 9.1.2. Let S be the set $\Sigma_{\mathrm{bad}}(K,1_K)$ as in (9.5.1.1), applied to $f^\infty = 1_K$. For all primes $p \notin S$ we have*

$$\log\zeta_p(\mathbf{IH}^*(\overline{\mathrm{Sh}_K},\mathbb{V}),s) = \sum_{\psi\in\widetilde{\Psi}(G^*)_{\mathbb{V}}}\ \sum_{\pi^\infty\in\widetilde{\Pi}^\infty_\psi(G)} \dim(\dot{\pi}^\infty)^K \cdot \sum_{\nu\in S^D_\psi} m(\pi^\infty,\psi,\nu)(-1)^n\nu(s_\psi)\log L_p(\mathcal{M}_p(\psi,\nu),s)$$

with notations explained below.

– *The set $\widetilde{\Psi}(G^*)_{\mathbb{V}}$ is as in Definition 9.6.1.*

– *The number $m_\psi \in \{1,2\}$ is defined in (9.2.2.2). In the odd case it is always* 1.

– *For each $\psi \in \widetilde{\Psi}(G^*)_{\mathbb{V}}$, $\pi^\infty = (\pi_v)_v \in \widetilde{\Pi}^\infty_\psi(G)$, and $\nu \in S^D_\psi$, the number $m(\pi^\infty,\psi,\nu) \in \{0,1\}$ is defined as follows. Fix an arbitrary $\pi_\infty \in \Pi^{\mathrm{AJ}}_{\psi_\infty}(G_\infty)$. On*

S_ψ we have the character:

$$s \longmapsto \nu(s)^{-1}\epsilon_\psi(s)\langle s, \lambda_{\pi_\infty}\rangle\langle s, \pi_\infty\rangle_{\mathrm{AJT}} \prod_{v\neq\infty} \langle s, \pi_v\rangle,$$

where $\langle s, \lambda_{\pi_\infty}\rangle$ is defined on p. 195 of [**Kot90**]*, and ϵ_ψ is as in (9.2.2.5). We define $m(\pi^\infty, \psi, \nu)$ to be 1 if this character is trivial and 0 otherwise.*

– The number $\nu(s_\psi)$ is 1 or -1 since $s_\psi^2 = 1$; see (9.2.2.4) for the definition of $s_\psi \in S_\psi$.

In particular, we have

$$\log \zeta^S(\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})), s) = \sum_{\psi\in\widetilde{\Psi}(G^*)_{\mathbb{V}}} \sum_{\pi^\infty\in\widetilde{\Pi}^\infty_\psi(G)} \dim(\dot{\pi}^\infty)^K \cdot \sum_{\nu\in S_\psi^D} m(\pi^\infty, \psi, \nu)(-1)^n\nu(s_\psi) \log L^S(\mathcal{M}(\psi, \nu), s),$$

for s in some right half plane. This expresses $\zeta^S(\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})), s)$ as a finite product of integral powers of $L^S(\mathcal{M}(\psi, \nu), s)$ for various ψ and ν, and gives a meromorphic continuation of $\zeta^S(\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})), s)$ to the whole $\mathbb{C}$.

Proof. — This immediately follows from Theorem 9.6.4 applied to $f^\infty dg^\infty = \mathrm{vol}(K)^{-1}1_K dg^\infty$. □

Remark 9.7.6. — Theorem 9.7.5 can be slightly generalized as follows. We can replace the completely symmetric K by a more general neat compact open subgroup K' of $G(\mathbb{A}_f)$ stable under ϑ^∞, and replace S by a sufficiently large, finite set of primes depending on K'. For the proof of this generalization, we can take a completely symmetric K contained in K' (see Remark 9.6.3), and apply Theorem 9.6.4 to K and the element $\mathrm{vol}(K')^{-1}1_{K'}dg^\infty$ of $\mathcal{H}(G(\mathbb{A}_f) /\!\!/ K)_\mathbb{Q}$.

9.8. More refined decompositions

9.8.1. — Throughout we assume the setting of Theorem 9.6.4. In particular we fix $\mathbb{V}$ as in §9.5.1 and assume that K is completely symmetric. By Remark 9.6.3, this assumption on K is harmless for the understanding of $\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})$ for general K. We also keep assuming Hypothesis 9.1.2 without further mentioning.

In the sequel, we write $\mathbf{IH}^j$ for $\mathbf{IH}^j(\overline{\mathrm{Sh}_K}, \mathbb{V})$. This is non-zero only for $0 \leq j \leq 2\dim \mathrm{Sh}_K = 2n$. We fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$, and do not distinguish between representations over $\mathbb{C}$ and over $\overline{\mathbb{Q}}_\ell$, nor between $\mathbb{C}$-valued functions and $\overline{\mathbb{Q}}_\ell$-valued functions. Nevertheless, we remember that $\Gamma_\mathbb{Q}$-representations on vector spaces over $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$ are always continuous with respect to the ℓ-adic topology. Let $\mathcal{H}_K := \mathcal{H}(G(\mathbb{A}_f) /\!\!/ K)_\mathbb{Q} \otimes_\mathbb{Q} \overline{\mathbb{Q}}_\ell$.

We shall apply Theorem 9.6.4 to obtain information about more refined decompositions of $\mathbf{IH}^j$ as a $\mathcal{H}_K \times \Gamma_{\mathbb{Q}}$-module. Ideally, one would like to decompose $\mathbf{IH}^j$ into π^∞-isotypic components $\mathbf{IH}^j[\pi^\infty]$, for π^∞ running through all irreducible admissible representations of $G(\mathbb{A}_f)$, and to describe the Galois module structure of each $\mathbf{IH}^j[\pi^\infty]$. However there are the following two technical obstructions (which can be overcome in the odd case, as we shall eventually see):

(1) In the even case, each element of a global packet $\widetilde{\Pi}_\psi^\infty(G)$ as in §9.4.6 does not give rise to a well-defined isomorphism class of $G(\mathbb{A}_f)$-representations, but rather it gives rise to an ϑ^∞-orbit of such isomorphism classes. This obstruction is intrinsic in the endoscopic classification in [**Art13**] and [**Taï19**]. As a result, we are only able to describe the Galois module structure for the direct sum of $\mathbf{IH}^j[\pi^\infty]$ over all π^∞ in the same ϑ^∞-orbit, as opposed to each individual $\mathbf{IH}^j[\pi^\infty]$. We mention that in the even case the need to assume that $\mathbb{V}$ is of the special form as in §9.5.1 also stems from the same obstruction in the endoscopic classification.

(2) In both the odd and even cases, for a general $\psi \in \widetilde{\Psi}_2(G^*)$ it is not known (although expected, as would follow from the Ramanujan–Petersson conjecture for general linear groups) that the localization ψ_v is bounded on WD_v for all finite places v. As a result of this drawback, the $G(\mathbb{Q}_v)$-representations in the local packet $\widetilde{\Pi}_{\psi_v}(G_v)$ are not known to be irreducible.

We make several comments on (2). Recall that for any $\psi \in \widetilde{\Psi}_2(G^*)$, the localization ψ_v of ψ lies in $\Psi^+_{\mathrm{unit}}(G^*_v)$. For arbitrary $\psi_v \in \Psi^+_{\mathrm{unit}}(G^*_v)$ (which may not arise as the localization of a global parameter), Arthur has conjectured that the $G^*(\mathbb{Q}_v)$-representations in the local packet $\widetilde{\Pi}_{\psi_v}(G^*_v)$ are irreducible. See [**Art13**, §§1.3–1.5, Conjecture 8.3.1] for more details. This conjecture would imply that the $G(\mathbb{Q}_v)$-representations in $\widetilde{\Pi}_{\psi_v}(G_v)$ are irreducible. In the even case, if ψ is "trivial on SL_2" in the sense that $\psi = \boxplus_i \pi_i[d_i]$ with all d_i equal to 1, then this conjecture has been proved(13) by B. Xu [**Xu18**, Appendix].

We can sometimes circumvent Arthur's conjecture by using known cases of the Ramanujan–Petersson conjecture. To wit, assume that $\psi = \boxplus_i \pi_i[d_i] \in \widetilde{\Psi}_2(G^*)$ satisfies the following condition:

(†) The constituents π_i, which we recall are self-dual unitary cuspidal automorphic representations of GL_{N_i} over $\mathbb{Q}$, are all regular C-algebraic or regular L-algebraic.(14)

Then we know that ψ_v is bounded on WD_v for all finite places v, since the Ramanujan–Petersson conjecture for π_i is known at all v. Indeed, let π_i' be the twist of π_i

(13) We thank the referee for pointing this out to us.

(14) The meaning of "regular C-algebraic" here is that the infinitesimal character of $\pi_{i,\infty}$ should be regular C-algebraic as in §9.3.4. In the more classical literature this condition is usually referred to as "regular algebraic". The meaning of "regular L-algebraic" is that the infinitesimal character of $\pi_{i,\infty}$ should be the Weyl orbit of a regular *integral* character of a maximal torus. The two notions are the same for GL_{N_i} precisely when N_i is odd.

by $\mathrm{GL}_{N_i}(\mathbb{A}) \to \mathbb{R}^\times, g \mapsto |\det(g)|^{1/2}$ if π_i is L-algebraic and N_i is even, and let $\pi_i' = \pi_i$ in all the other cases. Then π_i' regular C-algebraic, cuspidal, and essentially self-dual, and by the work of a long list of authors culminating in Caraiani's work [**Car12**, Thm. 1.2], $(\pi_i')_v$ is essentially tempered for all finite places v (cf. [**BLGGT14**, Thm. 2.1.1] for the essentially self-dual case, as well as a list of references). It then follows that $\pi_{i,v}$ is tempered for all finite places v, for instance by the unitarity of the central character. In conclusion, if ψ satisfies (†), we know that all the $G(\mathbb{Q}_v)$-representations in $\widetilde{\Pi}_{\psi_v}(G_v)$ are irreducible for all finite places v.

9.8.2. — Fix $\psi = \boxplus_i \pi_i[d_i] \in \widetilde{\Psi}(G^*)_{\mathbb{V}}$. We investigate when ψ satisfies (†) in §9.8.1. Let N_i be the integer such that π_i is a self-dual cuspidal automorphic representation of GL_{N_i}.

For any positive integer r, let T_r denote the diagonal matrix in GL_r, and identify $X^*(T_r)$ with $\mathbb{Z}^r$ as usual. The half sum of the standard system of positive roots is $(\frac{r-1}{2}, \frac{r-3}{2}, \cdots, \frac{1-r}{2})$. Hence an infinitesimal character $\mu \in (X^*(T_r) \otimes \mathbb{C})/\mathfrak{S}_r = \mathbb{C}^r/\mathfrak{S}_r$ for GL_r is C-algebraic if and only if it lies in $\mathbb{Z}^r/\mathfrak{S}_r$ when r is odd, and lies in $((\frac{1}{2}, \cdots, \frac{1}{2}) + \mathbb{Z}^r)/\mathfrak{S}_r$ when r is even.

When G is an odd special orthogonal group, we choose a Borel pair in $G_{\mathbb{C}}$, and identify the based root datum with $\mathrm{BRD}(\mathsf{B}_m)$ (see §1.2.5). Let ρ be the half sum of the positive roots, and let λ be the highest weight of $\mathbb{V}^*$. Thus

$$\lambda = x_1\epsilon_1 + \cdots + x_m\epsilon_m$$

with $x_i \in \mathbb{Z}$ satisfying $x_1 \geq x_2 \geq \cdots \geq x_m \geq 0$, and

$$\rho = (m-1)\epsilon_1 + (m-2)\epsilon_2 + \cdots + \epsilon_{m-1} + \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_m).$$

Under $\mathrm{Std}_G : \widehat{G} \to \widehat{\mathrm{GL}_{2m}}$, the infinitesimal character $\lambda + \rho$ of ψ_∞ gives rise to the infinitesimal character

$$(m-1+\frac{1}{2}+x_1, m-2+\frac{1}{2}+x_2, \cdots, \frac{1}{2}+x_m, -(\frac{1}{2}+x_m), \cdots, -(m-1+\frac{1}{2}+x_1))$$

for GL_{2m}. We see that it is always regular. It immediately follows that the infinitesimal character of each $\pi_{i,\infty}$ must be regular. Moreover, if d_i is odd, then the infinitesimal character of $\pi_{i,\infty}$ must lie in $((\frac{1}{2}, \cdots, \frac{1}{2}) + \mathbb{Z}^{N_i})/\mathfrak{S}_{N_i}$, so π_i is C-algebraic if and only if N_i is even. In fact this is automatic, since for d_i odd $\widehat{G_{\pi_i}}$ must be symplectic; see (9.2.2.3). If d_i is even, then the infinitesimal character of $\pi_{i,\infty}$ must lie in $\mathbb{Z}^{N_i}/\mathfrak{S}_{N_i}$, so π_i is L-algebraic. We conclude that (†) holds automatically.

When G is an even special orthogonal group, we choose a Borel pair in $G_{\mathbb{C}}$, and identify the based root datum with $\mathrm{BRD}(\mathsf{D}_m)$. Let ρ be the half sum of the positive roots, and let λ_0 be the highest weight of $\mathbb{V}_{\mathbb{C}}^*$ (resp. $\mathbb{V}_{0,\mathbb{C}}^*$) in the even symmetric case (resp. the even composite case). (See §9.5.1 for this dichotomy.) Thus

$$\lambda_0 = x_1\epsilon_1 + \cdots + x_m\epsilon_m$$

with $x_i \in \mathbb{Z}$ satisfying $x_1 \geq x_2 \geq \cdots \geq x_{m-1} \geq |x_m|$, and

$$\rho = (m-1)\epsilon_1 + (m-2)\epsilon_2 + \cdots + \epsilon_{m-1}.$$

We have $x_m = 0$ in the even symmetric case, and $x_m \neq 0$ in the even composite case. Under $\mathrm{Std}_G : \widehat{G} \to \widehat{\mathrm{GL}_{2m}}$, the infinitesimal character $\lambda_0 + \rho$ gives rise to the infinitesimal character

$$(m-1+x_1, m-2+x_2, \cdots, x_m, -x_m, \cdots, -(m-1+x_1))$$

of GL_{2m}. We see that in the even symmetric case, we cannot guarantee that the infinitesimal characters of $\pi_{i,\infty}$ are regular, whereas in the even composite case this is guaranteed. Moreover, by a similar analysis as in the odd case, π_i is C-algebraic if d_i and N_i are even, and π_i is L-algebraic if d_i is odd. Now when d_i is even, $\widehat{G_{\pi_i}}$ must be symplectic, so N_i is automatically even. We conclude that (†) automatically holds in the even composite case.

We summarize the above discussion in §9.8.1 and §9.8.2 in the following lemma.

Lemma 9.8.3. — *Let $\psi = \boxplus_i \pi_i[d_i] \in \widetilde{\Psi}(G^*)_{\mathbb{V}}$. In the odd case and the even composite case, the $G(\mathbb{Q}_v)$-representations in $\widetilde{\Pi}_{\psi_v}(G_v)$ are irreducible for all finite places v. If we are in the even symmetric case and all d_i are equal to 1, then the same conclusion also holds.* □

9.8.4. — As in [**Art13**, §3.4] we define a set $\widetilde{\mathcal{C}}_{\mathbb{A}}(G^*)$ of *Hecke systems* for G^* modulo a certain equivalence relation. Here a Hecke system for G^* is a family $(c_v)_v$, where v runs through all primes outside an unspecified finite set of primes containing all the ramified primes for G^*, and each c_v is a semi-simple conjugacy class in ${}^L G_v^*$ (where we take ${}^L G_v^*$ to be $\widehat{G^*} \rtimes \mathrm{Gal}(\mathbb{Q}_v^{\mathrm{ur}}/\mathbb{Q}_v)$ here for convenience) whose projection to $\mathrm{Gal}(\mathbb{Q}_v^{\mathrm{ur}}/\mathbb{Q}_v)$ is the Frobenius. Two such families $(c_v)_v$ and $(d_v)_v$ are said to be *equivalent* and thus define the same element of $\widetilde{\mathcal{C}}_{\mathbb{A}}(G^*)$, if for almost all v, the conjugacy classes c_v and d_v are in the same $\mathrm{Aut}({}^L G_v^*)$-orbit, or equivalently, the images of c_v and d_v under ${}^L G_v^* \to {}^L G^* \xrightarrow{\mathrm{Std}_{G^*}} \mathrm{GL}_N(\mathbb{C})$ are conjugate. (The equivalence of the two conditions follows easily from the description of $\mathrm{Aut}({}^L G_v^*)$ in Remark 9.2.6 and the fact that two elements of $\mathrm{O}_N(\mathbb{C})$ are conjugate if and only if they are conjugate in $\mathrm{GL}_N(\mathbb{C})$.)

Recall that as a fundamental construction in [**Art13**], we have a canonical injection

$$\begin{aligned} \widetilde{\Psi}(G^*) &\longrightarrow \widetilde{\mathcal{C}}_{\mathbb{A}}(G^*) \\ \psi &\longmapsto c(\psi) \end{aligned} \tag{9.8.4.1}$$

whose well-definedness is guaranteed by [**Art13**, Thm. 1.3.2, Thm. 1.4.1]. This map has the following characterization: Let $\psi \in \widetilde{\Psi}(G^*)$, and for almost all primes v for which ψ_v is unramified, denote by π_v the unique element of $\widetilde{\Pi}_{\psi_v}(G_v)$ that is a ϑ_v-orbit of $\mathcal{G}(\mathbb{Z}_v)$-unramified representations (see §9.4.6). Then for almost all v, for every $\dot{\pi}_v \in \pi_v$ the Satake parameter of the $\mathcal{H}(G(\mathbb{Q}_v) /\!\!/ \mathcal{G}(\mathbb{Z}_v))$-module $\dot{\pi}_v^{\mathcal{G}(\mathbb{Z}_v)}$ (which is 1-dimensional over $\mathbb{C}$, cf. Lemma 9.2.12) belongs to the $\mathrm{Aut}({}^L G_v^*)$-orbit of the component of $c(\psi)$ at v.

As in Definition 9.6.2, we have $K = \prod_v K_v$. We have a canonical (finite) direct sum decomposition of $\mathcal{H}_K \times \Gamma_{\mathbb{Q}}$-modules:

$$\mathbf{IH}^j = \bigoplus_{c \in \widetilde{\mathcal{C}}_{\mathbb{A}}(G^*)} \mathbf{IH}^j_c,$$

where $\mathbf{IH}^j_c$ is characterized by the property that for almost all primes v for which K_v is hyperspecial, the action of the unramified Hecke algebra $\mathcal{H}(G(\mathbb{Q}_v) /\!\!/ K_v)$ on $\mathbf{IH}^j_c$ is via characters which correspond under the Satake isomorphism to elements of the $\mathrm{Aut}({}^L G^*)$-orbit of the component of c at v.

We denote by $\mathrm{Irr}(G(\mathbb{A}_f))$ the set of isomorphism classes of irreducible admissible representations of $G(\mathbb{A}_f)$ (over $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$). For each $c \in \widetilde{\mathcal{C}}_{\mathbb{A}}(G^*)$ and $\tau \in \mathrm{Irr}(G(\mathbb{A}_f))$, let

$$W^j_c(\tau) := \mathrm{Hom}_{\mathcal{H}_K}(\tau^K, \mathbf{IH}^j_c).$$

We then have direct sum decompositions of $\mathcal{H}_K \times \Gamma_{\mathbb{Q}}$-modules

$$\mathbf{IH}^j_c = \bigoplus_{\tau \in \mathrm{Irr}(G(\mathbb{A}_f))} \tau^K \otimes_{\overline{\mathbb{Q}}_\ell} W^j_c(\tau),$$

$$\mathbf{IH}^j = \bigoplus_{c \in \widetilde{\mathcal{C}}_{\mathbb{A}}(G^*)} \bigoplus_{\tau \in \mathrm{Irr}(G(\mathbb{A}_f))} \tau^K \otimes_{\overline{\mathbb{Q}}_\ell} W^j_c(\tau), \tag{9.8.4.2}$$

where on the right hand sides $\mathcal{H}_K$ acts on τ^K and $\Gamma_{\mathbb{Q}}$ acts on $W^j_c(\tau)$. Here we have used the fact that $\mathbf{IH}^j$ is a semi-simple $\mathcal{H}_K$-module, which follows from the "Matsushima formula" for L^2-cohomology [**BC83**, Thm. 4.5] and Zucker's conjecture comparing L^2-cohomology with intersection cohomology, proved by Looijenga [**Loo88**], Saper–Stern [**SS90**], and Looijenga–Rapoport [**LR91**].

Theorem 9.8.5. — *Assume Hypothesis 9.1.2. Let $c \in \widetilde{\mathcal{C}}_{\mathbb{A}}(G^*)$. The following statements hold.*

(1) *If c is not in the image of $\widetilde{\Psi}(G^*)_{\mathbb{V}}$ under the map (9.8.4.1), then*

$$\mathbf{IH}^j_c = 0$$

for all j.

(2) *Assume that $c = c(\psi)$ for $\psi \in \widetilde{\Psi}(G^*)_{\mathbb{V}}$. The for almost all primes p and all integers a we have*

$$\sum_j (-1)^j \,\mathrm{Tr}(\mathrm{Frob}_p^a \mid \mathbf{IH}^j_c) \tag{9.8.5.1}$$

$$= m_\psi \sum_{\pi^\infty \in \widetilde{\Pi}^\infty_\psi(G)} \dim(\dot{\pi}^\infty)^K \sum_{\nu \in S^D_\psi} m(\pi^\infty, \psi, \nu)(-1)^n \nu(s_\psi) \,\mathrm{Tr}(\mathrm{Frob}_p^a \mid \mathcal{M}_p(\psi, \nu)),$$

where the terms on the right hand side are defined in the same way as in Theorem 9.7.5, with $\mathcal{M}_p(\psi,\nu)$ defined in §9.7.3.

(3) *Keep the assumption in (2), and assume that $\mathbf{IH}_c^j \neq 0$ for some j. Write $\psi = \boxplus_{i\in I}\pi_i[d_i]$. Then for each $i \in I$ and for almost all primes p, $\pi_{i,p}$ is tempered.*[15]

(4) *Keep the assumption in (2), and assume that the conclusion of Lemma 9.8.3 holds for ψ. Thus each $\pi^\infty \in \widetilde{\Pi}_\psi^\infty(G)$ determines a ϑ^∞-orbit $[\pi^\infty]$ in $\mathrm{Irr}(G(\mathbb{A}_f))$, as in §9.4.6. Let $\tau_0 \in \mathrm{Irr}(G(\mathbb{A}_f))$ be such that $\tau_0^K \neq 0$ and $\tau_0 \notin [\pi^\infty], \forall \pi^\infty \in \widetilde{\Pi}_\psi^\infty(G)$. Then*

$$W_c^j(\tau_0) = 0$$

for all j. Moreover, for each $\pi^\infty \in \widetilde{\Pi}_\psi^\infty(G)$, we have

$$(9.8.5.2)\quad \sum_j (-1)^j \operatorname{Tr}(\mathrm{Frob}_p^a \mid \bigoplus_{\tau\in[\pi^\infty]} \dim(\tau^K)\cdot W_c^j(\tau)) \\ = m_\psi \dim(\dot{\pi}^\infty)^K \sum_{\nu\in S_\psi^D} m(\pi^\infty,\psi,\nu)(-1)^n \nu(s_\psi) \operatorname{Tr}(\mathrm{Frob}_p^a \mid \mathcal{M}_p(\psi,\nu)),$$

for almost all primes p and all integers a.

Proof. — Throughout the proof we use the following notations: We fix the Haar measure dg^∞ on $G(\mathbb{A}_f)$ giving volume 1 to K. Let $\widetilde{\mathcal{H}} = (\mathcal{H}_K)^{\vartheta^\infty}$. Then $\widetilde{\mathcal{H}}$ is a $\mathbb{C}$-subalgebra of $\mathcal{H}_K$ with unit $1_K dg^\infty$. By the same argument as in the proof of Lemma 9.5.3, we know that as a $\mathbb{C}$-vector space $\widetilde{\mathcal{H}}$ is generated by elements of the form $(\prod_v f_v)dg^\infty$, where the product is over all primes v, $f_v \in C_c^\infty(G(\mathbb{Q}_v) /\!\!/ K_v)^{\theta_G}$ for all v, and $f_v = 1_{K_v}$ for almost all v. For any finite set of primes S such that K_v is hyperspecial for all $v \notin S$, we let $\widetilde{\mathcal{H}}^S$ be the $\mathbb{C}$-vector subspace of $\widetilde{\mathcal{H}}$ spanned by elements of the form $(\prod_v f_v)dg^\infty$, where $f_v \in C_c^\infty(G(\mathbb{Q}_v)/\!\!/K_v)^{\theta_G}$ for all v, and the set of v such that $f_v \neq 1_{K_v}$ is finite and disjoint from S. Then $\widetilde{\mathcal{H}}^S$ is a commutative unital subring of $\widetilde{\mathcal{H}}$, identified with the restricted tensor product of the ϑ_v-fixed subrings of the unramified Hecke algebras $C_c^\infty(G(\mathbb{Q}_v) /\!\!/ K_v)$ over all $v \notin S$.

For each $\psi \in \widetilde{\Psi}(G^*)_{\mathbb{V}}$ and each $\pi^\infty \in \widetilde{\Pi}_\psi^\infty(G)$, recall from §9.4.6 that the $G(\mathbb{A}_f)$-representation $\dot{\pi}^\infty$ is the restricted tensor product of $\dot{\pi}_v$ over all primes v, where for each v we choose a member $\dot{\pi}_v \in \pi_v$. The $\widetilde{\mathcal{H}}$-module $(\dot{\pi}^\infty)^K$ depends only on π^∞, not on the extra choices. We henceforth denote it $(\pi^\infty)^K$.

(1) By the finiteness statement in Theorem 9.6.4, on the RHS of (9.6.4.1) only a finite subset $\Psi_0 \subset \widetilde{\Psi}(G^*)_{\mathbb{V}}$ would potentially contribute non-trivially, and for each $\psi \in \Psi_0$ only a finite subset $\mathcal{U}_\psi \subset \widetilde{\Pi}_\psi^\infty(G)$ would potentially contribute non-trivially. Moreover Ψ_0 and $(\mathcal{U}_\psi)_{\psi\in\Psi_0}$ are independent of $f^\infty dg^\infty, p, a$. We may and shall take each $\mathcal{U}_\psi$ such that its members π^∞ satisfy $(\pi^\infty)^K \neq 0$.

Suppose c is as in (1) and $\mathbf{IH}_c^j \neq 0$ for some j. Let S be a finite set of primes such that K_v is hyperspecial for all $v \notin S$. Then the set of characters through which $\widetilde{\mathcal{H}}^S$ acts

[15] We thank the anonymous referee for suggesting this result to us.

on $\mathbf{IH}_c^j$ (i.e., the isomorphism classes of the simple $\widetilde{\mathcal{H}}^S$-submodules of $\mathbf{IH}_c^j$) is disjoint from the set of characters through which $\widetilde{\mathcal{H}}^S$ acts on $\mathbf{IH}_{c'}^{j'}$ for all j' and all $c' \neq c$, and on $(\pi^\infty)^K$ for all $\pi^\infty \in \coprod_{\psi\in\Psi_0} \mathcal{U}_\psi$. Indeed, this follows from the observation that for any two characters $\chi_1, \chi_2 : C_c^\infty(G(\mathbb{Q}_v) /\!\!/ K_v) \to \mathbb{C}$ having the same restriction to the ϑ_v-fixed subring, χ_1 and χ_2 must be related by ϑ_v, and hence the Satake parameters of χ_1 and χ_2 must be related by $\mathrm{Aut}({}^L G_v^*)$. The observation itself follows from the identity $\chi_1 + \theta_G(\chi_1) = \chi_2 + \theta_G(\chi_2)$ (which holds since for all $F \in C_c^\infty(G(\mathbb{Q}_v) /\!\!/ K_v)$, $F + \theta_G^* F$ lies in the ϑ_v-fixed subring) and the linear independence of characters. Since all these $\widetilde{\mathcal{H}}^S$-modules are finite-dimensional over $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ and there are only finitely many of them which are non-zero, there exists $f^\infty dg^\infty \in \widetilde{\mathcal{H}}^S \subset \widetilde{\mathcal{H}}$ that acts as the identity on $\mathbf{IH}_c^j$ for all j, as zero on $\mathbf{IH}_{c'}^{j'}$ for all j' and all $c' \neq c$, and as zero on $(\pi^\infty)^K$ for all $\pi^\infty \in \coprod_{\psi\in\Psi_0} \mathcal{U}_\psi$. We then apply Theorem 9.6.4 (generalized in the obvious manner from ϑ^∞-fixed elements of $\mathcal{H}(G(\mathbb{A}_f) /\!\!/ K)_{\mathbb{Q}}$ to elements of $\widetilde{\mathcal{H}}$) to $f^\infty dg^\infty$ and obtain

$$\sum_j (-1)^j \,\mathrm{Tr}(\mathrm{Frob}_p^a \mid \mathbf{IH}_c^j) = 0$$

for all sufficiently large primes p and all integers a. By Chebotarev's density theorem and the Brauer–Nesbitt theorem, this implies that in the Grothendieck group of $\Gamma_{\mathbb{Q}}$-representations over $\overline{\mathbb{Q}}_\ell$ we have

$$\sum_j (-1)^j [\mathbf{IH}_c^j] = 0.$$

By a purity result of Pink [**Pin92a**, Prop. 5.6.2] applied to our Shimura datum $\mathbf{O}(V)$ of abelian type, and by the purity of intersection cohomology, we know that for almost all primes p the action of Frob_p on $\mathbf{IH}^j$ has weight j. (Note that the weight cocharacter of the Shimura datum which appears in [**Pin92a**, §5.4] is trivial in our case.) It then follows that there is no cancellation between $[\mathbf{IH}_c^j]$ for different j in the Grothendieck group. Hence $\mathbf{IH}_c^j = 0$ for all j, which proves (1).

(2) Similarly as in the proof of (1), the set of characters through which $\widetilde{\mathcal{H}}^S$ acts on $\mathbf{IH}_c^j$ for all j and on $(\pi^\infty)^K$ for all $\pi^\infty \in \mathcal{U}_\psi$, is disjoint from the set of characters through which $\widetilde{\mathcal{H}}^S$ acts on $\mathbf{IH}_{c'}^{j'}$ for all j' and all $c' \neq c$ and on $(\pi^\infty)^K$ for all $\pi^\infty \in \coprod_{\psi'\in\Psi_0-\{\psi\}} \mathcal{U}_{\psi'}$. Thus we can find $f^\infty dg^\infty \in \widetilde{\mathcal{H}}^S \subset \widetilde{\mathcal{H}}$ which acts as the identity on $\mathbf{IH}_c^j$ for all j, as the identity on $(\pi^\infty)^K$ for all $\pi^\infty \in \mathcal{U}_\psi$, as zero on $\mathbf{IH}_{c'}^{j'}$ for all j' and all $c' \neq c$, and as zero on $(\pi^\infty)^K$ for all $\pi^\infty \in \coprod_{\psi'\in\Psi_0-\{\psi\}} \mathcal{U}_{\psi'}$. Applying Theorem 9.6.4 to $f^\infty dg^\infty$ then gives the desired result.

(3) For almost all p, part (2) gives a multiplicative relation

$$\det(T - \mathrm{Frob}_p \mid \mathcal{M}_p(\psi,\nu))^{k_\nu} = \prod_j \det(T - \mathrm{Frob}_p \mid \mathbf{IH}_c^j)^{(-1)^j}$$

for each $\nu \in S_\psi^D$, where k_ν is an integer (independent of p). By the purity results used in the proof of (1) and by our assumption that $\mathbf{IH}_c^j \neq 0$ for some j, we conclude that

$k_\nu \neq 0$ (since the right hand side of the above relation cannot be 1). It then follows from the above relation that Frob_p acts on $\mathcal{M}_p(\psi, \nu)$ with integer weights for each ν. On the other hand we have an isomorphism of WD_p-representations

$$\bigoplus_{\nu \in S_\psi^D} \mathcal{V}_p(\psi, \nu) \cong \bigoplus_{i \in I} \mathcal{V}(\pi_{i,p})^{(d_i)},$$

where $\mathcal{V}_p(\psi, \nu)$ is as in §9.7.3, and $\mathcal{V}(\pi_{i,p})$ is the WD_p-representation corresponding to $\pi_{i,p}$ under the local Langlands correspondence. Clearly $\pi_{i,p}$ is unitary since π_i is. By [**Sha74**], $\pi_{i,p}$ is generic. Hence by [**JS81**, Cor. 2.5], all eigenvalues λ of Frob_p on $\mathcal{V}(\pi_{i,p})$ satisfy $p^{-1/2} < |\lambda| < p^{1/2}$. Therefore if $\pi_{i,p}$ is not tempered for one i, then there is at least one eigenvalue of Frob_p on $\bigoplus_\nu \mathcal{V}_p(\psi, \nu)$ whose absolute value is not an integer power of $p^{1/2}$. This contradicts with the fact that Frob_p acts on $\mathcal{M}_p(\psi, \nu) = \|\cdot\|^{-n/2} \mathcal{V}_p(\psi, \nu)$ with integer weights for each ν. We have proved (3).

(4) Pick S large enough such that $\widetilde{\mathcal{H}}^S$ acts on $(\pi^\infty)^K$ for all $\pi^\infty \in \mathcal{U}_\psi$ through a common character $\chi^S : \widetilde{\mathcal{H}}^S \to \mathbb{C}$. Since τ_0 is an irreducible admissible $G(\mathbb{A}_f)$-representation, we know that $\widetilde{\mathcal{H}}^S$ must act on τ_0^K via a character χ_0^S (as opposed to several different characters). Assume for the sake of contradiction that $W_c^j(\tau_0) \neq 0$. Then up to enlarging S we must have $\chi^S = \chi_0^S$, by the definition of $\mathbf{IH}_c^j$. In the following we assume that this is the case. We have $\tau_0 = \bigotimes_v' \tau_{0,v}$, where each $\tau_{0,v}$ is an irreducible admissible representation of $G(\mathbb{Q}_v)$. Write G_S for $\prod_{v \in S} G(\mathbb{Q}_v)$, and write K_S for $\prod_{v \in S} K_v$. By a similar argument as in the proof of (1), our assumption that $\chi^S = \chi_0^S$ implies that for each $v \notin S$, the ϑ_v-orbit of the isomorphism class of the irreducible admissible $G(\mathbb{Q}_v)$-representation $\tau_{0,v}$ agrees with the ϑ_v-orbit arising from every $\pi^\infty \in \mathcal{U}_\psi$. Therefore our assumption on τ_0 implies that the ϑ_S-orbit of the isomorphism class of the irreducible admissible G_S-representation $\bigotimes_{v \in S} \tau_{0,v}$ is disjoint from the ϑ_S-orbit arising from any $\pi^\infty \in \mathcal{U}_\psi$. We can therefore find $f_S \in C_c^\infty(G_S /\!\!/ K_S) = \bigotimes_{v \in S} C_c^\infty(G(\mathbb{Q}_v) /\!\!/ K_v)$ such that (for a certain normalization of Haar measure) it acts as the identity on every ϑ_S-translate of $(\bigotimes_{v \in S} \tau_{0,v})^{K_S}$ and as zero on $(\bigotimes_{v \in S} \dot{\pi}_v)^{K_S}$ for all $\pi^\infty = (\pi_v)_v \in \mathcal{U}_\psi$ and all choices $(\dot{\pi}_v \in \pi_v)_{v \in S}$. Note that the defining property of f_S is invariant under the action of ϑ_S on $C_c^\infty(G_S /\!\!/ K_S)$. Hence we can replace f_S by its average under the finite group ϑ_S, and assume that f_S is fixed by ϑ_S.

After suitable scaling, the element $(f_S \cdot \prod_{v \notin S} 1_{K_v}) dg^\infty \in \widetilde{\mathcal{H}}$ acts as the identity on every ϑ^∞-translate of τ_0^K and as zero on $(\pi^\infty)^K$ for all $(\pi^\infty) \in \mathcal{U}_\psi$. By a similar argument, we can also construct an element of $\widetilde{\mathcal{H}}$ which acts as the identity on every ϑ^∞-translate of τ_0^K and as zero on τ^K for every $\tau \in \mathrm{Irr}(G(\mathbb{A}_f))$ such that τ is not isomorphic to a ϑ^∞-translate of τ_0 and $\tau^K \neq 0, W_c^j(\tau) \neq 0$. We have a third element of $\widetilde{\mathcal{H}}$, as constructed in the proof of (2), which acts as the identity on $\mathbf{IH}_c^j$ for all j, as zero on $\mathbf{IH}_{c'}^{j'}$ for all j' and all $c' \neq c$, and as zero on $(\pi^\infty)^K$ for all $\pi^\infty \in \coprod_{\psi' \in \Psi_0 - \{\psi\}} \mathcal{U}_{\psi'}$. Multiplying these three elements together, we obtain an element of $\widetilde{\mathcal{H}}$ which acts on

$\mathbf{IH}^j$ for each j as the projection to $\bigoplus_{\tau\in[\tau_0]} \tau^K \otimes W_c^j(\tau)$ with respect to (9.8.4.2), and acts as zero on $(\pi^\infty)^K$ for all $\pi^\infty \in \coprod_{\psi\in\Psi_0} \mathcal{U}_\psi$. Here $[\tau_0]$ denotes the ϑ^∞-orbit of τ_0 in $\mathrm{Irr}(G(\mathbb{A}_f))$. Applying Theorem 9.6.4 to this element we obtain

$$\sum_j (-1)^j \operatorname{Tr}(\mathrm{Frob}_p^a \mid \bigoplus_{\tau\in[\tau_0]} \tau^K \otimes W_c^j(\tau)) = 0$$

for almost all primes p and all integers a. By a similar argument as in (1), this implies that $\tau^K \otimes W_c^j(\tau) = 0$ for all $\tau \in [\tau_0]$, and in particular $W_c^j(\tau_0) = 0$, as desired.

Finally we prove (9.8.5.2). Since different elements $\pi^\infty \in \mathcal{U}_\psi$ give rise to disjoint ϑ^∞-orbits $[\pi^\infty]$, essentially the same argument as before gives us an element of $\widetilde{\mathcal{H}}$ which acts on $\mathbf{IH}^j$ for each j as the projection to $\bigoplus_{\tau\in[\pi^\infty]} \tau^K \otimes W_c^j(\tau)$ with respect to (9.8.4.2), acts as zero on $(\pi^{\infty,\prime})^K$ for all $\pi^{\infty,\prime} \in (\coprod_{\psi'\in\Psi_0-\{\psi\}} \mathcal{U}_{\psi'}) \sqcup (\mathcal{U}_\psi - \{\pi^\infty\})$, and acts as the identity on $(\pi^\infty)^K$. Applying Theorem 9.6.4 to this element we obtain (9.8.5.2). □

Remark 9.8.6. — Part (3) of Theorem 9.8.5 proves the Ramanujan–Petersson conjecture for π_i for almost all primes. As we have discussed in §9.8.1 and §9.8.2, this is known in the odd case and in the even composite case (where the conjecture is known for all primes). In the even symmetric case, however, the infinitesimal character of $\pi_{i,\infty}$ can be non-regular, and thus $\pi_i \otimes |\det|^\alpha$ is not cohomological for any $\alpha \in \mathbb{C}$. For such π_i our result proves new instances of the conjecture. We postpone a more systematic treatment to future work.

9.8.7. — By utilizing Theorem 9.8.5 (3), we can separate the contributions of different degrees j to the right hand sides of (9.8.5.1) and (9.8.5.2) as follows.

Let $\psi = \boxplus_{i\in I} \pi_i[d_i] \in \widetilde{\Psi}(G^*)_{\mathbb{V}}$, and keep the notation in §9.7.3 for ψ. Let $\nu \in S_\psi^D$. Recall from §9.7.3 that there is a subset I_ν of I of cardinality at most 2 such that $\mathcal{V}_p(\psi,\nu) = \bigoplus_{i\in I_\nu} \mathcal{V}(\pi_{i,p})^{(d_i)}$ for all primes p. Recall that $n = \dim \mathrm{Sh}_K$. For each integer j, define

$$\mathcal{M}_p(\psi,\nu,j) := \bigoplus_{\substack{i\in I_\nu \\ d_i-1\geq |n-j| \\ d_i-1\equiv n-j \mod 2}} \|\cdot\|^{-j/2}\, \mathcal{V}(\pi_{i,p}).$$

Thus

$$\mathcal{M}_p(\psi,\nu) = \bigoplus_{j\in\mathbb{Z}} \mathcal{M}_p(\psi,\nu,j). \tag{9.8.7.1}$$

Corollary 9.8.8. — *Let $c = c(\psi)$. For each integer j, we have*

$$(9.8.8.1)\quad (-1)^j \operatorname{Tr}(\mathrm{Frob}_p^a \mid \mathbf{IH}_c^j)$$
$$= m_\psi \sum_{\pi^\infty \in \widetilde{\Pi}_\psi^\infty(G)} \dim(\dot{\pi}^\infty)^K \sum_{\nu \in S_\psi^D} m(\pi^\infty, \psi, \nu)(-1)^n \nu(s_\psi) \operatorname{Tr}(\mathrm{Frob}_p^a \mid \mathcal{M}_p(\psi, \nu, j))$$

for almost all primes p and all integers a. If all d_i are 1, then

$$\mathbf{IH}_c^j = 0$$

for all $j \neq n$. If we assume that the conclusion of Lemma 9.8.3 holds for ψ, then for each integer j and each $\pi_0^\infty \in \widetilde{\Pi}_\psi^\infty(G)$, we have

$$(-1)^j \operatorname{Tr}(\mathrm{Frob}_p^a \mid \bigoplus_{\tau \in [\pi_0^\infty]} \dim(\tau^K) \cdot W_c^j(\tau))$$
$$= m_\psi \dim(\dot{\pi}_0^\infty)^K \sum_{\nu \in S_\psi^D} m(\pi_0^\infty, \psi, \nu)(-1)^n \nu(s_\psi) \operatorname{Tr}(\mathrm{Frob}_p^a \mid \mathcal{M}_p(\psi, \nu, j)),$$

for almost all primes p and all integers a.

Proof. — By Theorem 9.8.5 (3), we know that for almost all primes p, Frob_p acts on $\mathcal{M}_p(\psi, \nu, j)$ with weight j. By the purity results used in the proof of Theorem 9.8.5 (1), Frob_p acts on $\mathbf{IH}^j$ with weight j, for almost all p. The first and third statements in the corollary follow from these two facts, the decomposition (9.8.7.1), and the two formulas (9.8.5.1) and (9.8.5.2). For the second statement, for $j \neq n$ we have $\mathcal{M}_p(\psi, \nu, j) = 0$ for all $\nu \in S_\psi^D$. Applying (9.8.8.1) to $a = 0$ gives the result. □

9.8.9. — Keep the notation of §9.8.7, and assume that we are in the odd case or the even composite case. From the discussion in §9.8.2, one easily sees that for each $i \in I$ and $j \in \mathbb{Z}$ such that $d_i - 1 \equiv n - j \mod 2$, the cuspidal automorphic representation $\pi_i \otimes |\det|^{-j/2}$ of GL_{N_i} is essentially self-dual and regular L-algebraic. Thus the semi-simple ℓ-adic $\Gamma_{\mathbb{Q}}$-representation associated to $\pi_i \otimes |\det|^{-j/2}$ is known to exist and satisfies local-global compatibility; see for instance [**BLGGT14**, Thm. 2.1.1].[16] It follows that for each $j \in \mathbb{Z}$ and $\nu \in S_\psi^D$, there is a semi-simple ℓ-adic $\Gamma_{\mathbb{Q}}$-representation $\mathcal{M}(\psi, \nu, j)$, obtained by taking a direct sum of the ones just mentioned over $i \in I_\nu$ such that $d_i - 1 \geq |n - j|$ and $d_i - 1 \equiv n - j \mod 2$, such that for every prime $p \neq \ell$ the localization of $\mathcal{M}(\psi, \nu, j)$ gives the WD_p-representation $\mathcal{M}_p(\psi, \nu, j)$ up to semi-simplification.

Corollary 9.8.10. — *Let $c = c(\psi)$, and let $\pi_0^\infty \in \widetilde{\Pi}_\psi^\infty(G)$. Assume that we are in the odd case or the even composite case. Up to semi-simplification, the $\Gamma_{\mathbb{Q}}$-representations*

[16] In that reference, π is assumed to be a regular C-algebraic cuspidal essentially self-dual representation of GL_n, but the Galois representation is associated to $\pi \otimes |\det|^{(1-n)/2}$, which is regular L-algebraic.

$\mathbf{IH}_c^j$ *and* $\bigoplus_{\tau\in[\pi_0^\infty]}\dim(\tau^K)\cdot W_c^j(\tau)$ *are isomorphic to the virtual representations*

$$m_\psi \bigoplus_{\pi^\infty\in\widetilde{\Pi}_\psi^\infty(G)} \dim(\dot\pi^\infty)^K \bigoplus_{\nu\in S_\psi^D}(-1)^{j+n}\nu(s_\psi)m(\pi^\infty,\psi,\nu)\mathcal{M}(\psi,\nu,j)$$

and

$$m_\psi \dim(\dot\pi_0^\infty)^K \bigoplus_{\nu\in S_\psi^D}(-1)^{j+n}\nu(s_\psi)m(\pi_0^\infty,\psi,\nu)\mathcal{M}(\psi,\nu,j)$$

respectively. In the odd case, the semi-simplification of $W_c^j(\pi_0^\infty)$ *is isomorphic to*

$$\bigoplus_{\nu\in S_\psi^D}(-1)^{j+1}\nu(s_\psi)m(\pi_0^\infty,\psi,\nu)\mathcal{M}(\psi,\nu,j).$$

Proof. — This follows from Lemma 9.8.3, Corollary 9.8.8, Chebotarev's density theorem, and the Brauer–Nesbitt theorem. □

Remark 9.8.11. — In the even symmetric case, for $\psi=\boxplus_{i\in I}\pi_i[d_i]\in\widetilde{\Psi}(G^*)_{\mathbb{V}}$, the infinitesimal character of $\pi_{i,\infty}$ can be non-regular. Thus the conjectural ℓ-adic $\Gamma_{\mathbb{Q}}$-representation associated to (an L-algebraic twist of) π_i has not been constructed. To this end our Corollary 9.8.8 can be utilized for the construction of such a Galois representation. We will investigate this on another occasion.

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