Introduction to p-adic Geometry and Perfectoid Spaces

1 Adic spaces

As a rough idea, adic spaces are glued from affinoid adic spaces, resembling the situation that schemes are glued from affine schemes. For a *Huber pair* (A, A^+), which is a pair of topological rings $A^+ \subset A$ satisfying certain conditions, we can construct the affinoid adic spectrum

$$\mathrm{Spa}(A, A^+) := \{\text{continuous valuations } |\cdot| \colon A \to \Gamma \cup \{0\} \text{ such that } |A^+| \leqslant 1\}/\sim,$$

where Γ is a totally ordered abelian group written multiplicatively, such as $(\mathbb{R}_{>0}, \times)$. The set $Y = \mathrm{Spa}(A, A^+)$ is equipped with the so-called adic topology and it has a structure presheaf $\mathcal{O}_Y$. It is in general a subtle question as to determine when $\mathcal{O}_Y$ is a sheaf. When it is a sheaf, we say that the Huber pair (A, A^+) is *sheafy*. Roughly, adic spaces are glued from $\mathrm{Spa}(A, A^+)$ where (A, A^+) are sheafy Huber pairs.

1.1 Huber rings

We now discuss the conditions we impose on the pair (A, A^+).

Definition 1.1.1. Let A be a topological ring.

(1) We say A is *Huber* if there is an open subring $A_0 \subset A$ such that the topology on A_0 is I-adic, for some finitely generated ideal I of A_0. Recall that the I-adic topology on A_0 is the unique topology such that $\{I^n\}$ is an open neighborhood basis of $0 \in A_0$. If A is Huber, then any choice of A_0 as above (which is non-unique) is called a *ring of definition*, and I is called an *ideal of definition*.

(2) We say A is *Tate* if it is Huber and there is $\varpi \in A^\times$ which is topologically nilpotent, i.e., $\varpi^n \to 0$ as $n \to \infty$. Such a choice of ϖ is called a *pseudo-uniformizer*.

Here are some simple observations about Tate rings. For a Tate ring A, we choose a pseudo-uniformizer ϖ and a ring of definition $A_0 \subset A$. Then $\varpi^n \in A_0$ for $n \gg 0$ since A_0 is open. In this case, we may replace ϖ by ϖ^n and assume $\varpi \in A_0$. Then we have the following.

Lemma 1.1.2. *The topology on A_0 is ϖA_0-adic. Moreover, we have $A = A_0[\varpi^{-1}]$.*

Proof. For any open neighborhood U of 0 in A_0, we show that $(\varpi A_0)^m \subset U$ for all sufficiently large m. we may assume $U = I^n$ for some ideal of definition $I \subset A_0$. We see from Definition 1.1.1(2) that $\varpi^m \to 0$, and hence $\varpi^m \in U = I^n$ for $m \gg 0$. Since I^n is an ideal, this further implies $(\varpi A_0)^m \subset U$. On the other hand, for any $m \geqslant 0$, we need $(\varpi A_0)^m = \varpi^m A_0$ to be open. But this is true because the multiplication-by-ϖ^m map is a homeomorphism $A \xrightarrow{\sim} A$. □

Conversely, start with any ring A_0 and the nonzero-divisor $\varpi \in A_0$, we define $A = A_0[\varpi^{-1}]$ and put topology on A by declaring that A_0 is open, where the topology on A_0 is ϖ-adic. In this case, A is a priori a topological group and we can check that the multiplication is continuous. Then A is Tate with A_0 is being ring of definition and ϖ_0 being a pseudo-uniformizer.

Definition 1.1.3. A *non-archimedean field* is a field K equipped with the absolute value $|\cdot| \colon K \to \mathbb{R}_{\geqslant 0}$, such that

- $|\cdot|$ takes at least 3 different values, and

- the topology on K induced by $|\cdot|$ is non-archimedean and complete. (We do not require that $|\cdot|$ is discretely valued.)

Example 1.1.4 (Tate rings)**.** Let K be a non-archimedean field.

(1) The ring K is Tate with a ring of definition $\mathcal{O}_K = \{x \in K : |x| \leqslant 1\}$. Moreover, for any $\varpi \in K^\times$, it is a pseudo-uniformizer if and only if $|\varpi| < 1$.

(2) Let A be a topological K-algebra, where we require that the map $K \to A$ is a homeomorphism to its image. Then A is Tate if and only if there is an open subring $A_0 \subset A$ such that $\{xA_0 : x \in K^\times\}$ is a neighborhood basis of 0 in A. (Note that each xA_0 is automatically open in A, since multiplication by x is a homeomorphism.) In this case, any pseudo-uniformizer ϖ in K is also a pseudo-uniformizer in A.

(3) Let $(A, \|\cdot\|)$ be a normed K-algebra, namely A is a K-algebra and $\|\cdot\| : A \to \mathbb{R}_{\geqslant 0}$ is a map satisfying that

- $\|x\| = 0$ if and only if $x = 0$, (this is called the norm property; if "only if" does not hold, then we call $\|\cdot\|$ a *semi-norm*)
- $\|x + y\| \leqslant \max(\|x\|, \|y\|)$, (strong triangle inequality)
- $\|\lambda x\| = |\lambda| \cdot \|x\|$ for any $\lambda \in K$ and $x \in A$, (homogeneity)
- $\|xy\| \leqslant \|x\| \cdot \|y\|$, (sub-multiplicativity)
- $\|1\| \leqslant 1$.

We define the topology on A by the metric $(x, y) \mapsto \|x - y\|$. Then A is Tate by the criterion in (2). Indeed, we can take A_0 as in (2) to be $\{a \in A : \|a\| \leqslant 1\}$.

Definition 1.1.5 (Power-bounded elements)**.**

(1) In a topological ring A, a subset $S \subset A$ is *bounded* if for any neighborhood U of 0, there is a neighborhood V of 0 such that $S \cdot V \subset U$.

(2) An element $a \in A$ is called *power-bounded* if $\{a^n : n \geqslant 1\}$ is a bounded subset of A. We denote

$$A^\circ := \{\text{power-bounded elements}\},$$
$$A^{00} := \{\text{topologically nilpotent elements}\}.$$

Exercise 1.1.6. *Prove the following assertions.*

(1) *For any topological ring A, we always have $A^\circ \supset A^{00}$.*

(2) *If A is Huber, then A° is a subring of A.*

(3) *If A is discrete, then every subset is bounded, and $A^\circ = A$.*

(4) *If A is Tate with a ring of definition A_0 and a pseudo-uniformizer ϖ, then $S \subset A$ is a bounded subset if and only if $S \subset \varpi^{-n} A_0$ for some $n \geqslant 1$.*

(5) *Let K be a non-archimedean field, and let $(A, \|\cdot\|)$ be a normed K-algebra. Then $S \subset A$ is a bounded subset if and only if $\sup_{s \in S} \|s\| < \infty$. If $\|\cdot\|$ is further multiplicative (i.e. $\|xy\| = \|x\| \cdot \|y\|$ for all $x, y \in A$), then we have $A^\circ = \{a \in A : \|a\| \leqslant 1\}$.*

(6) *(Gauss Lemma.) Let $(A, \|\cdot\|)$ be a normed K-algebra. Let $B = \{a \in A : \|a\| \leqslant 1\}$, and $J = \{a \in A : \|a\| < 1\}$. Then B is a subring, and J is an ideal of B. Assume that the image of $\|\cdot\| : A \to \mathbb{R}_{\geqslant 0}$ is equal to the image of $|\cdot| : K \to \mathbb{R}_{\geqslant 0}$. Then the ideal J is prime if and only if $\|\cdot\|$ is multiplicative.*

Proposition 1.1.7. *Suppose A is a Huber ring and $B \subset A$ is a subring. Then the following are equivalent:*

(1) *B is a ring of definition in A;*

(2) *B is open and bounded.*

Proof. We leave the implication (1) $\Rightarrow$ (2) as an exercise. For (2) $\Rightarrow$ (1), we need that the topology of B is J-adic for some finitely generated ideal $J \subset B$. Choose $A_0 \subset A$ a ring of definition and $I \subset A_0$ an ideal of definition. Take a finite set $T \subset A_0$ generating I as an ideal in A_0. For each integer $n \geqslant 1$, we write T^n for the set of all monomials of degree n formed by elements of T.

Since B is assumed to be open, we have $I^k \subset B$ for some $k \gg 0$ as $\{I^k\}_k$ is a neighborhood basis of 0 in A. (Here I^k denotes the k-th power of I as an ideal of A_0.) We fix such a k and define $J := T^k B$, which is a finitely generated ideal of B. It remains to prove that the topology on B is J-adic.

For any $n \geqslant 1$, we have $J^n = T^{nk}B$. (Here J^n denotes the n-th power of J as an ideal of B.) Since $B \supset I^n$, we have $J^n \supset T^{nk}I^n = I^{nk+n}$, which implies that J^n is open.

Conversely, for any neighborhood U of 0 in B, we need to show that a sufficiently high power of J is contained in U. As B is assumed to be bounded, there is a neighborhood V of 0 in A such that $V \cdot B \subset U$ by Definition 1.1.5(1). There is an integer n such that $I^n \subset V$, and hence $I^n \cdot B \subset U$. But $I^n \cdot B \supset J^n$, which completes the proof. $\square$

Corollary 1.1.8. *Suppose A is a Huber ring. Then A is bounded if and only the topology on A is adic defined by some finitely generated ideal.*

Note that if A is Huber with a ring of definition A_0, then A_0 is bounded (which is one direction of Proposition 1.1.7). For any $a \in A_0$, the set $\{a^n\}$ is contained in A_0, and hence bounded. It follows that $A_0 \subset A^\circ$. In particular, A° is an open subring of A. Moreover, we have the following.

Exercise 1.1.9. *When A is a Huber ring, A° is the union of all possible rings of definition in A. Moreover, this union is filtered, i.e., the union of any two rings of definition is contained in a third.*

Definition 1.1.10. Let A be a Huber ring. We call A *uniform* if the subring of power-bounded elements A° is bounded, i.e., A° itself is a ring of definition.

Example 1.1.11. Any normed K-algebra $(A, \|\cdot\|)$ with $\|\cdot\|$ multiplicative is uniform since $A^\circ = \{a \in A \colon \|a\| \leqslant 1\}$ is bounded (cf. Exercise 1.1.6(5)).

Example 1.1.12. The following are some examples and non-examples of Huber and Tate rings.

(1) A discrete ring A is always Huber with the ring of definition A and ideal of definition (0).

(2) Any non-archimedean field K is Tate. Note that $K^\circ = \mathcal{O}_K$ and $K^{00} = \mathfrak{m}_K$. The ring of integers $\mathcal{O}_K$ is also Huber *but not Tate.*

(3) Continue with (2). The ring of formal power series $\mathcal{O}_K[\![T_1, \cdots, T_n]\!]$ is equipped with the adic topology defined by $(\varpi, T_1, \cdots, T_n)$, where $\varpi \in \mathfrak{m}_K$. It is Huber and bounded, *but not Tate.*

(4) Similarly, for any discrete ring R, the ring of formal power series $R[\![T_1, \cdots, T_n]\!]$ is equipped with the $(T_1, \cdots, T_n)$-adic topology. It is Huber and bounded, *but not Tate.*

Example 1.1.13 (Affinoid algebra)**.** For any non-archimedean field $(K, |\cdot|)$, consider

$$A = K\langle T_1, \cdots, T_n\rangle = \{f(T_1, \cdots, T_n) \in K[\![T_1, \cdots, T_n]\!] \colon \text{coefficients of } f \text{ tend to } 0\}.$$

It is equipped with the Gauss norm

$$\|\cdot\| \colon A \longrightarrow \mathbb{R}_{\geqslant 0}, \qquad \sum_{\underline{i} \in \mathbb{N}^n} a_{\underline{i}} T^{\underline{i}} \longmapsto \sup_{\underline{i}} |a_{\underline{i}}|_K.$$

Then $(A, \|\cdot\|)$ is a normed K-algebra, and hence Tate. Moreover, in the notation of Exercise 1.1.6(6), we have

$$B = \{f \in K\langle T_1, \cdots, T_n\rangle \colon \text{all coefficients of } f \text{ are in } \mathcal{O}_K\}$$

and

$$J = \{f \in K\langle T_1, \cdots, T_n\rangle \colon \text{all coefficients of } f \text{ are in } \mathfrak{m}_K\}.$$

Note that $B/J \cong (\mathcal{O}_K/\mathfrak{m}_K)[T_1, \cdots, T_n]$, which is an integral domain. Hence by that exercise, $\|\cdot\|$ is multiplicative. Then by Exercise 1.1.6(5) and Example 1.1.11, A is uniform and we have

$$A^\circ = B = \{f \in K\langle T_1, \cdots, T_n\rangle \colon \text{all coefficients of } f \text{ are in } \mathcal{O}_K\}.$$

This ring is also denoted by $\mathcal{O}_K\langle T_1, \cdots, T_n\rangle$. Note that this is the ϖ-adic completion of $\mathcal{O}_K[T_1, \cdots, T_n]$ for any pseudo-uniformizer $\varpi \in \mathfrak{m}_K$.

The objects in Example 1.1.12 and 1.1.13 are all uniform.

Example 1.1.14. Let $A = \mathbb{Q}_p[T]/T^2$. Define Gauss norm on A as before. Then A is a normed $\mathbb{Q}_p$-algebra and hence Tate. However, $A^\circ = \mathbb{Z}_p + \mathbb{Q}_p T$ is unbounded. So A is non-uniform. (Note that the Gauss norm on A is no longer multiplicative.)

1.2 Definition of adic spectra

Definition 1.2.1. Suppose A is a Huber ring.

(1) A subring $A^+ \subset A$ is called a *ring of integral elements* if it is open and integrally closed in A and if it satisfies $A^+ \subset A^\circ$.

(2) When $A^+ \subset A$ is a ring of integral elements, we call the pair (A, A^+) a *Huber pair*.

Our goal is now to define $\mathrm{Spa}(A, A^+)$ for a Huber pair (A, A^+).

Definition 1.2.2. Let A be a topological ring. By a *valuation* on A we mean a pair (v, Γ), where

- Γ is a totally ordered abelian group written multiplicatively.
- $v \colon A \to \Gamma \cup \{0\}$ is a map such that
  - $v(ab) = v(a) \cdot v(b)$,
  - $v(a + b) \leqslant \max(v(a), v(b))$, and
  - $v(1) = 1$, $v(0) = 0$.

  Here 0 is a formal symbol outside Γ (whose neutral element is denoted by 1), and we keep the conventions $\gamma > 0$ and $\gamma \cdot 0 = 0 \cdot \gamma = 0$ for all $\gamma \in \Gamma$.

We call v *continuous* if for any $a \in A$ the set $\{b \in A \colon v(b) < v(a)\}$ is open.

Exercise 1.2.3. *Given a valuation (v, Γ) on A as above, we have the following.*

(1) *Γ is torsion-free, and thus for any $a \in A$, the condition $v(a^n) = 1$ for some $n \geqslant 1$ implies $v(a) = 1$. In particular, $v(-1) = 1$.*

(2) *For any $a, b \in A$ such that $v(a) \neq v(b)$, we always have $v(a + b) = \max(v(a), v(b))$.*

(3) *The continuity of v is equivalent to the following condition: For any converging sequence (a_n) in A with limit a,*

   (a) If $v(a) \neq 0$, then $v(a) = v(a_n)$ for $n \gg 0$;

   (b) If $v(a) = 0$, then for any $b \in A$ such that $v(b) \neq 0$, we have $v(a_n) < v(b)$ for $n \gg 0$.

Fact 1.2.4. *Let (v, Γ) and (v', Γ') be two valuations on A. Then the following are equivalent:*

(1) *For any $a, b \in A$, $v(a) \leqslant v(b)$ if and only if $v'(a) \leqslant v'(b)$.*

(2) *There exist subgroups $\Gamma_1 < \Gamma$ and $\Gamma_1' < \Gamma'$ such that $\Gamma_1 \cup \{0\}$ and $\Gamma_1' \cup \{0\}$ contain the images of v and v' respectively, and there exists an isomorphism $\Gamma_1 \cong \Gamma_1'$ such that the resulting bijection $\Gamma_1 \cup \{0\} \cong \Gamma_1' \cup \{0\}$ takes v to v'.*

Definition 1.2.5. Say v and v' are *equivalent* via $\sim$ in the case (1) or (2) of Fact 1.2.4.

Definition 1.2.6 (Adic spectrum)**.** Let (A, A^+) be a Huber pair. Define its *adic spectrum* to be

$$\mathrm{Spa}(A, A^+) = \{(v, \Gamma) \text{ continuous valuation on } A\colon v(A^+) \leqslant 1\}/\sim .$$

The topology on $\mathrm{Spa}(A, A^+)$ is generated by sets of form

$$U\left(\frac{f}{g}\right) = \{v \in \mathrm{Spa}(A, A^+)\colon v(f) \leqslant v(g) \neq 0\}$$

for some fixed $f, g \in A$.

For any finite subset $T \subset A$ and $g \in A$, we write

$$U\left(\frac{T}{g}\right) := \bigcap_{t \in T} U\left(\frac{t}{g}\right).$$

This is an open set in $\mathrm{Spa}(A, A^+)$. If $T = \{t_1, \cdots, t_n\}$, we also write

$$U\left(\frac{t_1, \cdots, t_n}{g}\right) := U\left(\frac{T}{g}\right).$$

Definition 1.2.7. A subset of $\mathrm{Spa}(A, A^+)$ is called *rational* if it is of form

$$U\left(\frac{T}{g}\right)$$

where T is a finite subset of A generating an open ideal in A, namely such that TA is open. (If A is Tate, any open ideal must be equal to A.)

Lemma 1.2.8. *Rational subsets form a basis of topology of* $\mathrm{Spa}(A, A^+)$ *and the basis is stable under finite intersections.*

Proof. From Definition 1.2.6, we see the topology of $\mathrm{Spa}(A, A^+)$ is generated by open subsets of form $U(f/g)$. We need to show that $U(f/g)$ is a union of rational subsets. For this, let A_0 be a ring of definition, $I \subset A_0$ be an ideal of definition, and T be a finite set of generators of I as an ideal of A_0. If $v \in U(f/g)$, then $v(g) \neq 0$, and hence by the continuity of v we know that $\{h \in A\colon v(h) < v(g)\}$ is an open neighborhood of 0. Then for sufficiently large k we have $T^k \subset I^k \subset \{h \in A\colon v(h) < v(g)\}$. In other words, $v \in U(T^k/g)$. Hence we have

$$U\left(\frac{f}{g}\right) = \bigcup_{k \geqslant 1} U\left(\frac{\{f\} \cup T^k}{g}\right).$$

It remains to show that (f, T^k) is an open ideal of A for each fixed k. This is clear as it contains I^k.

We now check that the intersection of two rational subsets is rational. Let T_1, T_2 be finite subsets of A generating open ideals, and let $g_1, g_2 \in A$. Then we need to check that $U(T_1/g_1) \cap U(T_2/g_2)$ is rational. This set is equal to

$$U\left(\frac{\{t_1t_2, t_1g_2, t_2g_1\colon t_1 \in T_1, t_2 \in T_2\}}{g_1g_2}\right).$$

Here in the "numerator" we could have deleted the elements t_1t_2, but then the remaining elements may not generate an open ideal of A. The point here is that the elements t_1t_2 for $t_1 \in T_1$ and $t_2 \in T_2$ already generate an open ideal of A, the verification of which we leave as an exercise. □

Remark 1.2.9. An open subset of $\mathrm{Spa}(A, A^+)$ of the form

$$U\left(\frac{T}{g}\right)$$

for a finite set $T \subset A$ may or may not be a rational subset, if we do not require that T generates an open ideal. (It could still be rational, since it may have some other presentation $U(T'/g')$ where T' is a finite set generating an open ideal.) When it is not rational, it could be non-quasi-compact.

1.3 Geometry of adic spectra

Theorem 1.3.1 (Huber, cf. [Hub93, Theorem 3.5(i)]). *Let (A, A^+) be a Huber pair. The adic spectrum $X = \mathrm{Spa}(A, A^+)$ is a spectral topological space, i.e., it satisfies*

(a) *X has basis of topology $\{T_i\}_{i \in I}$ satisfying that each T_i is quasi-compact and that for any $i, j \in I$ there is $k \in I$ such that $T_i \cap T_j = T_k$;*

(b) *X is quasi-compact;*

(c) *each irreducible closed subset of X has a unique generic point. (This property is called "sober".)*

Moreover, the rational subsets of X form such a basis and in particular every rational subset is quasi-compact.

Remark 1.3.2. A topological space is spectral in the sense of Theorem 1.3.1 if and only if it is homeomorphic to $\mathrm{Spec}\, B$ for some ring B.

Fact 1.3.3. *A topological space $(X, \mathcal{T})$ is spectral if*

- *$(X, \mathcal{T})$ is T_0, and*
- *there exists a different topology $\mathcal{T}'$ on the same underlying set X, such that $(X, \mathcal{T}')$ is quasi-compact and there is a collection $\{U_i\}$ of clopens[1] in $\mathcal{T}'$ generating $\mathcal{T}$.*

Moreover, in this case, every U_i is quasi-compact for the topology induced from $\mathcal{T}$.

Proof Sketch of Huber's Theorem 1.3.1. Let $\mathrm{Cont}(A)$ be the set of equivalence classes of continuous valuations on A, and let $\mathrm{Spv}(A)$ be the set of equivalence classes of all valuations on A. We then have

$$\mathrm{Spa}(A, A^+) \subset \mathrm{Cont}(A) \subset \mathrm{Spv}(A).$$

For any ideal I of A such that $\sqrt{I}$ is equal to $\sqrt{J}$ for some finitely generated ideal J, Huber defines a subset $\mathrm{Spv}(A, I) \subset \mathrm{Spv}(A)$, which is independent of the topology on A. This definition is technical and we omit it. We take $I := A^{00} \cdot A$ to be the ideal generated by A^{00}. In this case, we have

$$\mathrm{Cont}(A) \subset \mathrm{Spv}(A, I).$$

We define the topology on $\mathrm{Spv}(A)$ and $\mathrm{Spv}(A, I)$ in the same way as in Definition 1.2.6. On $\mathrm{Spv}(A, I)$, one can further define rational subsets

$$\widetilde{U}\left(\frac{T}{g}\right),$$

in the same way as in Definition 1.2.7, except that one replaces the condition "TA is open" by the condition "$I \subset \sqrt{TA}$". (For our particular I, these two conditions are equivalent, but the second condition is in terms of only the pair (A, I), not the topology on A.)

We will need auxiliary topologies on each of the three spaces $\mathrm{Spa}(A, A^+)$, $\mathrm{Spv}(A, I)$, $\mathrm{Spv}(A)$. Let $\mathrm{Spa}(A, A^+)'$ denote the new topology on $\mathrm{Spa}(A, A^+)$ which is the coarsest such that every rational subset is clopen. (This is more refined than the original topology on $\mathrm{Spa}(A, A^+)$.) Similarly, we define $\mathrm{Spv}(A, I)'$ using the notion of rational subsets of $\mathrm{Spv}(A, I)$. In addition, we define a new topology on $\mathrm{Spv}(A)$, denoted by $\mathrm{Spv}(A)'$, to be generated by subsets of the form $\{v \colon v(f) \leqslant v(g)\}$ or $\{v \colon v(f) < v(g)\}$ for $f, g \in A$. (Note that $\mathrm{Spv}(A)'$ is more refined than the original topology on $\mathrm{Spv}(A)$, since the latter is generated by $\{v \colon v(f) \leqslant v(g)\} \cap \{v \colon v(0) < v(g)\}$.)

It is easily checked that $\mathrm{Spv}(A)$ and all its subspaces are T_0. Thus in view of Lemma 1.2.8 and Fact 1.3.3, in order to prove the theorem it suffices to prove that $\mathrm{Spa}(A, A^+)'$ is quasi-compact. This is proved in the following three steps.

Step I. Reduce to showing that $\mathrm{Spv}(A, I)'$ is quasi-compact.

It turns out that for any $v \in \mathrm{Spv}(A, I)$, v is continuous if and only if $v(f) < 1 = v(1)$ for all $f \in A^{00}$. (This equivalence is not true for a general element $v \in \mathrm{Spv}(A)$.) Such elements v form the complement of the rational subset $\widetilde{U}(1/f)$ in $\mathrm{Spv}(A, I)$. Also, for v to lie in $\mathrm{Spa}(A, A^+)$, we have the condition

[1] A clopen is a subset that is simultaneously closed and open.

$v(f) \leqslant 1 = v(1)$ imposed for every $f \in A^+$. For each f, this condition defines the rational subset $\widetilde{U}(\{f,1\}/1) \subset \mathrm{Spv}(A,I)$. Consequently, $\mathrm{Spa}(A,A^+)$ is the intersection of certain subsets of $\mathrm{Spv}(A,I)$ which are either rational subsets or complements of rational subsets of $\mathrm{Spv}(A,I)$. Thus the image of the inclusion $\mathrm{Spa}(A,A^+)' \to \mathrm{Spv}(A,I)'$ is closed. Moreover, this inclusion is a homeomorphism onto its image, since every rational subset of $\mathrm{Spa}(A,A^+)$ is the intersection of $\mathrm{Spa}(A,A^+)$ with a rational subset of $\mathrm{Spv}(A,I)$.

Step II. Reduce to showing that $\mathrm{Spv}(A)'$ is quasi-compact.

This is done through constructing and studying a natural retraction $r\colon \mathrm{Spv}(A) \to \mathrm{Spv}(A,I)$ for the inclusion $\mathrm{Spv}(A,I) \hookrightarrow \mathrm{Spv}(A)$. We need the following property: For any rational subset $U = \widetilde{U}(T/g) \subset \mathrm{Spv}(A,I)$ with $\sqrt{TA} \supset I$, we have $r^{-1}(U) = \{v \in \mathrm{Spv}(A) \colon \forall f \in T,\ v(f) \leqslant v(g) \neq 0\}$. (This seemingly harmless statement is not true without the condition $\sqrt{TA} \supset I$.) It easily follows that r induces a continuous map $\mathrm{Spv}(A)' \to \mathrm{Spv}(A,I)'$. Since r is surjective and $\mathrm{Spv}(A)'$ is quasi-compact, we conclude that $\mathrm{Spv}(A,I)'$ is quasi-compact.

Step III. Prove that $\mathrm{Spv}(A)'$ is quasi-compact.

Each $v \in \mathrm{Spa}(A)$ gives rise to a binary relation $|$ on A, namely $f|g$ if and only if $v(f) \geqslant v(g)$. The resulting map

$$\mathrm{Spv}(A) \longrightarrow \{0,1\}^{A\times A}$$

is injective with closed image. Here we equip the target with the product topology of the discrete topology on $\{0,1\}$, which is quasi-compact by Tychonoff's theorem. Clearly the subspace topology on $\mathrm{Spv}(A)$ defined by the above injection is exactly $\mathrm{Spv}(A)'$. $\square$

Remark 1.3.4. Similar methods of proof also show that $\mathrm{Cont}(A)$ and $\mathrm{Spv}(A)$ are spectral.

Theorem 1.3.5 (Huber's reconstruction theorem)**.** *Suppose (A,A^+) is a Huber pair.*

(1) $\mathrm{Spa}(A,A^+) = \emptyset$ *if and only if* $A = \overline{\{0\}}$, *i.e. the maximal Hausdorff quotient of A is* 0.

(2) *Given any $f \in A$, we have $f \in A^+$ if and only if $v(f) \leqslant 1$ for all $v \in \mathrm{Spa}(A,A^+)$.*

Remark 1.3.6. We have the following strengthening of Theorem 1.3.5(2). For any subset S of A, define

$$X_S = \{v \in \mathrm{Cont}(A) \colon v(f) \leqslant 1 \text{ for all } f \in S\}.$$

Then we have inverse bijections

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{all possible rings } A^+ \subset A \\ \text{of integral elements}\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{c}\text{all possible subsets } X \subset \mathrm{Cont}(A) \text{ of form} \\ X = X_S, \text{ where } S \text{ is any subset of } A\end{array}\right\} \\
A^+ & \longmapsto & \mathrm{Spa}(A,A^+) \\
\{f \in A \colon v(f) \leqslant 1 \text{ for all } v \in X\} & \longleftarrow\!\shortmid & X.
\end{array}$$

Example 1.3.7. We analyze $\mathrm{Spa}(A,A^+)$ when A is a field.

(1) Let K be a field. Recall that a subring $R \subset K$ is called a *valuation subring* if we have $\forall f \in K^\times, f \in R$ or $f^{-1} \in R$. Then there are inverse bijections

$$\begin{array}{ccc}
\mathrm{Spv}(K) = \{\text{all valuations } v\colon K \to \Gamma \cup \{0\}\}/\sim & \longleftrightarrow & \{\text{valuation subrings } R \subset K\} \\
v & \longmapsto & R_v = \{f \in K \colon v(f) \leqslant 1\} \\
(v_R, \Gamma_R) & \longleftarrow\!\shortmid & R.
\end{array}$$

Here, as for the image of R, we write

- $\Gamma_R = K^\times/R^\times$ as an abelian group, together with the total order defined by the rule $f \leqslant g$ if and only if $f/g \in R$.
- $v_R\colon K \to (K^\times/R^\times) \cup \{0\}$, the natural projection map.

(2) Consider the Huber pair (K, K^+) for a discrete field K. Here K^+ is any integrally closed subring of K. Then the bijection in (1) restricts to a bijection

$$\operatorname{Spa}(K, K^+) \longleftrightarrow \{R \subset K \text{ valuation subring such that } R \supset K^+\}.$$

Now we work with the special case that K^+ is itself a valuation ring. As an exercise, check that any valuation subring of K is integrally closed in K. We have the following facts:

(a) Any subring R of K containing K^+ is automatically a valuation subring. All valuation subrings of K are local.

(b) There are inverse bijections

$$\begin{aligned} \{R \text{ valuation subring of } K \colon R \supset K^+\} &\longleftrightarrow \operatorname{Spec} K^+ \\ R &\longmapsto \mathfrak{m}_R \\ (K^+)_{\mathfrak{p}} &\longleftarrow\!\shortmid\ \mathfrak{p} \end{aligned}$$

Here $\mathfrak{m}_R$ denotes the unique maximal ideal of R.

In this special case where $K^+ \subset K$ is a valuation subring, we have

$$\operatorname{Spa}(K, K^+) \cong \operatorname{Spec} K^+.$$

This is in fact a homeomorphism.

(3) Let K be a non-archimedean field with a ring of integral elements K^+. Further, assume K^+ is a valuation subring, i.e. K^+ is an arbitrary open bounded valuation subring of K. If v is any valuation on K such that $v(K^+) \leqslant 1$, corresponding to the valuation ring R with the unique maximal ideal $\mathfrak{m}_R$, then v is continuous if and only if $\mathfrak{m}_R \supset K^{00}$. Hence

$$\begin{aligned} \operatorname{Spa}(K, K^+) &\cong \{R \text{ valuation subring of } K \colon R \supset K^+,\ \mathfrak{m}_R \supset K^{00}\} \\ &\cong \operatorname{Spec}(K^+/K^{00}). \end{aligned}$$

Note that this formula is also true in case where K is discrete with the valuation subring K^+, by the discussion in (2).

Definition 1.3.8 (Affinoid field)**.** A Huber pair (A, A^+) is called an *affinoid field* if A is a field whose topology is either discrete or non-archimedean (induced by $|\cdot| \colon A \to \mathbb{R}_{\geqslant 0}$) and if A^+ is a valuation subring.

Our previous discussion shows the following:

Proposition 1.3.9. *Let* (A, A^+) *be an affinoid field. Then* $\operatorname{Spa}(A, A^+)$ *is homeomorphic to* $\operatorname{Spec}(A^+/A^{00})$.

Example 1.3.10. (1) By Example 1.3.7(3),

$$\operatorname{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) \cong \operatorname{Spec}(\mathbb{Z}_p/p\mathbb{Z}_p) = \{*\}.$$

More precisely, in this case, the single point is the class of the usual absolute value $|\cdot|_p$.

(2) Consider $v \in \operatorname{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$. We see $v^{-1}(0) \subset \mathbb{Z}_p$ is always a prime ideal $\operatorname{supp}(v)$.

- ⋄ If $\operatorname{supp}(v) = (0)$, then v extends to a continuous valuation of $\mathbb{Q}_p$. This forces v to be $|\cdot|_p$ as in (1).
- ⋄ If $\operatorname{supp}(v) = (p)$, then v factors through $\overline{v} : \mathbb{F}_p \to \Gamma \cup \{0\}$, where $\overline{v} \in \operatorname{Spa}(\mathbb{F}_p, \mathbb{F}_p) = \{*\}$. (Here $\mathbb{F}_p$ has the discrete topology. As an exercise, check that $\operatorname{Spv}(\mathbb{F}_p) = \operatorname{Spa}(\mathbb{F}_p, \mathbb{F}_p)$.) If we write v_0 for the single point of $\operatorname{Spa}(\mathbb{F}_p, \mathbb{F}_p)$, then $v_0(0) = 0$ and $v_0(x) = 1$ for $x \neq 0$.

Combining these, we see

$$\operatorname{Spa}(\mathbb{Z}_p, \mathbb{Z}_p) = \{|\cdot|_p, v_0\}$$

where $|\cdot|_p$ is the usual p-adic absolute value and

$$v_0(x) = \begin{cases} 1, & p \nmid x, \\ 0, & \text{otherwise.} \end{cases}$$

1.4 The adic closed unit disc

Let $(K, |\cdot|_K)$ be an algebraically closed non-archimedean field. Consider the uniform Tate ring

$$A = K\langle T\rangle = \left\{ \sum_{n\geqslant 0} a_n T^n \colon a_n \in K,\ a_n \to 0 \text{ as } n \to \infty \right\},$$

whose topology is given by the multiplicative Gauss norm $\|f\| = \max_{n\geqslant 1} |a_n|_K$. We have

$$A^\circ = \mathcal{O}_K\langle T\rangle = \left\{ \sum_{n\geqslant 0} a_n T^n \colon a_n \in \mathcal{O}_K,\ a_n \to 0 \text{ as } n \to \infty \right\}.$$

(See Example 1.1.13 for details.)

Exercise 1.4.1. *In any normed K-algebra $(A, \|\cdot\|)$ such that $\|\cdot\|$ is multiplicative, the set $\{a \in A \colon \|a\| \leqslant 1\}$ is an integrally closed subring.*

By the exercise $(A, A^+) = (K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$ is a Huber pair. We then consider $X = \operatorname{Spa}(A, A^+)$. The goal is to describe its points. For any $x \in \mathcal{O}_K$ and $r \in [0,1]$, we introduce the notations

$$B_{x,r} = \{y \in K \colon |y - x|_K \leqslant r\},$$
$$B^\circ_{x,r} = \{y \in K \colon |y - x|_K < r\}.$$

Note that $B_{x,r} \subset B_{0,1}$ always holds. For any $f \in A$ and any $x \in B_{0,1}$, we can evaluate $f(x) \in B_{0,1}$.

Exercise 1.4.2 (Maximum modulus principle)**.** *Fix an element $f \in A$. Let $x \in B_{0,1}$ and $r \in [0,1]$.*

(1) *If $r \in |K^\times|_K$, then the function $B_{x,r} \to \mathbb{R}_{\geqslant 0}$ via $y \mapsto |f(y)|_K$ has a maximum, and it is reached on $B_{x,r} - B^0_{x,r}$.*

(2) *For all $r \in [0,1]$, define*

$$v_{x,r}(f) := \sup_{y\in B_{x,r}} |f(y)|_K.$$

Check that $v_{x,r} \in X$ is a continuous valuation on A such that $v_{x,r}(A^+) \leqslant 1$. Further, if we have $f(T) = \sum_n a_n(T - x)^n$, then

$$v_{x,r}(f) = \max_n |a_n|_K \cdot r^n.$$

(3) *For any $f_1, f_2 \in A$, we always have*

$$v_{x,r}(f_1 f_2) = v_{x,r}(f_1) \cdot v_{x,r}(f_2).$$

(4) *We have $v_{x,r} \in \operatorname{Spa}(A, A^+)$.*

We are going to give a complete classification of points in $X = \operatorname{Spa}(A, A^+)$, with $A = K\langle T\rangle \supset A^+ = A^\circ = \mathcal{O}_K\langle T\rangle$, into certain types. It easily follows from the above exercise that for each $x \in \mathcal{O}_K$ and $r \in [0,1]$, the function $v_{x,r} \colon A \to \mathbb{R}_{\geqslant 0}$ is a continuous valuation such that $v_{x,r}(f) \leqslant 1$ for all $f \in A^+$. Thus $v_{x,r}$ can be viewed as an element of X.

Construction 1.4.3 (Points in the adic closed unit disc)**.** *Assume $\Gamma = \mathbb{R}_{>0}$ for (I)–(IV) in the following.*

(I) *Classical points: $v_{x,0} \colon f \mapsto |f(x)|_K$ for $x \in \mathcal{O}_K$.*

(II) *$v_{x,r}$ with $x \in \mathcal{O}_K$ and $r \in (0,1] \cap |K^\times|_K$.*

(III) *$v_{x,r}$ with $x \in \mathcal{O}_K$, for $r \in (0,1]$ but $r \notin |K^\times|_K$. Note that for $r = 1$, the point $v_{x,r}$ is independent of x, and is nothing but the Gauss norm $\|\cdot\|$ on A. We denote this point by β, called the Gauss point.*

(IV) The points of this type only exist when K is not spherically complete. In this case, we can have a family $\mathcal{F} = (B_{x_1,r_1} \supset B_{x_2,r_2} \supset \cdots)$ of closed balls with descending radii r_i such that $\bigcap_n B_{x_n,r_n} = \emptyset$. (It follows that $\lim_i r_i > 0$.) Then we can construct $v_{\mathcal{F}} \in X$ by

$$v_{\mathcal{F}}(f) = \inf_n v_{x_n,r_n}(f).$$

Note that this can happen when $K = \mathbb{C}_p$.

In contrast with $\Gamma = \mathbb{R}_{>0}$ before, we now assume $\Gamma = \gamma^{\mathbb{Z}} \times \mathbb{R}_{>0}$ for some generator γ such that $r < \gamma < 1$ for all $r \in (0,1)$. That is, we equip the abelian group $\Gamma \cong \mathbb{Z} \times \mathbb{R}_{>0}$ with the total order such that $(\gamma^n, r) \leqslant (\gamma^m, s)$ if and only if $r < s$ or ($r = s$ and $n \geqslant m$).

(V) Rank 2 points: Start with a point $v_{x,r} \in X$ of type (II) or (III) for $x \in \mathcal{O}_K$ and $r \in (0,1)$. We have

$$v_{x,r}\left(\sum_n a_n(T-x)^n\right) = \sup_n |a_n| \cdot r^n.$$

We are to modify this point by $\varepsilon \in \{\pm 1\}$, to get $v_{x,r,\varepsilon} \in X$ defined by

$$v_{x,r,\varepsilon}\left(\sum_n a_n(T-x)^n\right) = \sup_n |a_n| \cdot (r \cdot \gamma^{\varepsilon})^n \in (\gamma^{\mathbb{Z}} \times \mathbb{R}_{>0}) \cup \{0\}.$$

More precisely, the new points of type (V) arises from the following recipe:

(i) If $r \notin |K^\times|_K$, i.e. $v_{x,r}$ is of type (III), then we do not get new points because $v_{x,r,\varepsilon} \sim v_{x,r}$, i.e., they are equivalent valuations.

(ii) If $r \in |K^\times|_K$, then $v_{x,r,\varepsilon} \not\sim v_{x,r}$. We see $v_{x,r,+1}$ depends on (x,r) only via $B^\circ_{x,r}$, and $v_{x,r,-1}$ depends on (x,r) via $B_{x,r}$.

Now for the type (II) point $v_{x,1} = \beta$ (i.e., the Gauss point), we have a new type (V) point $v_{x,1,+1}$ by the same recipe as above, for each $x \in \mathcal{O}_K$. The point $v_{x,r,+1}$ depends on x only via $B^\circ_{x,1}$. Note that the valuation $v_{x,1,-1}$ sends $T \in A^+$ to γ^{-1}, which is > 1. This violates the defining condition for $\mathrm{Spa}(A, A^+)$. Hence we do not have points in $\mathrm{Spa}(A, A^+)$ of the form $v_{x,1,-1}$.

In fact, for $v_{x,r}$ of type (II), the closure of $\{v_{x,r}\}$ in X consists of $v_{x,r}$ and certain type (V) points. More precisely:

- *When $r \in (0,1)$,*

$$\overline{\{v_{x,r}\}} = \{v_{x,r}\} \sqcup \{v_{x',r,+1} \colon B_{x,r} = B_{x',r},\ \text{i.e. } x' \in B_{x,r}\} \sqcup \{v_{x,r,-1}\}.$$

Moreover, for $x', x'' \in B_{x,r}$, we have $v_{x',r,+1} = v_{x'',r,+1}$ if and only if $B^\circ_{x',r} = B^\circ_{x'',r}$, if and only if $|x' - x''| < r$. Thus if we choose an element $u \in K$ such that $|u| = r$ and denote by l the residue field of K, then the set $\{v_{x',r,+1} \colon x' \in B_{x,r}\}$ is in bijection with $\mathbb{A}^1(l)$ under the map $v_{x',r,+1} \mapsto$ the residue class of $(x'-x)/u \in \mathcal{O}_K$. This bijection extends to a bijection

$$\overline{\{v_{x,r}\}} \xrightarrow{\sim} \mathbb{P}^1_l,$$

where we send $v_{x,r}$ to the generic point and send $v_{x,r,-1}$ to the closed point ∞. The above bijection turns out to be a homeomorphism.

- *When $r = 1$, we have $v_{x,r} = \beta$, and*

$$\overline{\{\beta\}} = \{\beta\} \sqcup \{v_{x',1,+1} \colon x' \in \mathcal{O}_K\} \cong \mathbb{A}^1_l.$$

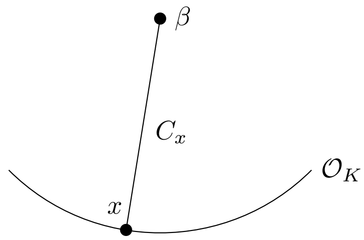
Such points in Construction 1.4.3 form $\mathrm{Spa}(A, A^+)$ whose geometry can be illustrated as a tree. A more precise description is as follows.

Construction 1.4.4 (Visualizations of points of each type in Construction 1.4.3)**.** *Let $x \in \mathcal{O}_K$ and consider the curve*

$$C_x := \{v_{x,r} : r \in [0,1]\} \subset X.$$

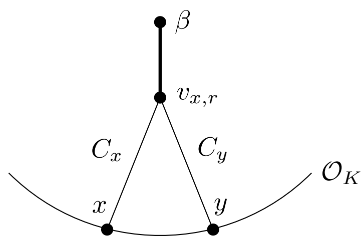
For $r = 0$, $v_{x,r}$ is a classical point; for $r = 1$, $v_{x,r} = \beta$ is the Gauss point. We thus regard C_x as a curve going from β to the classical point $v_{x,0}$ as r decreases. However, we caution that the natural map $[0,1] \xrightarrow{\sim} C_x$ is not a homeomorphism.

(I) In the following picture, the curve below denotes the set of all classical points.



Each $x \in \mathcal{O}_K$ corresponds to a point $v_{x,1}$ of type (I) via C_x. Hence the classical points are "endpoints" of the tree X, and the set of them can be identified with $\mathcal{O}_K$.

(II) We consider $v_{x,r}$ by fixing $0 \neq r \in |K^\times|_K$ and say $r = |x - y|$ with $x \neq y$. Then $B_{x,r} = B_{y,r}$ and hence $v_{x,r} = v_{y,r}$. Thus the two curves C_x and C_y meet at $v_{x,r}$. Moreover, for $r' \in [0, r)$, we have $B_{x,r'} \neq B_{y,r'}$ and so $v_{x,r'} \neq v_{y,r'}$, which means that C_x and C_y actually depart each other at $v_{x,r}$ when they go downward. Thus the point $v_{x,r}$ of type (II) is a branching point.



Note that the branching dos not happen along the upward direction, i.e., $v_{x,r'} = v_{y,r'}$ for all $r' \in [r, 1]$. In other words, the segment of C_x from β to $v_{x,r}$ is equal to the segment of C_y from β to $v_{x,r}$.

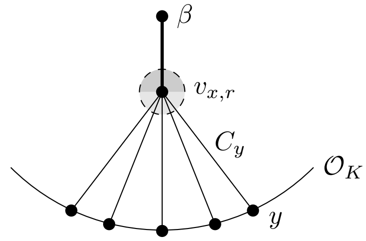
(III) Any point $v_{x,r}$ of type III is not a branching point. Therefore, the points of type (II) and (III) are seen as "limbs" of the tree X.

(IV) By Construction 1.4.3, each point $v_{\mathcal{F}}$ of type (IV) is represented by a path on the tree starting at β, passing through infinitely many branching points, but never reaching an end point. We can write this path as

$$(\beta \xrightarrow{\quad C_{x_1} \quad} v_{x_1,r_1} \xrightarrow{\quad C_{x_2} \quad} v_{x_2,r_2} \xrightarrow{\qquad\quad} \cdots)$$

where the symbol C_{x_i} on the i-th edge means that the part of the path from $v_{x_{i-1},r_{i-1}}$ to v_{x_i,r_i} agrees with the corresponding segment of the curve C_{x_i}. Therefore, the point $v_{\mathcal{F}}$ in the tree X looks like "the limiting point of ramification points on limbs".

(V) Again, start with a point $v_{x,r}$ of type (II) for $r < 1$. For each distinct ray C_y passing through $v_{x,r}$ downward, we uniquely have $v_{y,r,+1}$. On the other hand, from $v_{x,r}$ we have a unique direction going upward, corresponding to $v_{y,r,-1}$.



Therefore, we see points of type (V) of rank 2 lie in the closure of points of type (II), or more heuristically, they are "infinitesimally close to the limbs" of the tree X. For $r = 1$, namely for $v_{x,r} = \beta$, the visualization is similar, except that we do not have the "upward direction", corresponding to the fact that we do not have the point $v_{x,r,-1}$.

Proposition 1.4.5. *All points of X are given by Construction 1.4.3.*

Proof. **Step I.** Use the completion containment

$$(K[T], \mathcal{O}_K[T]) \subset (K[T]^\wedge, \mathcal{O}_K[T]^\wedge) = (K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$$

to get a natural map (by restricting the valuations)

$$\operatorname{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle) \longrightarrow \operatorname{Spa}(K[T], \mathcal{O}_K[T]).$$

It is a general fact about completions of Huber pairs that the above is a homeomorphism. We will discuss this general fact later in Proposition 1.5.8.

Step II. Classify points of rank 1.

The rank-one points are by definition continuous valuations $v\colon K[T] \to \mathbb{R}_{\geqslant 0}$ such that $v(\mathcal{O}_K[T]) \leqslant 1$. Fix such a v. We define $r_x := v(T - x)$ for any $x \in \mathcal{O}_K$. Since $T - x \in \mathcal{O}_K[T]$, we have $r_x \in [0, 1]$. Thus we can form the ball $B_{x,r_x} \subset B_{0,1}$. Using axioms for v, one can show for any $x, y \in \mathcal{O}_K$ that

$$r_x \leqslant r_y \implies B_{x,r_x} \subset B_{y,r_y}.$$

One can also show that for any $f \in K[T]$,

$$v(f) = \inf_{x\in\mathcal{O}_K} v_{x,r_x}(f) = \inf_{x\in\mathcal{O}_K} \sup_{y\in B_{x,r_x}} |f(y)|_K$$

by reducing to the case where f is a linear polynomial. Combining these two observations, we can find a decreasing family $(B_{x_1,r_1} \supset B_{x_2,v_2} \supset \cdots)$ such that

$$v(f) = \inf_n v_{x_n,r_n}(f).$$

This forces such v to be one of the points of type (I)–(IV).

Step III. Classify the rest of the point of X.

In general, for any Huber pair (A, A^+) and any $(v : A \to \Gamma \cup \{0\}) \in \operatorname{Spa}(A, A^+)$, the subset

$$\operatorname{supp}(v) := v^{-1}(0)$$

of A is a prime ideal. (Moreover, the map $\operatorname{supp}\colon \operatorname{Spa}(A, A^+) \to \operatorname{Spec} A, v \mapsto \operatorname{supp}(v)$ is continuous.) We define the *residue field* of v to be the residue field of $\operatorname{supp}(v)$, namely

$$k(v) := \operatorname{Frac}(A/\operatorname{supp}(v)).$$

This v induces a valuation $k(v) \to \Gamma\cup\{0\}$, along which the inverse image $k(v)^+$ of $\{\gamma \in \Gamma\colon \gamma \leqslant 1\}\cup\{0\}$ is a valuation subring of $k(v)$. Consequently, $k(v)^+$ is a local ring with residue field

$$\kappa(v) = \{x \in k(x)\colon v(x) \leqslant 1\}/\{x \in k(x)\colon v(x) < 1\} = k(v)^+/\{x \in k(x)\colon v(x) < 1\}.$$

Note that the natural image of A^+ in $k(v)$ is contained in $k(v)^+$.

Fact 1.4.6. *Suppose for $X = \mathrm{Spa}(A, A^+)$, the Huber pair (A, A^+) is Tate. Then:*

(1) *If $v, w \in X$ such that $v \rightsquigarrow w$, i.e. $w \in \overline{\{v\}}$, then $\mathrm{supp}(w) = \mathrm{supp}(v)$.*

(2) *Fix a point v. Then*

$$\begin{aligned}\overline{\{v\}} &\xrightarrow{\sim} \{R \text{ valuation subring of } k(v)\colon \mathrm{im}(A^+ \to k(v)^+) \subset R \subset k(v)^+\} \\ &\xrightarrow{\sim} \{\text{valuation subrings of } \kappa(v) \text{ containing } \mathrm{im}(A^+ \to \kappa(v))\} \\ &\xrightarrow{\sim} \mathrm{Spa}(\kappa(v), \mathrm{im}(A^+ \to \kappa(v))).\end{aligned}$$

Here the first isomorphism sends w to $k(w)^+$, which is a valuation subring of $k(w)$ and we have $k(w) = k(v)$ by part (1). The second isomorphism is induced by the projection $k(v)^+ \to \kappa(v)$. In the third line, we equip $\kappa(v)$ with the discrete topology. Moreover, the composition of the above three bijections is a homeomorphism.

(3) *For each $w \in X$ there exists a unique point of rank 1, say $v \in X$, such that $w \in \overline{\{v\}}$.*

Back to considering the adic closed unit disc $\mathrm{Spa}(K[T], \mathcal{O}_K[T])$. We have already classified all rank 1 points. By Fact 1.4.6 (3), in order to classify all points, we only need to compute the closure of every rank 1 point. This closure can be computed using Fact 1.4.6 (2). Now all points of rank 1 are closed except for those of type (II). Take $v_{x,r}$ of type (II) with $r < 1$. Then we have $\kappa(v_{x,r}) = l(T)$ and $\mathrm{im}(A^+ \to \kappa(v_{x,r})) = l$. Here l is the residue field of K. Thus we have

$$\overline{\{v_{x,r}\}} \cong \mathrm{Spa}(l(T), l) \cong \mathbb{P}^1_l.$$

By unwrapping the constructions it is not hard to see that the type (V) points actually realize this bijection, in the way described in Construction 1.4.3. Similarly, for $r = 1$, we have $\kappa = l(T)$ and $\mathrm{im}(A^+) = l[T]$. The situation is different because

$$\mathrm{Spa}(l(T), l[T]) \cong \mathbb{A}^1_l.$$

□

1.5 Completion

Definition 1.5.1. Let A be a topological ring. Say A is *complete* if A is Hausdorff and every Cauchy filter basis has a limit in A.

Fact 1.5.2 (Universal property of completions)**.** *There is a continuous map $A \to \widehat{A}$ such that $\widehat{A}$ is complete and this map is universal for all continuous maps $A \to B$ for complete B. The A-algebra $\widehat{A}$ is called the completion of A; it is unique up to unique isomorphism as a topological A-algebra.*

Example 1.5.3. If A has I-adic topology for an ideal I, then $\widehat{A}$ is the I-adic completion of A, i.e., $\widehat{A} = \varprojlim_n A/I^n$. However, we caution that $\widehat{A}$ may not be I-adically complete as an A-module, that is, we may not have $\widehat{A} \cong \varprojlim_n \widehat{A}/I^n\widehat{A}$. This is not a contradiction with the fact that $\widehat{A}$ is complete. When we say that $\widehat{A}$ is complete, it is with respect to some intrinsic topology on $\widehat{A}$ on account of $\widehat{A}$ being a completion of A; in the current case the topology of $\widehat{A}$ is the inverse limit topology coming from the discrete topology on A/I^n for all n. This topology is in general different from the I-adic topology on $\widehat{A}$. All these problems go away when I is finitely generated, as we will see below.

The following facts can be more important.

Fact 1.5.4. *If A has I-adic topology for a finitely generated ideal I, then the topology on $\widehat{A}$ is $(I \cdot \widehat{A})$-adic.*

This can be further extended to the case where A is Huber with a ring of definition A_0 and an ideal of definition I. Denote by $\psi\colon A \to \widehat{A}$ the completion map, and let $\widehat{A}_0$ be the closure of $\psi(A_0)$ in $\widehat{A}$, equipped with the subspace topology inherited from $\widehat{A}$. The notation $\widehat{A}_0$ clashes with the notation for the completion of A_0, but the two are actually isomorphic as stated below.

Fact 1.5.5. *The map $\psi\colon A_0 \to \widehat{A}_0$ identifies the topological ring $\widehat{A}_0$ with the completion of A_0. In particular, by Fact 1.5.4, the topology on $\widehat{A}_0$ is $(\psi(I)\cdot\widehat{A}_0)$-adic. Here note that $\psi(I)\cdot\widehat{A}_0$ is a finitely generated ideal in $\widehat{A}_0$. Moreover, $\widehat{A}_0$ is an open subring of $\widehat{A}$. In particular, $\widehat{A}$ is Huber with a ring of definition $\widehat{A}_0$ and an ideal of definition $\psi(I)\cdot\widehat{A}_0$.*

Remark 1.5.6. Notation as above, we have a natural isomorphism

$$\widehat{A} \cong \widehat{A}_0 \otimes_{A_0} A.$$

One can actually use the right hand side to *construct* $\widehat{A}$, and then show that it satisfies the characterizing universal property.

Fact 1.5.7. *If A is Huber with completion map $\psi : A \to \widehat{A}$, then we have a bijection.*

$$\begin{aligned} \{\textit{open subrings of } A\} &\xleftrightarrow{\sim} \{\textit{open subrings of } \widehat{A}\} \\ R &\longmapsto \overline{\psi(R)} \end{aligned}$$

Moreover, $\overline{\psi(R)}$ is identified with the completion of R itself. Under this bijection, A° is sent to $(\widehat{A})^\circ$. This bijection also restricts to the following bijections:

$$\{\textit{rings of definition of } A\} \xleftrightarrow{\sim} \{\textit{rings of definition of } \widehat{A}\}$$

and

$$\{\textit{rings of integral elements of } A\} \xleftrightarrow{\sim} \{\textit{rings of integral elements of } \widehat{A}\}.$$

By virtue of the last bijection, starting with any Huber pair (A, A^+) we can take its completion $(\widehat{A}, \widehat{A}^+)$ which is another Huber pair, where $\widehat{A}$ is the completion of A, and $\widehat{A}^+$ is the closure of the image of A^+ in $\widehat{A}$, itself identified with the completion of A^+. By restricting the valuations, we get from the process above a natural map

$$\varphi\colon \operatorname{Spa}(\widehat{A}, \widehat{A}^+) \longrightarrow \operatorname{Spa}(A, A^+).$$

The following proposition claims the homeomorphism which appeared in the proof of Proposition 1.4.5 before.

Proposition 1.5.8. *The map $\varphi\colon \operatorname{Spa}(\widehat{A}, \widehat{A}^+) \longrightarrow \operatorname{Spa}(A, A^+)$ is a homeomorphism that preserves rational subsets.*

To prove Proposition 1.5.8, we need the following two lemmas.

Lemma 1.5.9 (Minimum modulus principle)**.** *Suppose (A, A^+) is a Huber pair and take a quasi-compact subset $C \subset \operatorname{Spa}(A, A^+)$. Suppose $f \in A$ is such that $v(f) \neq 0$ for all $v \in C$. Then there is a neighborhood U of 0 in A such that for all $g \in U$ we have*

$$v(f) > v(g), \quad \forall v \in C.$$

Proof. Choose a ring of definition $A_0 \subset A$ and an ideal of definition $I \subset A_0$. Write $I = T \cdot A_0$ for some finite set $T \subset A_0$. For each integer $n \geqslant 1$, the set

$$X_n = \{v \in \operatorname{Spa}(A, A^+)\colon v(t) \leqslant v(f) \neq 0 \text{ for all } t \in T^n\}$$

is open. For any $v \in C$, since v is continuous, we see $v(f) \neq 0$ implies that $\{g \in A\colon v(g) < v(f)\}$ is an open neighborhood of 0, and hence contains $I^n \supset T^n$ for some $n \gg 0$. This means that $v \in X_n$. Thus we have

$$C \subset \bigcup_{n \geqslant 1} X_n.$$

Note each $t \in T$ satisfies $t^n \to 0$, so for any continuous valuation v on A we must have $v(t) < 1$. (Indeed, for sufficiently large n, t^n will lie in $\{f\colon v(f) < v(1) = 1\}$, which is an open neighborhood of 0 in A. Hence $v(t^n) < 1$ and so $v(t) < 1$.) Thus $X_n \subset X_{n+1} \subset \cdots$. It then follows from the quasi-compactness of C that $C \subset X_n$ for some $n \gg 0$. If so, we can take $U = I^{n+1}$ and then $v(g) < v(f)$ for all $g \in U$ and all $v \in C$. (For the strict inequality again use $v(t) < 1$ for any $t \in T$.) $\square$

Example 1.5.10. To illustrate the situation of Lemma 1.5.9, we consider the adic closed unit disc. Let $(K, |\cdot|)$ be an algebraically closed non-archimedean field. Take

$$X = \mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle).$$

Then the following subset is open in X:

$$C = X - \{v_{0,0}\} = \{v \in X : v(T) \neq 0\}.$$

Take $f = T$. We have $v_{0,|p^{n+1}|} \in C$ for all $n \in \mathbb{N}$. Also note that

$$v_{0,|p^{n+1}|}(p^n) = |p^n| > v_{0,|p^{n+1}|}(T) = |p^{n+1}|.$$

Since $p^n \to 0$ in A, this means there does not exist an open neighborhood U of 0 in A satisfying the conclusion of Lemma 1.5.9 for $f = T$. Thus C is not quasi-compact. This C is an example of an open subset of $\mathrm{Spa}(A, A^+)$ which is not quasi-compact; in particular it is not a rational subset.

Lemma 1.5.11. *Suppose (A, A^+) is a complete Huber pair. Take*

$$U = U\left(\frac{f_1, \cdots, f_n}{g}\right) \subset \mathrm{Spa}(A, A^+)$$

as a rational subset with open ideal $(f_1, \ldots, f_n)A$. Then there is a neighborhood V of 0 in A satisfying the following condition: For all $f'_i \in f_i + V$ with $i = 1, \ldots, n$, and for all $g' \in g + V$, we still have

$$U = U\left(\frac{f'_1, \cdots, f'_n}{g'}\right)$$

and $(f'_1, \ldots, f'_n)A$ is an open ideal as well.

Proof. We need to make use of the following

> *Claim.* Let B be a ring, and $J = r_1B + \cdots + r_mB$ a finitely generated ideal. Assume that B is J-adically complete. Then for any $r'_1 \in r_1 + J^2, \cdots, r'_m \in r_m + J^2$, we have $J = r'_1B + \cdots + r'_mB$.

To prove the claim, we need to show that the B-module map $\phi : B^m = B^{\oplus m} \to J, (b_1, \cdots, b_m) \mapsto \sum b_i r'_i$ is surjective. Both sides are J-adically complete, so it suffices to show that ϕ induces a surjection between the graded modules associated to the J-adic filtrations on the two sides. (See [Bou64, III.2, No 8, Cor.2].) Thus we need to show for each $i \geqslant 0$ that $\phi : J^iB^m/J^{i+1}B_m \to J^iJ/J^{i+1}J$ is surjective. For this, clearly it is enough to show that $\{r'_1, \cdots, r'_m\} \cup J^2$ generates J as a B-module. This is true since the ideal generated by $\{r'_1, \cdots, r'_m\} \cup J^2$ is contained in the ideal generated by $\{r_1, \cdots, r_m\}$, which is J. The claim is proved.

We now start the proof of the lemma.

Choose a ring of definition A_0 in A and an ideal of definition $I = (r_1, \ldots, r_m)A_0 \subset A_0$. Since $(f_1, \cdots, f_n)A$ is open, we may assume that $I \subset (f_1, \cdots, f_n)A$. We then write

$$r_j = \sum_{i=1}^{n} a_{ij} f_i, \quad a_{ij} \in A.$$

Let V_0 be a neighborhood of 0 in A such that $a_{ij}V_0 \subset I^2$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Then for arbitrary $f'_i \in f_i + V_0$, if we define

$$r'_j = \sum_{i=1}^{n} a_{ij} f'_i,$$

we get $r'_j \in r_j + I^2$ and so $(r'_1, \ldots, r'_m)A_0 = I$ by the claim. But this implies that $(f'_1, \ldots, f'_n)A$ is again an open ideal in A.

We now show that if we further shrink V_0, then we can ensure that $U = U(\frac{f'_1, \ldots, f'_n}{g'})$ for arbitrary $f'_i, g' \in V_0$.

Set $f_0 = g$, and $U_i = U(\frac{f_0, f_1 \dots, f_n}{f_i})$ for $i = 0, \cdots, n$. Thus $U = U_0$. Since U_i is rational, it is quasi-compact. Since f_i is non-zero on U_i, using Minimum modulus principle (Lemma 1.5.9) we find a neighborhood V of 0 in A such that for any $f \in V$, any $0 \leqslant i \leqslant n$, and any $v \in U_i$, we have $v(f_i) > v(f)$. We may and shall assume that $V \subset V_0 \cap A^{00}$.

We now show that for arbitrary $f'_0 \in f_0 + V, \cdots, f'_n \in f_n + V$, we have

$$U\left(\frac{f'_0, \cdots, f'_n}{f'_0}\right) = U_0.$$

Clearly this will imply the lemma. Denote the left hand side by U'_0.

We prove $U'_0 \supset U_0$. Let $v \in U_0$. Then $v(f_0) > v(f_i - f'_i)$ for all $i = 0, \cdots, n$, since $f_i - f'_i \in V$. We then have

$$v(f'_i) \leqslant \max(v(f_i), v(f_i - f'_i)) \leqslant v(f_0) = v(f'_0 - f'_0 + f_0) = v(f'_0).$$

Hence $v \in U'_0$.

Finally, we prove $U'_0 \subset U_0$. Let $v \in U'_0$ and suppose $v \notin U_0$. We analyze the situation in two cases. Case one, $v(f_i) = 0$ for all $i = 0, \cdots, n$. In this case, $\mathrm{supp}(v)$ is an ideal containing all f_i and hence it is open. Since $f_0 - f'_0 \in V \subset A^{00}$, we have $f_0 - f'_0 \in \sqrt{\mathrm{supp}(v)} = \mathrm{supp}(v)$. Thus $v(f'_0) = 0$, a contradiction with the assumption that $v \in U'_0$.

Case two, $v(f_i) \neq 0$ for some i. We may assume that i is such that $v(f_i) = \max(v(f_0), \cdots, v(f_n))$. Since $v \notin U_0$, we have either $v(f_0) = 0$, or $v(f_j) > v(f_0)$ for some j. In all cases, we have $v(f_i) > v(f_0)$. In particular $i \neq 0$. Then $v(f'_0) \leqslant \max(v(f_0), v(f_0 - f'_0)) < v(f_i) = v(f'_i)$, a contradiction. □

Now we are ready to prove that $\varphi\colon \mathrm{Spa}(\widehat{A}, \widehat{A}^+) \to \mathrm{Spa}(A, A^+)$ is a homeomorphism which preserves rational subsets.

Proof of Proposition 1.5.8. It is easy to show that φ is bijective. Then it remains to show that φ preserves rational subsets, since after that it will automatically follow that φ is a homeomorphism. It is not hard to show that φ^{-1} maps a rational subset of $\mathrm{Spa}(A, A^+)$ to a rational subset of $\mathrm{Spa}(\widehat{A}, \widehat{A}^+)$. Thus it remains to show the following:

- If $U = U(\{f_1, \dots, f_n\}/g)$ is rational in $\mathrm{Spa}(\widehat{A}, \widehat{A}^+)$ and $(f_1, \dots, f_n)\widehat{A}$ is open, then $\varphi(U)$ is rational in $\mathrm{Spa}(A, A^+)$.

The natural map $\iota\colon A \to \widehat{A}$ has dense image, so by Lemma 1.5.11, without loss of generality, we may assume $f_i = \iota(h_i)$ and $g = \iota(k)$ with $k \in A$ and $h_i \in A$ for $i = 1, \dots, n$. This further implies

$$\varphi(U) = U\left(\frac{h_1, \cdots, h_n}{k}\right).$$

So the proof would be complete if $(h_1, \dots, h_n)$ is an open ideal in A.

In general, we observe that U is quasi-compact since it is rational, and $\iota(k) = g$ is non-zero on U. By Lemma 1.5.9, there is a neighborhood V of 0 in $\widehat{A}$ such that for all $y \in V$ we have $v(y) < v(g)$ for all $v \in U$.

Recall from Fact 1.5.5 that if we take a ring of definition $A_0 \subset A$ and an ideal of definition $I \subset A_0$, then $\widehat{A}_0$ is a ring of definition in $\widehat{A}$ whose topology is $(\iota(I) \cdot \widehat{A}_0)$-adic. Therefore, up to replacing I by a power of itself, we can choose I such that $\iota(I) \subset V$. Thus for each $y \in I$, $v(\iota(y)) < v(g)$ holds for any $v \in U$. This implies that $v(y) < v(k)$ for all $v \in \varphi(U)$. Consequently,

$$\varphi(U) = U\left(\frac{h_1, \cdots, h_n}{k}\right) = U\left(\frac{h_1, \cdots, h_n, T}{k}\right),$$

in which T is a finite set of generators of the ideal I in A_0. Then $\{h_1, \cdots, h_n\} \cup T$ generate an open ideal in A, and so $\varphi(U)$ is rational. □

Proposition 1.5.12. *Let $U \subset \mathrm{Spa}(A, A^+)$ be a rational subset. Then there is*

- *a complete Huber pair (A_U, A_U^+) associated to U, and*
- *a map of Huber pairs $\varphi_0\colon (A, A^+) \to (A_U, A_U^+)$,*

such that

(1) *The map* $\mathrm{Spa}(A_U, A_U^+) \to \mathrm{Spa}(A, A^+)$ *induced by* φ_0 *is a homeomorphism* $\mathrm{Spa}(A_U, A_U^+) \xrightarrow{\sim} U$, *and this homeomorphism preserves rational subsets. (Here rational subsets of* U *refer to those rational subsets of* $\mathrm{Spa}(A, A^+)$ *that are contained in* U.)

(2) *For each complete Huber pair* (B, B^+) *together with a map* $\varphi\colon (A, A^+) \to (B, B^+)$, *if the induced map* $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ *factors through* U, *then* φ *factors uniquely through* φ_0, *i.e.,*

$$\begin{tikzcd}
(A, A^+) \arrow[r, "\varphi"] \arrow[dr, "\varphi_0"'] & (B, B^+) \\
& (A_U, A_U^+) \arrow[u, dashed, "\exists!"']
\end{tikzcd}$$

Proof. We only sketch the construction of (A_U, A_U^+). It is not *a priori* clear that the construction is independent of various choices. However, one proves that the construction indeed satisfies the universal property as in the proposition, from which uniqueness follows.

Choose a ring of definition $A_0 \subset A$ and an ideal of definition $I \subset A_0$. We construct (A_U, A_U^+) as follows. Write

$$U = U\left(\frac{T}{g}\right)$$

where T is a finite subset of A generating an open ideal. We admit the following claim.

> *Claim.* On $A[1/g]$, there exists a unique ring topology such that $A_0[T/g]$ is open with $(I \cdot A_0[T/g])$-adic topology.

Here existence and uniqueness of a *group topology* (for addition) satisfying the same conditions is clear. The key point is to show that this group topology is also a ring topology, i.e., that multiplication is continuous. Granting the claim, we see $A[1/g]$, equipped with the prescribed topology, is a Huber ring; we then take its completion and denote the completion by $A\langle T/g\rangle$. The integral closure of $A^+[T/g]$ in $A[1/g]$ is a ring of integral elements in the Huber ring $A[1/g]$. Hence the completion of it is a ring of integral elements in $A\langle T/g\rangle$. We denote this ring of integral elements by $A\langle T/g\rangle^+$. In conclusion we have obtained a complete Huber pair $(A\langle T/g\rangle, A\langle T/g\rangle^+)$, and this is our desired (A_U, A_U^+). ☐

Remark 1.5.13 (Universal property of the A-algebra $A\langle T/g\rangle$). The map $A \to A\langle T/g\rangle$ is universal among the continuous maps $\varphi\colon A \to B$ with complete B and such that $\varphi(g)$ is invertible and $\varphi(T)/\varphi(g)$ is power-bounded as a set in B. Here, a subset $S \subset B$ is called power-bounded if $\bigcup_n S^n$ is bounded.

1.6 The structure presheaf and adic spaces

Fix a Huber pair (A, A^+) and let $X = \mathrm{Spa}(A, A^+)$.

Definition 1.6.1. Let $W \subset X = \mathrm{Spa}(A, A^+)$ be an open subset. Define

$$\mathcal{O}_X(W) := \varprojlim_{\substack{U \subset W \\ \text{rational}}} A_U, \qquad \mathcal{O}_X^+(W) := \varprojlim_{\substack{U \subset W \\ \text{rational}}} A_U^+.$$

Here in both projective limits, the transition maps in the projective system arise from the fact that, for two rational sets $U' \subset U$, the Huber pair $(A_{U'}, A_{U'}^+)$ is an algebra over (A_U, A_U^+).

We equip $\mathcal{O}_X(W)$ and $\mathcal{O}_X^+(W)$ with the inverse limit topology coming from the topologies on A_U and A_U^+. Inverse limits of complete rings are complete. Hence $\mathcal{O}_X(W)$ and $\mathcal{O}_X^+(W)$ are complete rings. Thus $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are presheaves sending open sets in X to complete topological rings. However, they are not sheaves in general.

For a rational subset $U \subset X$, we have $\mathcal{O}_X(U) = A_U$ and $\mathcal{O}_X^+(U) = A_U^+$. Since rational subsets form a basis of the topology, the stalks of $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are given by

$$\mathcal{O}_{X,x} := \varinjlim_{\substack{x\in U\subset X \\ U \text{ rational}}} \mathcal{O}_X(U) = \varinjlim_{U} A_U, \qquad \mathcal{O}_{X,x}^+ := \varinjlim_{U} A_U^+.$$

Take $x \in X$ with corresponding valuation $v\colon A \to \Gamma \cup \{0\}$. For each rational subset U of X containing x, one checks that v extends uniquely to a continuous valuation $v\colon A_U \to \Gamma \cup \{0\}$. Thus we obtain a canonical valuation

$$v_x\colon \mathcal{O}_{X,x} \longrightarrow \Gamma \cup \{0\}.$$

Therefore, for a general open subset $W \subset X$, and for any $f \in \mathcal{O}_X(W)$, the valuation $v(f)$ for $v \in W$ makes sense. Namely, we define $v(f)$ to be the canonical valuation on $\mathcal{O}_{X,v}$ applied to the image of f in $\mathcal{O}_{X,v}$. The following notation is standard: We often write $|f(v)|$ for $v(f)$, even though $f(v)$ does not make sense, and $|f(v)|$ is an element of a group that depends on v.

Fact 1.6.2. *Consider $X = \mathrm{Spa}(A, A^+)$. We naturally view $\mathcal{O}_X^+$ as a sub-presheaf of $\mathcal{O}_X$.*

(1) *For any open subset W, we have*

$$\mathcal{O}_X^+(W) = \{f \in \mathcal{O}_X(W)\colon v(f) \leqslant 1 \text{ for all } v \in W\}.$$

In particular, $\mathcal{O}_X^+$ is determined by $\mathcal{O}_X$ together with the canonical valuations on the stalks of $\mathcal{O}_X$. It is automatically a sheaf whenever $\mathcal{O}_X$ is a sheaf.

(2) *For each $x \in X$, $\mathcal{O}_{X,x}$ is a local ring whose residue field is isomorphic to*

$$k(x) = \mathrm{Frac}(A/\operatorname{supp}(x)).$$

(3) *We have*

$$\mathcal{O}_{X,x}^+ = \{f \in \mathcal{O}_{X,x}\colon v_x(f) \leqslant 1\}.$$

Moreover, it is a local ring with maximal ideal $\{f \in \mathcal{O}_{X,x}^+\colon v_x(f) < 1\}$. Its residue field is identified with the residue field of $k(x)^+$. (Recall that $k(x)^+$ is the valuation subring of $k(x)$ corresponding to the valuation on $k(x)$ induced by v_x.)

Definition 1.6.3. A Huber pair (A, A^+) is called *sheafy* if the presheaf $\mathcal{O}_X$ on $X = \mathrm{Spa}(A, A^+)$ is a sheaf.

Definition 1.6.4 (Adic spaces)**.** We define a category $\mathscr{V}$ as follows.

- Each object of $\mathscr{V}$ is a tuple $(X, \mathcal{O}_X, \{v_x\}_{x\in X})$, where
  - X is a topological space,
  - $\mathcal{O}_X$ is a sheaf of topological rings such that $\mathcal{O}_{X,x}$ is a local ring for all $x \in X$, and
  - v_x for each $x \in X$ is an equivalence class of valuations on $\mathcal{O}_{X,x}$ such that $\operatorname{supp}(v_x)$ is the maximal ideal of $\mathcal{O}_{X,x}$.
- Each morphism of $\mathscr{V}$ is the data consisting of
  - a map $f\colon X \to Y$ of topological spaces, together with
  - a map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of topological rings, satisfying the property that the induced maps between stalks of $\mathcal{O}_X$ and $\mathcal{O}_Y$ are compatible with the canonical valuations.

An *adic space* is an object of $\mathscr{V}$ that is locally isomorphic to $\mathrm{Spa}(A, A^+) \in \mathscr{V}$ for a sheafy Huber pair (A, A^+). Morphisms between adic spaces are by definition morphisms in $\mathscr{V}$.

Remark 1.6.5. Since inverse limits of complete topological rings are complete, if $(X, \mathcal{O}_X)$ is an adic space then $\mathcal{O}_X$ is a sheaf of complete topological rings.

Remark 1.6.6. The functor

$$\begin{aligned}(\text{Sheafy Huber pairs})^{\mathrm{op}} &\longrightarrow (\text{Adic spaces})\\ (A, A^+) &\longmapsto \mathrm{Spa}(A, A^+).\end{aligned}$$

is in fact fully faithful.

In the sequel, we shall write $\mathrm{Spa}(A)$ for $\mathrm{Spa}(A, A^+)$ in the occasion where $A^+ = A^\circ$.

We now investigate on sufficient conditions to guarantee that a Huber pair is sheafy. First observe that whether a Huber pair is sheafy is insensitive to completion, so we may just work with complete Huber pairs. There indeed exist non-sheafy Huber pairs, see the examples by Buzzard–Verberkmoes and Mihara.

Theorem 1.6.7 (Huber)**.** *A complete Huber pair (A, A^+) is sheafy in the following cases.*

(1) *A is discrete.*

(2) *A has a noetherian ring of definition.*

(3) *A is Tate and strongly noetherian, i.e., for each $n \geqslant 1$,*

$$A\langle T_1, \ldots, T_n\rangle = \{f \in A[\![T_1, \ldots, T_n]\!] \colon \text{coefficients of } f \text{ tend to } 0 \text{ in } A\}$$

is noetherian.

Moreover, in all of these classical cases, Tate Acyclicity holds! Namely, for the affinoid space $X = \mathrm{Spa}(A, A^+)$, the augmented Čech complex for $\mathcal{O}_X$ with respect to any rational cover is exact. Consequently, the higher sheaf cohomology vanishes: $H^i(U, \mathcal{O}_X) = 0$ for all $i > 0$ and any rational subset $U \subseteq X$.

Definition 1.6.8. Let $f\colon A \to B$ be a continuous map between Huber rings A and B. It is of *topologically finite type* (TFT) if

(i) B is complete,

(ii) there are rings of definition $A_0 \subset A$ and $B_0 \subset B$ such that $f(A_0) \subset B_0$,

(iii) B is a finitely generated algebra over $f(A) \cdot B_0$, and

(iv) there is $n \in \mathbb{N}$ and a surjective open continuous map

$$\widehat{A}_0\langle x_1, \ldots, x_n\rangle \longrightarrow B_0$$

of A_0-algebras.

Proposition 1.6.9. *If $f\colon A \to B$ is of topological finite type, and A satisfies* (2) *or* (3) *of Theorem 1.6.7, then B also satisfies* (2) *or* (3) *respectively.*

Proposition 1.6.10. *Every non-archimedean field K is (Tate and) strongly noetherian.*

1.7 Stably uniform affinoids

The perfectoid rings we want to study are huge, infinitely ramified, and highly non-Noetherian, so they fail the sufficient conditions in Huber's theorem. Nevertheless, they satisfy a non-Noetherian criterion which we now discuss.

In the typical examples of non-sheafy Huber pairs, there exists a global section restricts to 0 on an open cover but is globally non-zero. This failure stems from power-bounded elements failing to be topologically bounded.

Definition 1.7.1. A complete Tate Huber pair (R, R^+) is *stably uniform*, if for every rational subset $W \subset \mathrm{Spa}(R, R^+)$, the complete Tate ring $\mathcal{O}_X(W)$ is uniform.

Theorem 1.7.2 (Buzzard-Verberkmoes, 2014)**.** *If (R, R^+) is a stably uniform complete Tate Huber pair, then $\mathcal{O}_X$ is a sheaf on $X = \mathrm{Spa}(R, R^+)$. Moreover, for any rational cover, the augmented Čech complex is exact. Consequently, Tate acyclicity holds: $H^i(U, \mathcal{O}_X) = 0$ for all $i > 0$ and any rational subset U.*

Sketch of proof. By Huber's reduction argument (similar to the proof of Tate acyclicity for rigid analytic spaces), the theorem reduces to checking that for a basic Laurent cover: $X = U \cup V$, where $U = \{x \in X : |t(x)| \leq 1\}$ and $V = \{x \in X : |t(x)| \geq 1\}$ for some $t \in R$, the sequence

$$0 \to \mathcal{O}_X(X) \to \mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \xrightarrow{(x,y)\mapsto y-x} \mathcal{O}_X(U \cap V)$$

is exact. In fact, in this reduction step, we need to possibly replace X by its rational subsets; this is OK since our assumption that (R, R^+) is stably uniform is by design invariant under passing to rational subsets.

Fix a ring of definition R_0 of R. Recall that Huber's rule dictates that for a rational subset $W = U(T/g)$, the ring $\mathcal{O}_X(W)$ is the completion of the algebraic localization $R[1/g]$ with respect to its topology which is defined by declaring $R_0[T/g]$ to be an open ring of definition. Let's apply this:

We have $U = U\left(\frac{t,1}{1}\right)$. The algebraic ring is $A = R[1/1] = R$, and the ring of definition is $A_0 = R_0[\frac{t}{1}, \frac{1}{1}] = R_0[t]$. Thus, $\mathcal{O}_X(U)$ is the completion of $A = R$ topologized by $A_0 = R_0[t]$.

We have $V = U\left(\frac{1,t}{t}\right)$. The algebraic ring is $B = R[1/t]$, and the ring of definition is $B_0 = \phi(R_0)[\frac{1}{t}, \frac{t}{t}] = \phi(R_0)[1/t]$, where ϕ denotes the natural map $R \to B$. Thus, $\mathcal{O}_X(V)$ is the completion of $B = R[1/t]$ topologized by $B_0 = \phi(R_0)[1/t]$.

We have $U \cap V = U\left(\frac{t,1}{1}\right) \cap U\left(\frac{1,t}{t}\right) = U\left(\frac{\{t,1\}\cdot\{1,t\}}{1\cdot t}\right) = U\left(\frac{1,t,t^2}{t}\right)$. The algebraic ring is $C = R[1/t]$. (Note that abstractly $C = B$). The ring of definition is $C_0 = \phi(R_0)\left[\frac{1}{t}, \frac{t}{t}, \frac{t^2}{t}\right] = \phi(R_0)[1/t, 1, t] = \phi(R_0)[t, 1/t]$. Thus, $\mathcal{O}_X(U \cap V)$ is the completion of $C = R[1/t]$ topologized by $C_0 = \phi(R_0)[t, 1/t]$.

Before completion, we form the split exact sequence of abstract abelian groups: $0 \to R \xrightarrow{\epsilon} A \oplus B \xrightarrow{\delta} C \to 0$, where $\epsilon(r) = (r, \phi(r))$ and $\delta(a, b) = b - \phi(a)$. Our goal is to show that after completion this sequence remains exact. In general, completing an exact sequence of topological abelian groups preserves exactness if and only if the transition maps are strict. (A map $f : G \to H$ is called strict if the topology on $\mathrm{im}(f)$ defined as a quotient of G agrees with that defined as a subgroup of H.) In the current situation, it suffices to show that the inverse image under ϵ of a ring of definition of $A \oplus B$ is bounded in R. Thus we only need to show that $A_0 \cap \phi^{-1}(B_0) = R_0[t] \cap \phi^{-1}(\phi(R_0)[1/t])$ is a bounded subset of R. By assumption, R° is bounded. Hence it remains to prove the following claim:

Claim: $R_0[t] \cap \phi^{-1}(\phi(R_0)[1/t]) \subseteq R^\circ$.

Proof: If r is in the intersection, then there exists a degree d_1 polynomial f_1 over R_0 such that $r = f_1(t)$, and a degree $\leq d_2$ polynomial f_2 over R_0 such that $t^{d_2} r = f_2(t)$. Let $N = d_1 + d_2$, and $\varpi \in R$ a pseudo-uniformizer such that $\varpi t \in R_0$. Then prove by induction on $m \geq 0$ that $\varpi^N g(t) r^m \in R_0$ for all deg $\leq N$ polynomials g over R_0. (Consider $g(t) = t^k$. If $k \leq d_2$, replace r by $f_1(t)$. If $k \geq d_2$, replace $t^{d_2} r$ by $f_2(t)$.) In particular, $\varpi^N r^m \in R_0$ for all m, so $r \in R^\circ$. □

1.8 Comparing with rigid analytic spaces and formal schemes

Let k be a complete non-archimedean field. Classical rigid geometry studies the maximal spectrum $\mathrm{Sp}(R)$ where R is an affinoid k-algebra topologically of finite type.

Theorem 1.8.1 (Huber)**.** *There is a fully faithful functor*

$$\{\textit{rigid analytic spaces over } k\} \longrightarrow \{\textit{adic spaces}\}$$

which on affinoids sends $\mathrm{Sp}(R)$ *to* $\mathrm{Spa}(R, R^\circ)$*. (This is a valid adic space because R is topologically of finite type, hence strongly noetherian, ensuring that (R, R°) is sheafy by case (3) of Theorem 1.6.7).*

We now consider formal schemes over $\mathcal{O}_k$. In order to ensure that $\mathcal{O}_k$ is Noetherian, we assume in addition that k is discretely valued. Suppose $\mathfrak{X}$ is a locally Noetherian formal scheme over $\mathrm{Spf}(\mathcal{O}_k)$. Then locally, $\mathfrak{X} = \mathrm{Spf}(A)$, where A is a Noetherian topological ring complete with respect to the I-adic topology for a (finitely generated) ideal I. Note that the pair (A, A) is a sheafy Huber pair by case (2) of Theorem 1.6.7.

Theorem 1.8.2 (Huber)**.** *There is a fully faithful functor:*

$$\{\textit{locally Noetherian formal schemes over } \mathcal{O}_k\} \longrightarrow \{\textit{adic spaces}\}$$

which on affine formal schemes sends $\mathrm{Spf}(A)$ *to* $\mathrm{Spa}(A, A)$*.*

We now have a much more straightforward point of view toward Raynaud's rigid generic fiber functor. Recall that this functor attaches a rigid analytic space X over k to each locally Noetherian formal scheme $\mathfrak{X}$ over $\mathcal{O}_k$. If we embed both categories into the category of adic spaces, then the relationship between X and $\mathfrak{X}$ is expressed by the simple fiber product relation:

$$X = \mathfrak{X} \times_{\mathrm{Spa}(\mathcal{O}_k,\mathcal{O}_k)} \mathrm{Spa}(k, \mathcal{O}_k).$$

(However, the category of adic spaces does not admit arbitrary fiber products.)

2 Perfectoid spaces

2.1 Perfectoid fields and tilting

The following result is an important motivation for the theory of perfectoid spaces.

Theorem 2.1.1 (Fontaine–Wintenberger). *Let K be the p-adic completion of $\mathbb{Q}_p(p^{1/p^\infty}) := \bigcup_{n\geq 1} \mathbb{Q}_p(p^{1/p^n})$. (Here, for each n, we choose a root p^{1/p^n} in $\overline{\mathbb{Q}}_p$ such that $(p^{1/p^{n+1}})^p = p^{1/p^n}$.) Then there is a canonical isomorphism between absolute Galois groups*

$$G_K \cong G_{K^\flat}$$

where $K^\flat$ is the t-adic completion of $\mathbb{F}_p((t))(t^{1/p^\infty}) := \bigcup_{n\geq 1} \mathbb{F}_p((t))(t^{1/p^n})$.

Remark 2.1.2. The theorem has its root in the following result of Deligne: For any integer $n \geq 1$, there is a canonical isomorphism

$$G_{\mathbb{Q}_p(p^{1/p^n})}/G^{(p)}_{\mathbb{Q}_p(p^{1/p^n})} \cong G_{\mathbb{F}_p((t))(t^{1/p^n})}/G^{(p)}_{\mathbb{F}_p((t))(t^{1/p^n})}$$

where the superscript (p) denotes the upper numbering ramification group.

The following definition captures the key features of K and $K^\flat$ above.

Definition 2.1.3 (Scholze). A *perfectoid field* is a nonarchimedean field $(K, |\cdot|)$ such that

1. $|K^\times|$ is a non-discrete subgroup of $\mathbb{R}_{>0}$.
2. The residue field of K is of characteristic $p > 0$. (In particular, $|p| < 1$).
3. $\mathrm{Frob} : \mathcal{O}_K/p \to \mathcal{O}_K/p$, $x \mapsto x^p$ is surjective.

In the following, p as in condition 2 is always fixed. Note that K is allowed to have characteristic 0 or p.

Remark 2.1.4. Over characteristic p, a perfectoid field is the same thing as a non-archimedean field K which is perfect, i.e., $\mathrm{Frob} : K \to K, x \mapsto x^p$ is a bijection. This is because $\mathrm{Frob} : K \to K$ is always injective, and it is surjective if and only if $\mathrm{Frob} : \mathcal{O}_K \to \mathcal{O}_K$ is surjective. Moreover, if $\mathrm{Frob} : K \to K$ is surjective, then $|K^\times|$ is p-divisible and hence non-discrete.

Example 2.1.5. Perfectoid fields of characteristic 0:

- $\widehat{\mathbb{Q}_p(p^{1/p^\infty})} := \left(\bigcup_{n\geq 1} \mathbb{Q}_p(p^{1/p^n})\right)^\wedge$
- $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})} := \left(\bigcup_{n\geq 1} \mathbb{Q}_p(\zeta_{p^n})\right)^\wedge$
- $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}}_p}$

Perfectoid fields of characteristic p:

- $\widehat{\mathbb{F}_p((t))(t^{1/p^\infty})} := \left(\bigcup_{n\geq 1} \mathbb{F}_p((t))(t^{1/p^n})\right)^\wedge$

- $\widehat{\overline{\mathbb{F}_p((t))}}$

.

Lemma 2.1.6. *Let $(K, |\cdot|)$ be a non-archimedean field of residue characteristic p. Assume that $|K^\times|$ is non-discrete, and that there exists $\varpi \in K$ such that $|p| \le |\varpi| < 1$ and such that* $\mathrm{Frob} : \mathcal{O}_K/\varpi \to \mathcal{O}_K/\varpi, x \mapsto x^p$ *is surjective. (Note that $\mathcal{O}_K/\varpi$ is a ring of characteristic p.) Then $|K^\times|$ is p-divisible. In particular, this applies to any perfectoid field K.*

Proof. For any non-discrete subgroup Γ of $\mathbb{R}_{>0}$ and any $0 < a < b$, we know that Γ is generated by $\Gamma \cap (a, b)$. Hence $|K^\times|$ is generated by $|t|$ with $t \in K$ such that $|\varpi| < |t| < 1$. For such a t, by assumption there exists $y \in \mathcal{O}_K$ such that $y^p \equiv t \pmod{\varpi}$. It then follows that $|y^p| = |y|^p = |t|$. $\square$

Lemma 2.1.7. *Let $(K, |\cdot|)$ be a non-archimedean field with residue characteristic p and such that $|K^\times|$ is non-discrete. Then the following conditions are equivalent:*

1. *K is perfectoid.*
2. *For any $\varpi \in K$ satisfying $|p| \le |\varpi| < 1$ we have* $\mathrm{Frob} : \mathcal{O}_K/\varpi \to \mathcal{O}_K/\varpi, x \mapsto x^p$ *is surjective. (Note that $\mathcal{O}_K/\varpi$ is a ring of characteristic p.)*
3. *For some $\varpi \in K$ satisfying $|p| \le |\varpi| < 1$ we have* $\mathrm{Frob} : \mathcal{O}_K/\varpi \to \mathcal{O}_K/\varpi, x \mapsto x^p$ *is surjective.*

Proof. Condition 1 implies condition 2 since $\mathcal{O}_K/\varpi$ is a quotient ring of $\mathcal{O}_K/p$. It remains to show that condition 3 implies condition 1. Let $y \in \mathcal{O}_K$. We need to find $x \in \mathcal{O}_K$ such that $x^p \equiv y \pmod{p}$. First find $x_1 \in \mathcal{O}_K$ such that $x_1^p \equiv y \pmod{\varpi}$. By Lemma 2.1.6, we can pick $\varpi_1 \in K$ such that $|\varpi_1|^p = |\varpi|$. Write $y - x_1^p = \varpi_1^p z$, with $z \in \mathcal{O}_K$. Find $x_2 \in \mathcal{O}_K$ such that $x_2^p \equiv z \pmod{\varpi}$. Then $x_1 + \varpi_1 x_2$ satisfies

$$(x_1 + \varpi_1 x_2)^p = x_1^p + p(\cdots) + \varpi_1^p x_2^p \equiv x_1^p + \varpi_1^p z = y \mod (\varpi^2, p).$$

If $|\varpi^2| \le |p|$, then the above congruence holds modulo p, so we have found x such that $x^p \equiv y \pmod{p}$. Otherwise, we can replace ϖ by ϖ^2 and replace x_1 by $x_1 + \varpi_1 x_2$, and iterate the process. Since one of $\varpi, \varpi^2, \varpi^4, \varpi^8, \ldots$ must have absolute value less than $|p|$, we will be able to find x after finitely many steps. $\square$

Let K be a perfectoid field. Our next goal is to construct its *tilt* $K^\flat$, which is another perfectoid field and its characteristic is p.

Pick $\varpi \in \mathcal{O}_K$ such that $|p| \le |\varpi| < 1$. Consider the inverse limit

$$\mathcal{O}_K^\flat := \varprojlim_{\mathrm{Frob}} \mathcal{O}_K/\varpi.$$

Thus its elements are sequences $(x_0, x_1, \ldots)$ with $x_i \in \mathcal{O}_K/\varpi$ such that $x_{i+1}^p = x_i$. Since Frob is a ring endomorphism of $\mathcal{O}_K/\varpi$, $\mathcal{O}_K^\flat$ is naturally a ring.

Remark 2.1.8. We call an $\mathbb{F}_p$-algebra perfect if the Frobenius endomorphism $x \mapsto x^p$ is an isomorphism. Note that $\mathcal{O}_K^\flat$ is perfect. Indeed, $\mathrm{Frob}^{-1}(x_0, x_1, \ldots) = (x_1, x_2, \ldots)$.

Lemma 2.1.9. *Let T be any ring, and p a prime number. Let ϖ be an element of T dividing p. Let $a, b \in T$ be such that $a \equiv b \pmod{\varpi}$. Then $a^{p^n} \equiv b^{p^n} \pmod{\varpi^{n+1}}$ for all $n \ge 0$.*

Proof. We prove by induction. The case $n = 0$ follows from our assumption. Suppose this holds for n. Write $a^{p^n} = b^{p^n} + c\varpi^{n+1}$. Then

$$a^{p^{n+1}} = (b^{p^n} + c\varpi^{n+1})^p = b^{p^{n+1}} + c^p \varpi^{p(n+1)} + \sum_{i=1}^{p-1} \binom{p}{i} \varpi^{i(n+1)} c^i b^{p^n(p-i)}.$$

Since $\varpi^{p(n+1)}$ is divisible by ϖ^{n+2}, and each $\binom{p}{i}\varpi^{i(n+1)}$ is divisible by $p\varpi^{n+1}$ which is divisible by ϖ^{n+2}, the above is $\equiv b^{p^{n+1}} \pmod{\varpi^{n+2}}$. $\square$

Proposition 2.1.10. *The following statements hold.*

1. *The natural map (induced by projection)* $\varprojlim_{x\mapsto x^p}\mathcal{O}_K \to \varprojlim_{\mathrm{Frob}}\mathcal{O}_K/\varpi$ *is an isomorphism of multiplicative monoids. In particular,* $\mathcal{O}_K^\flat$ *as a ring is independent of the choice of* ϖ *up to canonical ring isomorphism.*

2. $\mathcal{O}_K^\flat$ *is an integral domain. We denote its fraction field by* $K^\flat$*.*

3. *Denote the composition of the bijection* $\mathcal{O}_K^\flat \xrightarrow{\sim} \varprojlim_{x\mapsto x^p}\mathcal{O}_K$ *(by part 1) followed by the 0-th projection* $\varprojlim_{x\mapsto x^p}\mathcal{O}_K \to \mathcal{O}_K$ *by* $\sharp : \mathcal{O}_K^\flat \to \mathcal{O}_K, x\mapsto x^\sharp$*, which is also a multiplicative map. Define* $|x|^\flat := |x^\sharp|$ *for* $x\in\mathcal{O}_K^\flat$*. Then* $|\cdot|^\flat$ *is a valuation on* $\mathcal{O}_K^\flat$*. Moreover,* $|\mathcal{O}_K^\flat| = |\mathcal{O}_K| \subset \mathbb{R}_{\geq 0}$*.*

4. *The topology on* $\mathcal{O}_K^\flat$ *defined by* $|\cdot|^\flat$ *agrees with the inverse limit topology on* $\mathcal{O}_K^\flat = \varprojlim_{\mathrm{Frob}}\mathcal{O}_K/\varpi$ *coming from the discrete topology on each copy of* $\mathcal{O}_K/\varpi$*.*

5. *The bijection* $\mathcal{O}_K^\flat \xrightarrow{\sim} \varprojlim_{x\mapsto x^p}\mathcal{O}_K$ *in part 1 extends to a multiplicative bijection* $K^\flat \xrightarrow{\sim} \varprojlim_{x\mapsto x^p} K$*. For any* $\varpi^\flat \in \mathcal{O}_K^\flat$ *satisfying* $|p| \leq |\varpi^\flat|^\flat < 1$ *(which exists by part 3), we have* $K^\flat = \mathcal{O}_K^\flat[1/\varpi^\flat]$*. In particular, we can extend* $|\cdot|^\flat$ *to a non-archimedean absolute value on* $K^\flat$*.*

6. $(K^\flat, |\cdot|^\flat)$ *is a perfectoid field of characteristic* p*, and* $\mathcal{O}_K^\flat = \{x\in K^\flat \mid |x|^\flat \leq 1\}$*. Moreover,* $|K^{\flat,\times}|^\flat = |K^\times|$*.*

7. *Let* $\varpi^\flat \in \mathcal{O}_K^\flat$ *and* $\varpi\in\mathcal{O}_K$ *be such that satisfying* $|p| \leq |\varpi^\flat|^\flat = |\varpi| < 1$*. Then there is a natural ring isomorphism* $\mathcal{O}_K^\flat/\varpi^\flat \cong \mathcal{O}_K/\varpi$*. Moreover, the residue fields of* $\mathcal{O}_K^\flat$ *and* $\mathcal{O}_K$ *are naturally isomorphic.*

Proof. **(1)** Let $x = (x_0, x_1, \ldots) \in \varprojlim_{\mathrm{Frob}}\mathcal{O}_K/\varpi$. Choose lifts $\tilde{x}_n \in \mathcal{O}_K$ of x_n. For $m\geq 0$, let

$$x^{(m)} = \lim_{n\to\infty} \tilde{x}_{m+n}^{p^n} \in \mathcal{O}_K.$$

Claim: $x^{(m)}$ exists and is independent of choices of the lifts.
By Lemma 2.1.9, if $\tilde{x}'_{m+n}$ is another lift of x_{m+n}, then $\tilde{x}_{m+n}^{p^n} \equiv (\tilde{x}'_{m+n})^{p^n} \pmod{\varpi^{n+1}}$. This implies both that $\{\tilde{x}_{m+n}^{p^n}\}$ is a Cauchy sequence, and that the limit is independent of the choices of lifts.

Now it is easy to see that $(x^{(m+1)})^p = x^{(m)}$ and that $x^{(m)}$ lifts x_m. Thus the map $\varprojlim_{\mathrm{Frob}}\mathcal{O}_K/\varpi \to \varprojlim_{x\mapsto x^p}\mathcal{O}_K, x\mapsto (x^{(0)}, x^{(1)}, \ldots)$ is the desired inverse map.

For the "in particular" part, note that if ϖ' is such that $|p| \leq |\varpi| \leq |\varpi'| < 1$, then the natural projection $\mathcal{O}_K/\varpi \to \mathcal{O}_K/\varpi'$ induces a ring homomorphism $\varprojlim_{\mathrm{Frob}}\mathcal{O}_K/\varpi \to \varprojlim_{\mathrm{Frob}}\mathcal{O}_K/\varpi'$ which commutes with the bijections from both sides to $\varprojlim_{x\mapsto x^p}\mathcal{O}_K$. Hence this homomorphism must be a bijection.

(2) Suppose $x, y\in\mathcal{O}_K^\flat$ such that $xy = 0$. Then $|\varpi| \geq |x^{(m+n)}y^{(m+n)}| = |x^{(n)}y^{(n)}|^{p^{-m}}$ for all $m, n\geq 0$, which implies that $x^{(n)} = y^{(n)} = 0$ for all n.

(3) To show that $|\cdot|^\flat$ is a valuation, we only need to show that it satisfies the strong triangle inequality. (Other properties are straightforward). Let $x = (x_0,\ldots), y = (y_0,\ldots)\in\mathcal{O}_K^\flat$. Since $x^{(n)} + y^{(n)}$ is a lift of $x_n + y_n$, we have $(x+y)^\sharp = (x+y)^{(0)} = \lim_{n\to\infty} x^{(n)} + y^{(n)}$ by the proof of part 1. In particular,

$$\begin{aligned}
|x+y|^\flat = |(x+y)^{(0)}| &= \lim_{n\to\infty}|x^{(n)} + y^{(n)}|^{p^n} \\
&\leq \lim_{n\to\infty}(\max(|x^{(n)}|^{p^n}, |y^{(n)}|^{p^n})) \\
&= \lim_{n\to\infty}(\max(|x^{(0)}|, |y^{(0)}|)) \\
&= \max(|x|^\flat, |y|^\flat).
\end{aligned}$$

We now show $|\mathcal{O}_K^\flat| = |\mathcal{O}_K|$. Obviously $|\mathcal{O}_K^\flat| \subset |\mathcal{O}_K|$. Since $|\mathcal{O}_K|$ is generated (as a monoid) by $|t|$ such that $|\varpi| < |t| < 1$, it suffices to find $x\in\mathcal{O}_K^\flat$ such that $|x|^\flat = |t|$ for any such $t\in\mathcal{O}_K$. By the surjectivity of Frob : $\mathcal{O}_K/\varpi \to \mathcal{O}_K/\varpi$, we can find an element $x = (x_0, \ldots) \in \mathcal{O}_K^\flat = \varprojlim_{\mathrm{Frob}}\mathcal{O}_K/\varpi$

such that x_0 is the image of t. Since $x^{(0)} \equiv t \mod \varpi$ and $|t| > |\varpi|$, we have $|x|^\flat = |x^{(0)}| = |t|$, as desired.

(4) We need to show two things. First, for any $n \in \mathbb{N}$, there exists $\epsilon > 0$ such that for any $x = (x_0, x_1, \dots) \in \mathcal{O}_K^\flat$ satisfying $|x|^\flat < \epsilon$ we have $x_n = 0$. For this, we have $|x^{(n)}| = |x^{(0)}|^{p^{-n}} = (|x|^\flat)^{p^{-n}}$. Take ϵ such that $\epsilon^{p^{-n}} < |\varpi|$. Then for any x such that $|x|^\flat < \epsilon$ we have $|x^{(n)}| < |\varpi|$, and so $x_n = 0$.

Second, for any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that for any $x = (x_0, x_1, \dots) \in \mathcal{O}_K^\flat$ satisfying $x_n = 0$ we have $|x|^\flat < \epsilon$. We have $|x|^\flat = |x^{(0)}| = |x^{(n)}|^{p^n} \leq |\varpi|^{p^n}$. So we can take n sufficiently large such that $|\varpi|^{p^n} < \epsilon$.

(5) Write y_n for $(\varpi^\flat)^{(n)}$. We can extend the multiplicative bijection $\mathcal{O}_K^\flat \xrightarrow{\sim} \varprojlim_{x \mapsto x^p} \mathcal{O}_K$ to a multiplicative map

$$\mathcal{O}_K^\flat[1/\varpi^\flat] \to \varprojlim_{x \mapsto x^p} K, \quad x/(\varpi^\flat)^k \mapsto (x^{(n)}/y_n^k)_n.$$

This is easily seen to be a bijection preserving 0. Since every non-zero element of the right hand side has a multiplicative inverse, the same holds for the left hand side. Hence $\mathcal{O}_K^\flat[1/\varpi^\flat] = K^\flat$.

(6) To show that $\mathcal{O}_K^\flat = \{x \in K^\flat \mid |x|^\flat \leq 1\}$, it suffices to show that for any $x, y \in \mathcal{O}_K^\flat$ with $|x|^\flat \leq |y|^\flat$, we have $y | x$ in $\mathcal{O}_K^\flat$. Since $|x^{(n)}| = (|x|^\flat)^{p^{-n}} \leq |y^{(n)}| = (|y|^\flat)^{p^{-n}}$, we have $y^{(n)} | x^{(n)}$ in $\mathcal{O}_K$. Define $z = (x^{(n)}/y^{(n)})_n \in \varprojlim_{x \mapsto x^p} \mathcal{O}_K \cong \mathcal{O}_K^\flat$. We then have $x = yz$.

Clearly $K^\flat$ is of characteristic p. By part 4, the topology on $K^\flat$ defined by $|\cdot|^\flat$ is complete. To show that it is perfectoid, it suffices to show that $\mathrm{Frob} : \mathcal{O}_K^\flat \to \mathcal{O}_K^\flat$ is surjective, but this is clear.

The fact that $|K^{\flat,\times}|^\flat = |K^\times|$ follows from the last statement in part 3.

(7) We may assume that $\mathcal{O}_K^\flat$ is defined using ϖ. Write $\varpi^\flat = (\varpi_0^\flat, \varpi_1^\flat, \dots) \in \varprojlim_{\mathrm{Frob}} \mathcal{O}_K/\varpi$. We have

$$\mathcal{O}_K^\flat/\varpi^\flat \cong \varprojlim_{\mathrm{Frob}} (\mathcal{O}_K/\varpi)/\varpi_n^\flat \cong \varprojlim_{\mathrm{Frob}} \mathcal{O}_K/(\varpi, (\varpi^\flat)^{(n)}) = \varprojlim_{\mathrm{Frob}} \mathcal{O}_K/(\varpi^\flat)^{(n)},$$

where the last equality is because $(\varpi^\flat)^{(n)}$ divides ϖ in $\mathcal{O}_K$. In the last inverse limit, all the transition maps are isomorphisms, so the projection to the 0-th coordinate is an isomorphism onto $\mathcal{O}_K/(\varpi^\flat)^{(0)}$. But $(\varpi^\flat)^{(0)}$ and ϖ generate the same ideal in $\mathcal{O}_K$ since they have the same absolute value. Hence the last ring is $\mathcal{O}_K/\varpi$.

The residue field of $\mathcal{O}_K^\flat$ can be written as a direct limit of $\mathcal{O}_K^\flat/\varpi^\flat$ over all choices of $\varpi^\flat$ such that $|\varpi^\flat|^\flat < 1$. Similarly for the residue field of $\mathcal{O}_K$. Since $|K^{\flat,\times}|^\flat = |K^\times|$, and since the transition maps in the two direct limits are compatible with the isomorphisms $\mathcal{O}_K^\flat/\varpi^\flat \cong \mathcal{O}_K/\varpi$ established above, the two residue fields are isomorphic. □

Remark 2.1.11. If K is of characteristic $= p$, then $K^\flat \cong K$.

Example 2.1.12. If $K = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$, then $K^\flat = \widehat{\mathbb{F}_p((t))(t^{1/p^\infty})}$ (where $\wedge$ denotes the completion with respect to the natural topology on $\overline{\mathbb{Q}}_p$ or $\overline{\mathbb{F}_p((t))}$), where $t = (p \bmod p, p^{1/p} \bmod p, p^{1/p^2} \bmod p, \dots) \in \varprojlim_{\mathrm{Frob}} \mathcal{O}_K/p = \mathcal{O}_K^\flat$.

Proposition 2.1.13. *Let $(K, |\cdot|)$ be a perfectoid field. If $K^\flat$ is algebraically closed, so is K.*

Proof. Let $F(X) = X^d + a_{d-1}X^{d-1} + \cdots + a_0 \in K[X]$ be irreducible. We want to find a root of F. Since $K^\flat$ is algebraically closed, the abelian group $|K^\times| = |K^{\flat,\times}|^\flat$ is a $\mathbb{Q}$-vector space. Hence we can find $\alpha \in \mathcal{O}_K$ with $|\alpha| = |a_0|^{-1/d}$. Then by replacing X by αX, we reduce to the case where $|a_0| = 1$. But then it follows that $F(X) \in \mathcal{O}_K[X]$. This is because the irreducibility of F guarantees that its Newton polygon must be a straight line segment, and since the leading coefficient and constant term are both units the Newton polygon must be of slope zero, which implies that all a_i lie in $\mathcal{O}_K$. Fix $\varpi \in K$ and $\varpi^\flat \in K^\flat$ such that $|p| \leq |\varpi| = |\varpi^\flat|^\flat < 1$. Now, consider $F(X) \bmod \varpi$ as an element of $(\mathcal{O}_K/\varpi)[X] \cong (\mathcal{O}_K^\flat/\varpi^\flat)[X]$, and choose a monic degree d lift $G(X) \in \mathcal{O}_K^\flat[X]$. By assumption, $G(X)$ admits a root $y_1 \in \mathcal{O}_K^\flat$.

Consider

$$F_1(X) := \beta_1^{-d} F(\beta_1 X + y_1^\sharp)$$

where $\beta_1 \in \mathcal{O}_K$ is chosen such that $|\beta_1|^d = |F(y_1^\sharp)| \leq |\varpi|$ (inequality by construction of y). Note $F_1(X) \in K[X]$ is still irreducible monic of degree d and the constant term is a unit. As observed before, we have $F_1(X) \in \mathcal{O}_K[X]$. Repeating the process, we can find $y_2 \in \mathcal{O}_K^\flat$ such that $|F_1(y_2^\sharp)| \le |\varpi|$, and set

$$F_2(X) = \beta_2^{-d} F_1(\beta_2 X + y_2^\sharp) \quad \text{where } \beta_2 \in \mathcal{O}_K \text{ s.t. } |\beta_2|^d = |F_1(y_2^\sharp)|.$$

Keep repeating the process, and we get $y_n \in \mathcal{O}_K^\flat$ and $F_n \in \mathcal{O}_K[X]$ for all $n \in \mathbb{N}$. We have

$$|\varpi| \ge |F_n(y_{n+1}^\sharp)| = \left|\beta_n^{-d} F_{n-1}(\beta_n y_{n+1}^\sharp + y_n^\sharp)\right| = \left|\beta_n^{-d}\beta_{n-1}^{-d} F_{n-2}(\beta_n\beta_{n-1} y_{n+1}^\sharp + \beta_{n-1} y_n^\sharp + y_{n-1}^\sharp)\right| =$$

$$\cdots = \left|\prod_{i=1}^n \beta_i^{-d} F(\sum_{i=1}^{n+1} (\prod_{j=1}^{i-1} \beta_i) y_i^\sharp)\right|.$$

Let

$$x_n = \sum_{i=1}^{n+1} (\prod_{j=1}^{i-1} \beta_i) y_i^\sharp = y_1^\sharp + \beta_1 y_2^\sharp + \beta_1\beta_2 y_3^\sharp + \cdots + \beta_1\beta_2\cdots\beta_n y_{n+1}^\sharp \in \mathcal{O}_K.$$

Then

$$|F(x_n)| \le |\varpi| \prod_{i=1}^n |\beta_i|^d \le |\varpi|^{n+1},$$

which tends to 0 as $n \to \infty$. On the other hand $x = \lim_n x_n \in \mathcal{O}_K$ exists since $|x_n - x_{n-1}| = |\beta_1\beta_2\cdots\beta_n y_{n+1}^\sharp| \le |\varpi|^{n/d}$. Hence $x \in \mathcal{O}_K$ is a root of F. $\square$

Let K be a perfectoid field. We extend $\sharp : \mathcal{O}_K^\flat \to \mathcal{O}_K$ to a multiplicative map $K^\flat \to K$, still denoted by $\sharp$. This is nothing but the composition of the bijection $K^\flat \xrightarrow{\sim} \varprojlim_{x \mapsto x^p} K$ in Proposition 2.1.10 (5) followed by the 0-th projection.

Exercise 2.1.14. *The bijection $K^\flat \xrightarrow{\sim} \varprojlim_{x\mapsto x^p} K$ is given by the formula $x \mapsto (x^\sharp, (x^{1/p})^\sharp, (x^{1/p^2})^\sharp, \ldots)$. Here x^{1/p^n} is well defined since $K^\flat$ is a perfect field.*

Proposition 2.1.15. *For every continuous valuation $v : K \to \Gamma \cup \{0\}$ on K (where Γ is a totally ordered abelian group), the map $v \circ \sharp : K^\flat \to \Gamma \cup \{0\}$ is a continuous valuation. This construction induces a bijection from the set of continuous valuations on K to the analogous set for $K^\flat$. Moreover, let $K^+ \subset \mathcal{O}_K$ be an open valuation subring of K and let $K^{\flat+} := \varprojlim_{\mathrm{Frob}} K^+/\varpi$. Then $K^{\flat+} \subset \mathcal{O}_K^\flat$ is also an open valuation subring of $K^\flat$, and the valuation on K corresponding to K^+ is sent to the valuation on $K^\flat$ corresponding to $K^{\flat+}$ under the above bijection. We have a natural homeomorphism* $\mathrm{Spa}(K, K^+) \cong \mathrm{Spa}(K^\flat, K^{\flat+})$.

Sketch of proof. The statement that $v \circ \sharp$ is a valuation is proved in exactly the same way as part 3 of Proposition 2.1.10. To prove the continuity of $v \circ \sharp$ and the rest of the proposition, notice that we have a bijection between the set of continuous valuations on K with the set of open valuation subrings $K^+ \subset \mathcal{O}_K$, and similarly a bijection between the set of continuous valuations on $K^\flat$ with the set of open valuation subrings $K^{\flat+} \subset \mathcal{O}_K^\flat$. These two sets are respectively identified with the set of valuation subrings of $\mathcal{O}_K/\mathfrak{m}_K$ and the set of valuation subrings of $\mathcal{O}_K^\flat/\mathfrak{m}_{K^\flat}$. Both constructions $v \mapsto v \circ \sharp$ and $K^+ \mapsto K^{\flat+}$ correspond to the bijection between the last two sets induced by the natural isomorphism $\mathcal{O}_K/\mathfrak{m}_K \cong \mathcal{O}_K^\flat/\mathfrak{m}_{K^\flat}$. $\square$

Theorem 2.1.16 (General Fontaine–Wintenberger Theorem)**.** *Let K be a perfectoid field.*

1. *Let L/K be a finite extension. Equip L with the natural topology induced from K. Then L is a perfectoid field.*

2. *There is an equivalence of categories*

$$\{\textit{fin extensions of } K\} \xrightarrow{\sim} \{\textit{finite extensions of } K^\flat\}, \quad L \mapsto L^\flat$$

This equivalence preserves degrees. (As a corollary $G_K \cong G_{K^\flat}$).

We will prove the theorem in a much more general context.

2.2 Perfectoid algebras

Let K be a perfectoid field.

Definition 2.2.1. A *perfectoid K-algebra* is a Banach K-algebra R which is uniform and such that for some $\varpi \in K$ satisfying $|p| \le |\varpi| < 1$ the map $\mathrm{Frob} : R^\circ/\varpi \to R^\circ/\varpi, x \mapsto x^p$ is surjective.

Remark 2.2.2. Similar to Lemma 2.1.7, if R is a perfectoid K-algebra then $\mathrm{Frob} : R^\circ/\varpi \to R^\circ/\varpi$ is surjective for *all* $\varpi \in K$ satisfying $|p| \le |\varpi| < 1$.

Remark 2.2.3. (See [Bha17, Prop. 5.2.5]) The category of uniform Banach K-algebras (where morphisms are continuous K-algebra maps) is equivalent to the category of ϖ-adically complete, ϖ-torsion free $\mathcal{O}_K$-algebras A such that A is totally integrally closed in $A[1/\varpi]$, via the functor $R \mapsto R^\circ$. A quasi-inverse is given by $A \mapsto A[1/\varpi]$, together with the norm on $A[1/\varpi]$ defined by $\|f\| = \inf\{|t| \mid t \in K, f \in tA\}$. Here, we say A is totally integrally closed in $A[1/\varpi]$ if for every element $x \in A[1/\varpi]$ not in A, the set $x^{\mathbb{N}}$ is not contained in any finitely generated A-submodule of $A[1/\varpi]$. (Note that this may be strictly stronger than requiring that $A[x]$ is not finitely generated, since we do not assume that A is noetherian.)

Example 2.2.4. Let $\varpi \in K$ be such that $|p| \le |\varpi| < 1$. Then $R = K\langle T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}\rangle := \left(\bigcup_{m\ge 0} \mathcal{O}_K[T_1^{1/p^m}, \dots, T_n^{1/p^m}]\right)^\wedge [\varpi^{-1}]$ is a perfectoid K-algebra. Here $\wedge$ denotes ϖ-adic completion, and the definition of R is independent of the choice of ϖ.

Lemma 2.2.5. *If K is a perfectoid field of characteristic p, then a Banach K-algebra R is perfectoid if and only if it is perfect.*

Proof. Any uniform normed K-algebra R must be reduced. This is because: if there is a non-zero $x \in R$ such that $x^n = 0$ for some n, then $Kx \subset R^\circ$ but Kx is not bounded, a contradiction. It follows that every perfectoid K-algebra is perfect. Conversely, let R be a perfect Banach K-algebra. We only need to show that it is automatically uniform. The following argument is due to André, see [Bha17, Lem. 7.1.6].

Fix a ring of definition $R_0 \subset R$ (e.g. the closed unit ball). By the Banach Open Mapping Theorem, the continuous surjection $\mathrm{Frob} : R \to R$ (which is a bijection) must be open, so $\mathrm{Frob}(R_0)$ is open. Hence there exists a $\varpi \in R_0 \cap R^\times$ (or even $R_0 \cap K^\times$) with $\|\varpi\| < 1$ such that $\varpi R_0 \subset \mathrm{Frob}(R_0)$. Then

$$\mathrm{Frob}^{-1}(R_0) \subset \varpi^{-p^{-1}} R_0,$$

$$\mathrm{Frob}^{-2}(R_0) \subset \mathrm{Frob}^{-1}(\varpi^{-p^{-1}} R_0) = \varpi^{-p^{-2}} \mathrm{Frob}^{-1}(R_0) \subset \varpi^{-p^{-2}-p^{-1}} R_0,$$

$$\dots$$

$$\mathrm{Frob}^{-n}(R_0) \subset \varpi^{\sum_{k=1}^n -p^{-k}} R_0.$$

Since $\mathrm{Frob}^{-n}(R_0) \subset \mathrm{Frob}^{-n-1}(R_0)$ and $\sum_{k=1}^n p^{-k} \le 1$ for all $n \in \mathbb{N}$, we have

$$R_0' := \bigcup_{n\in\mathbb{N}} \mathrm{Frob}^{-n}(R_0) \subset \varpi^{-1} R_0.$$

In particular R_0' is bounded. We complete the proof by showing that $R^\circ \subset \varpi^{-1} R_0'$. For this, let $f \in R^\circ$. Then there exists $n_0 \ge 1$ such that $\varpi^{n_0} \cdot f^{\mathbb{N}} \subset R_0 \subset R_0'$. Since R_0' is by construction stable under Frob, for any $n \ge 1$ we have $R_0' \ni (\varpi^{n_0} f^{p^n})^{p^{-n}} = (\varpi^{n_0})^{p^{-n}} f$. Choosing n such that $p^n > n_0$, we have $\varpi^{n_0} | \varpi^{p^n}$ in R_0', and so $(\varpi^{n_0})^{p^{-n}} | \varpi$ in R_0' (since R_0' is stable under Frob). It follows that $\varpi f \in R_0'$. ☐

From a perfectoid K-algebra R, we can construct a perfectoid $K^\flat$-algebra $R^\flat$ by tilting. The construction is similar to the field case, and is summarized in the following proposition. The proof is similar to Proposition 2.1.10 and we omit it.

Proposition 2.2.6. *Fix $\varpi \in K$ such that $|p| \le |\varpi| < 1$. Let $R^{\circ\flat} := \varprojlim_{\mathrm{Frob}} R^\circ/\varpi$. This is naturally an $\mathcal{O}_K^\flat$-algebra. Let $R^\flat := R^{\circ\flat} \otimes_{\mathcal{O}_K^\flat} K^\flat$.*

1. *The natural map $\varprojlim_{x\mapsto x^p} R^\circ \to R^{\circ\flat}$ is a multiplicative bijection. Its inverse extends to a multiplicative bijection $R^\flat \xrightarrow{\sim} \varprojlim_{x\mapsto x^p} R$. In particular, the formations of $R^{\circ\flat}$ and $R^\flat$ are independent of the choice of ϖ.*
2. *Denote the composition of $R^\flat \xrightarrow{\sim} \varprojlim_{x\mapsto x^p} R$ with the 0-th projection by $\sharp : R^\flat \to R$. Then the pull-back of the norm on R along $\sharp$ is a norm on $R^\flat$. Under this norm, $R^\flat$ is a perfectoid $K^\flat$-algebra, and we have $(R^\flat)^\circ = R^{\circ\flat}$.*

Thus we obtain the tilting functor

$$\{\text{perfectoid } K\text{-algebras}\} \to \{\text{perfectoid } K^\flat\text{-algebras}\}, \quad R \mapsto R^\flat.$$

Here, the morphisms in the two categories are continuous K-algebra maps and continuous $K^\flat$-algebra maps respectively.

Theorem 2.2.7. *The tilting functor is an equivalence of categories .*

2.3 Witt vectors and the theta map

In order to prove Theorem 2.2.7, we need to answer the following question: If $R^\flat$ is the tilt of R, how do we recover R from $R^\flat$? The key tools to understand this come from the theory of Witt vectors, especially the *theta maps* and the notion of *primitive elements*. Throughout we fix a prime p.

Fact 2.3.1. *There exist unique polynomials $S_n(X_0,\dots,X_n,Y_0,\dots,Y_n)$ and $P_n(X_0,\dots,X_n,Y_0,\dots,Y_n)$ in $\mathbb{Z}[X_0,\dots,X_n,Y_0,\dots,Y_n]$ for $n \geq 0$ such that for every ring R, if we define addition and multiplication on $R^{\mathbb{N}}$ by*

$$\underline{x}+\underline{y} := (S_0(x_0,y_0), S_1(x_0,x_1,y_0,y_1),\dots), \quad \underline{x}\cdot\underline{y} := (P_0(x_0,y_0), P_1(x_0,x_1,y_0,y_1),\dots),$$

then $R^{\mathbb{N}}$ is a ring, and moreover the map

$$w_n : R^{\mathbb{N}} \to R, \quad \underline{x} \mapsto \sum_{i=0}^{n} p^i x_i^{p^{n-i}}$$

is a ring homomorphism for all $n \in \mathbb{N}$. We denote $R^{\mathbb{N}}$ with this ring structure by $W(R)$, called the ring of (p-typical) Witt vectors over R.

Indeed, if p is invertible in R, then we can solve for $x_0, x_1, \dots$ successively from $w_0(\underline{x}), w_1(\underline{x}), \dots$:

$$\begin{cases} w_0 = x_0, \\ w_1 = x_0^p + p x_1, \\ w_2 = x_0^{p^2} + p x_1^p + p^2 x_2, \\ \dots \end{cases} \Rightarrow \begin{cases} x_0 = w_0, \\ x_1 = p^{-1}(w_1 - x_0^p), \\ x_2 = p^{-2}(w_2 - x_0^{p^2} - p x_1^p), \\ \dots \end{cases}$$

Thus the map $(w_0, w_1, \dots) : R^{\mathbb{N}} \to R^{\mathbb{N}}$ is a bijection, and we obtain a ring structure on the left hand side by pulling back the natural ring structure on the right hand side. This allows one to uniquely determine the polynomials S_n and P_n over $\mathbb{Z}[1/p]$ (for instance, by considering the "universal case" where $R = \mathbb{Z}[1/p][X_n, n \in \mathbb{N}]$). There are then several ways to show S_n and P_n are actually defined over $\mathbb{Z}$ (see for instance [Ser13, §II.6]). These polynomials satisfy the suitable algebraic relations which guarantee that $W(R)$ is a ring and $W_n : W(R) \to R$ is a ring homomorphism for any ring R.

Fact 2.3.2. *The map $[\cdot] : R \to W(R), x \mapsto [x] := (x, 0, 0, \dots)$ is multiplicative. It is called the* Teichmüller map.

Fact 2.3.3. *If R is a perfect $\mathbb{F}_p$-algebra, then $W(R)$ is p-adically complete*[2] *and p-torsion free. Moreover, projection to the 0-th coordinate induces a ring isomorphism $W(R)/p \xrightarrow{\sim} R$, and the inverse isomorphism is the composition $R \xrightarrow{[\cdot]} W(R) \to W(R)/p$.*

[2] By complete we always mean complete and separated.

Remark 2.3.4. We call a ring S a *strict p-ring* if it is p-torsion free, p-adically complete, and such that S/p is perfect. The functor $S \mapsto S/p$ is an equivalence of categories from strict p-rings to perfect $\mathbb{F}_p$-algebras, and a quasi-inverse is given by $R \mapsto W(R)$. In particular, for a perfect $\mathbb{F}_p$-algebra R, $W(R)$ is the unique (up to isomorphism) p-torsion free and p-adically complete ring such that $W(R)/p \cong R$.

The Teichmüller map $[\cdot] : R \to W(R)$ is a multiplicative section of the reduction map $W(R) \to W(R)/p \cong R$. Even without knowing its construction, we can abstractly show its existence and uniqueness as follows.

Lemma 2.3.5. *Let R be a perfect $\mathbb{F}_p$-algebra.*

1. *Let S be a p-adically complete ring, and $\sharp : R \to S$ a multiplicative map. For any $n \in \mathbb{N}$, any $x \in R$, and any $s \in S$ such that $s \equiv (x^{p^{-n}})^\sharp \pmod{p}$, we have $x^\sharp \equiv s^{p^n} \pmod{p^{n+1}}$.*

2. *There eixsts a unique multiplicative section of the reduction map $W(R) \to W(R)/p \cong R$.*

Proof. Part 1 follows from the fact that $x^\sharp = ((x^{p^{-n}})^\sharp)^{p^n}$ and Lemma 2.1.9. The uniqueness in part 2 follows from part 1 and the p-adic completeness of $W(R)$. For the existence, we need to show that for $x \in R$ the sequence $\{\tilde{x}_n\}$, where $\tilde{x}_n \in W(R)$ is any lift of $x^{p^{-n}} \in R$, converges ot a limit which is independent of the choices of lifts. Using the p-adic completeness of $W(R)$, one proves this in exactly the same way as the claim in the proof of Proposition 2.1.10 (1). $\square$

Let R be a perfect $\mathbb{F}_p$-algebra. Since $W(R)$ is p-adically complete and since $[\cdot]$ is a section of the reduction mod p map, every element of $W(R)$ can be written uniquely as

$$\sum_{n=0}^{\infty} [a_n] p^n, \quad \text{where } a_n \in R.$$

Now let S be a p-adically complete ring, and $\sharp : R \to S$ a multiplicative map such that the composition $R \xrightarrow{\sharp} S \to S/p$ is a ring homomorphism. We define the *theta map*

$$\theta : W(R) \to S, \quad \sum_{n=0}^{\infty} [a_n] p^n \mapsto \sum_{n=0}^{\infty} a_n^\sharp p^n.$$

Proposition 2.3.6. *The map θ is a ring homomorphism.*

Remark 2.3.7. The proposition is saying that $(W(R), [\cdot])$ is the initial object in the category of all pairs $(S, \sharp : R \to S)$ as above, where morphisms $(S_1, \sharp_1) \to (S_2, \sharp_2)$ are p-adically continuous ring homomorphisms $S_1 \to S_2$ taking $\sharp_1$ to $\sharp_2$. It is of central importance in p-adic Hodge theory.

Proof. (See [Ked15, Lem. 1.1.6].) We prove by induction on $n \geq 1$ that θ is additive modulo p^n. The case $n = 1$ follows from our assumption on $\sharp$. Suppose we know this for n. Let $x, y \in W(R)$ and $z = x + y$. Write $x = [\bar{x}] + px_1, y = [\bar{y}] + py_1, z = [\bar{z}] + pz_1$, where $\bar{x}, \bar{y}, \bar{z}$ are the images of x, y, z modulo p, and $x_1, y_1, z_1 \in W(R)$. Since $[\bar{x}^{p^{-n}}] + [\bar{y}^{p^{-n}}]$ lifts $\bar{x}^{p^{-n}} + \bar{y}^{p^{-n}} = \bar{z}^{p^{-n}}$ mod p, we have

$$[\bar{z}] \equiv ([\bar{x}^{p^{-n}}] + [\bar{y}^{p^{-n}}])^{p^n} \pmod{p^{n+1}}$$

by Lemma 2.3.5 (1). Thus

$$pz_1 - px_1 - py_1 = [\bar{x}] + [\bar{y}] - [\bar{z}] \equiv -\sum_{i=1}^{p^n-1} \binom{p^n}{i} [\bar{x}^{p^{-n}}]^i [\bar{y}^{p^{-n}}]^{p^n-i} \pmod{p^{n+1}}.$$

Since each $\binom{p^n}{i}$ is divisible by p, it follows that

$$z_1 - x_1 - y_1 \equiv -\sum_{i=1}^{p^n-1} \frac{1}{p} \binom{p^n}{i} [\bar{x}^{p^{-n}i} \bar{y}^{p^{-n}(p^n-i)}] \pmod{p^n}.$$

By induction hypothesis,

$$\theta(z_1)-\theta(x_1)-\theta(y_1)\equiv\theta(z_1-x_1-y_1)\equiv-\sum_{i=1}^{p^n-1}\frac{1}{p}\binom{p^n}{i}\theta([\bar{x}^{p^{-n}i}\bar{y}^{p^{-n}(p^n-i)}])$$

$$\equiv-\sum_{i=1}^{p^n-1}\frac{1}{p}\binom{p^n}{i}(\bar{x}^{p^{-n}i}\bar{y}^{p^{-n}(p^n-i)})^\sharp \pmod{p^n}.$$

Clearly θ commutes with multiplication by p, so

$$\theta(pz_1)-\theta(px_1)-\theta(py_1)\equiv-\sum_{i=1}^{p^n-1}\binom{p^n}{i}((\bar{x}^{p^{-n}})^\sharp)^i((\bar{y}^{p^{-n}})^\sharp)^{p^n-i} \pmod{p^{n+1}}.$$

On the other hand, again by Lemma 2.3.5 (1), we have

$$\bar{z}^\sharp\equiv((\bar{x}^{p^{-n}})^\sharp+(\bar{y}^{p^{-n}})^\sharp)^{p^n} \pmod{p^{n+1}}.$$

Adding the above two congruences, we get

$$\bar{z}^\sharp+\theta(pz_1)\equiv\bar{x}^\sharp+\theta(px_1)+\bar{y}^\sharp+\theta(py_1) \pmod{p^{n+1}}.$$

But clearly $\theta(x)=\bar{x}^\sharp+\theta(px_1)$, and similarly for y and z. Thus we have shown that θ is additive modulo p^{n+1}. Since S is p-adically complete, it follows that θ is additive.

To show multiplicativity, again it suffices to show that θ is multiplicative modulo p^n for each n. For this, we only need to check that for finite sums $x=\sum_{n=0}^N[x_n]p^n$ and $y=\sum_{n=0}^N[y_n]p^n$, we have $\theta(xy)\equiv\theta(x)\theta(y)$. Using additivity, we compute

$$\theta(xy)=\sum_n\sum_{i,j,i+j=n}\theta([x_iy_j]p^n)=\sum_n\sum_{i,j,i+j=n}(x_iy_j)^\sharp p^n=\theta(x)\theta(y).$$

Obviously θ preserves 0 and 1. $\square$

Next we study the problem of equipping $W(R)$ with norms. Let R be a perfect $\mathbb{F}_p$-algebra, and let $\|\cdot\|:R\to[0,1]$ be a norm, i.e., a function satisfying

- *non-degeneracy*: $\|x\|=0\Leftrightarrow x=0$,
- *strong triangle inequality*: $\|x+y\|\le\max(\|x\|,\|y\|)$,
- *sub-multiplicativity*: $\|xy\|\le\|x\|\|y\|$.

Note that as a consequence of sub-multiplicativity and the assumption that $\|\cdot\|$ is bounded by 1 we have $\|uy\|=\|y\|$ for any $u\in W(R)^\times$. We further assume that $\|\cdot\|$ is

- *power multiplicative*: $\|x^n\|=\|x\|^n$.

For any $\alpha\in(0,1]$, define

$$N_\alpha:W(R)\to[0,1],\quad \sum_{n\ge0}[a_n]p^n\mapsto\sup_n\|a_n\|\alpha^n.$$

Note that for $\alpha<1$, this supremum is a maximum.

Proposition 2.3.8. *For $\alpha\in(0,1]$, the function N_α is a power multiplicative norm.*

See [Ked13, 4.1, 4.2] for a more general version. For the proof we need the following standard fact.

Lemma 2.3.9. *For any perfect $\mathbb{F}_p$-algebra R, if in $W(R)$ we have $\sum_n[x_n]p^n+\sum_n[y_n]p^n=\sum_n[z_n]p^n$, then each z_n is a universal polynomial over $\mathbb{F}_p$ in the variables $x_k^{p^{-i}}, y_k^{p^{-i}}\in R$ for $i\ge0$ and $0\le k\le n$. Moreover, this polynomial is homogeneous of degree 1, where $x_k^{p^{-i}}$ and $y_k^{p^{-i}}$ have fraction degree p^{-i}.*

Proof. We reduce to the universal case where

$$R = \mathbb{F}_p[x_k^{p^{-\infty}}, y_k^{p^{-\infty}}, k \in \mathbb{N}] = \bigcup_{i \geq 0} \mathbb{F}_p[x_k^{p^{-i}}, y_k^{p^{-i}}, k \in \mathbb{N}]$$

and

$$x = \sum_n [x_n]p^n, \quad y = \sum_n [y_n]p^n.$$

Then z_n, as an element of R, is a polynomial in $x_k^{p^{-i}}, y_k^{p^{-i}}$, and by functoriality this is the universal polynomial we are seeking. We now prove by induction that z_n only involves $x_k^{p^{-i}}, y_k^{p^{-i}}$ for $0 \leq k \leq n$. Reduction mod p gives $z_0 = x_0 + y_0$, so this holds for $n = 0$. For $n \geq 1$, let $\Phi : R \to R$ be the endomorphism sending $x_k^{p^{-i}}$ and $y_k^{p^{-i}}$ to themselves for $k < n$ and to zero for $k \geq n$. Then

$$\sum_{m \geq 0} [\Phi z_m]p^m = \sum_{m=0}^{n-1} [x_m]p^m + \sum_{m=0}^{n-1} [y_m]p^m \equiv x + y - [x_n]p^n - [y_n]p^n \equiv \left(\sum_{m=0}^{n} [z_m]p^m \right) - [x_n]p^n - [y_n]p^n \mod p^{n+1}.$$

By induction hypothesis, $\Phi z_m = z_m$ for $0 \leq m \leq n-1$, so $[z_n] = [\Phi z_n] + [x_n] + [y_n] \mod p$, i.e., $z_n = \Phi z_n + x_n + y_n$. This shows that z_n depends only on $x_k^{p^{-i}}, y_k^{p^{-i}}$ for $0 \leq k \leq n$.

Finally, to show the homogeneity, let $\lambda \in \mathbb{F}_p$ and let $\Psi : R \to R$ be the endomorphism sending $x_k^{p^{-i}} \mapsto \lambda^{p^{-i}} x_k^{p^{-i}}$ and $y_k^{p^{-i}} \mapsto \lambda^{p^{-i}} y_k^{p^{-i}}$. Then

$$\sum_n [\Psi z_n]p^n = \sum_n [\lambda x_n]p^n + \sum_n [\lambda y_n]p^n = \lambda \sum_n [z_n]p^n = \sum_n [\lambda z_n]p^n.$$

Hence $\Psi z_n = \lambda z_n$, which shows the homogeneity. □

Proof of Proposition 2.3.8. Note that for each $x \in W(R)$ we have $N_1(x) = \lim_{\alpha \to 1^-} N_\alpha(x)$, so the statement for N_1 follows from that for N_α with $\alpha < 1$. Now assume $\alpha \in (0, 1)$. The property $N_\alpha(x) = 0 \Leftrightarrow x = 0$ is clear.

For strong triangle inequality, let $x = \sum_n [x_n]p^n, y = \sum_n [y_n]p^n$, and $x + y = \sum_n [z_n]p^n$. By power multiplicativity of $\|\cdot\|$, we have $\|x_k^{p^{-i}}\| = \|x_k\|^{p^{-i}}$ and $\|y_k^{p^{-i}}\| = \|y_k\|^{p^{-i}}$. Then by Lemma 2.3.9,

$$\|z_n\| \leq \max_{\alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n \in \mathbb{Q}_{\geq 0}, \sum \alpha_k + \beta_k = 1} \|x_0\|^{\alpha_0} \cdots \|x_n\|^{\alpha_n} \|y_0\|^{\beta_0} \cdots \|y_n\|^{\beta_n} \leq \max_{0 \leq k \leq n} (\|x_k\|, \|y_k\|).$$

Hence

$$\alpha^n \|z_n\| \leq \max_{0 \leq k \leq n} (\alpha^n \|x_k\|, \alpha^n \|y_k\|) \leq \max_{0 \leq k \leq n} (\alpha^k \|x_k\|, \alpha^k \|y_k\|) \leq \max(N_\alpha(x), N_\alpha(y)).$$

It follows that $N_\alpha(x + y) \leq \max(N_\alpha(x), N_\alpha(y))$.

For sub-multiplicativity, let $x, y \in W(R)$. We need to show that $N_\alpha(xy) \leq N_\alpha(x)N_\alpha(y)$. We may assume that x, y, xy are all non-zero. Write $x = \sum_n [x_n]p^n$. There is $N \in \mathbb{N}$ such that $N_\alpha(x) = \|x_n\|\alpha^n$ for some $n \leq N$ and such that $N_\alpha(x) > \alpha^N$. It follows that for any $x' \in W(R)$, if $x' \equiv x \mod p^{N+1}$, then $N_\alpha(x) = N_\alpha(x')$. Using this observation we reduce to the case where x and y are finite sums: $x = \sum_{n=0}^N [x_n]p^n, y = \sum_{n=0}^N [y_n]p^n$. Then by strong triangle inequality for N_α we have

$$N_\alpha(xy) \leq \sup_{i,j} N_\alpha([x_i]p^i[y_j]p^j) = \sup_{i,j} \|x_i y_j\| \alpha^{i+j} \leq \sup_{i,j} \|x_i\| \|y_j\| \alpha^{i+j} = N_\alpha(x)N_\alpha(y).$$

Finally, we prove power multiplicity. Let $x = \sum [x_n]p^n$. We need to show that $N_\alpha(x^r) = N_\alpha(x)^r$. Let k be minimal such that $N_\alpha(x) = \|x_k\|\alpha^k$, and let $x' = \sum_{n \geq k} [x_n]p^n$. Then

$$N_\alpha(x'^r) \leq N_\alpha(x')^r \leq (\|x_k\|\alpha^k)^r = \|x_k^r\|\alpha^{kr} \leq N_\alpha(x'^r),$$

where the first inequality is by the sub-multiplicativity of N_α, and the last inequality is because x'^r is of the form

$$p^{kr}[x_k^r] + \sum_{n \geq kr+1} [\cdots]p^n.$$

Hence we have

$$N_\alpha(x'^r) = N_\alpha(x)^r.$$

We now show that $N_\alpha(x'^r) = N_\alpha(x^r)$, which will finish the proof of power multiplicativity. We have

$$N_\alpha(x^r - x'^r) \leq N_\alpha(x - x')N_\alpha(\sum_{i=0}^{r-1} x^i x'^{r-1+i}) \leq N_\alpha(x - x') \max_{0\leq i\leq r-1} N_\alpha(x)^i N_\alpha(x')^{r-1+i}$$

$$\leq N_\alpha(x - x')N_\alpha(x)^{r-1} < N_\alpha(x)^r,$$

where the last strict inequality is because $N_\alpha(x - x') < N_\alpha(x)$ by the choice of k. But $N_\alpha(x)^r = N_\alpha(x'^r)$, so $N_\alpha(x^r - x'^r) < N_\alpha(x'^r)$. It follows that $N_\alpha(x^r) = N_\alpha(x'^r)$, as desired. □

2.4 Primitive elements and untilting

Let L be a perfectoid field of characteristic p, and $(A, \|\cdot\|)$ a perfectoid L-algebra. It is a general fact that for any uniform Banach algebra over a non-archimedean field, we can find a power multiplicative norm on that algebra defining the same topology. Thus we may and shall assume that $\|\cdot\|$ on A is power multiplicative.

Definition 2.4.1. An element of $W(A^\circ)$ is called *primitive of degree* 1 (which we will simply call primitive) if it is of the form $z = [\bar{z}]+pu$ with $\bar{z} \in A^\circ$ topologically nilpotent (equivalently, $\|\bar{z}\| < 1$) and $u \in W(A^\circ)^\times$. An element of $W(A^\circ)$ is called *stable* if it is of the form $\sum_{n\geq 0}[a_n]p^n$ with $\|a_0\| \geq \|a_n\|$ for all $n \geq 1$.

The following lemma and its proof are of central importance in the theory.

Lemma 2.4.2 (Key Lemma, Euclidean Division)**.** *Let $z \in W(A^\circ)$ be a primitive element. Then every element of $W(A^\circ)$ is congruent to a stable element modulo z.*

Proof. For every element $y \in W(A^\circ)$, we write $\bar{y} \in W(A^\circ)/p = A^\circ$ for the reduction mod p. Write $z = [\bar{z}] + pu$. We introduce an "Euclidean division" algorithm on $W(A^\circ)$. Let $x \in W(A^\circ)$. We have

$$x = Q(x)z + R(x)$$

where

$$Q(x) := \frac{x - [\bar{x}]}{p}u^{-1}, \quad R(x) := [\bar{x}] - Q(x)[\bar{z}].$$

Here, if $x = \sum_{n\geq 0}[x_n]p^n$, then $\frac{x-[\bar{x}]}{p} = \sum_{n\geq 0}[x_{n+1}]p^n \in W(A^\circ)$.

Recall from Proposition 2.3.8 that

$$N_1(\sum[a_n]p^n) = \sup_n \|a_n\|$$

is a power multiplicative norm on $W(A^\circ)$. We have two possibilities for x. First, suppose

$$\|\bar{x}\| > \|\bar{z}\|N_1(x).$$

We have $\overline{Q(x)} = x_1\overline{u^{-1}}$, so

$$\|\overline{Q(x)}\bar{z}\| \leq \|x_1\|\|\bar{z}\| \leq N_1(x)\|\bar{z}\| < \|\bar{x}\|.$$

It follows that

$$\|\overline{R(x)}\| = \|\bar{x} - \overline{Q(x)}\bar{z}\| = \|\bar{x}\|.$$

On the other hand

$$N_1(Q(x)[\bar{z}]) = N_1(u^{-1}\sum_{n\geq 0}[x_{n+1}\bar{z}]p^n) = N_1(\sum_{n\geq 0}[x_{n+1}\bar{z}]p^n) \leq \|\bar{z}\|N_1(x),$$

so

$$N_1(R(x)) = N_1([\bar{x}] - Q(x)[\bar{z}]) \leq \max(\|\bar{x}\|, N_1(Q(x)[\bar{z}])) = \|\bar{x}\|.$$

In conclusion

$$N_1(R(x)) = \|\overline{R(x)}\| = \|\bar{x}\|,$$

and in particular $R(x)$ is stable.

Otherwise, suppose

$$\|\bar{x}\| \leq \|\bar{z}\| N_1(x).$$

Since we still have $N_1(Q(x)[\bar{z}]) \leq \|\bar{z}\| N_1(x)$, we have

$$N_1(R(x)) \leq \max(\|\bar{x}\|, N_1(Q(x)[\bar{z}])) \leq \|\bar{z}\| N_1(x).$$

Now define a sequence $x^{(0)}, x^{(1)}, x^{(2)}, \ldots$ in $W(A^\circ)$ by $x^{(0)} = x, x^{(l+1)} = R(x^{(l)})$. They are all congruent to x mod z. By the above analysis, either one of them is stable, in which case we are done, or we have

$$N_1(Q(x^{(l)})) \leq N_1(x^{(l)}) \leq \|\bar{z}\| N_1(x^{(l-1)}) \leq \|\bar{z}\|^l N_1(x)$$

for all $l \geq 1$. Suppose we are in the latter case. Then $N_1(Q(x^{(l)})) \to 0$ and $N_1(x^{(l)}) \to 0$ as $l \to \infty$. The topology on $W(A^\circ)$ defined by N_1 is clearly finer than that defined by N_α for $0 < \alpha < 1$, and the latter is the same as the $(p, [t])$-adic topology, where t is any pseudo uniformizer in L. Since A° is $[t]$-adically complete and $W(A^\circ)$ is p-adically complete, by induction and using the exact sequence $0 \to p^n W(A^\circ)/p^{n+1} W(A^\circ) \to W(A^\circ)/p^{n+1} \to W(A^\circ)/p^n \to 0$ we see that $W(A^\circ)$ is $(p, [t])$-adically complete. Hence $Q := \sum_{l \geq 0} Q(x^{(l)})$ converges in $W(A^\circ)$ with respect to the $(p, [t])$-adic topology. Clearly $x = Qz$, so x is congruent to the stable element 0 in this case. □

Remark 2.4.3. The lemma is analogous to the classical Weierstrass Division Theorem in commutative algebra: Let $(R, \mathfrak{m})$ be a complete local ring, and let $z \in R[[t]]$ be *regular of degree* d in the sense that $z = \sum_{n \geq 0} a_n t^n$ with $a_0, \ldots, a_{d-1} \in \mathfrak{m}$ and $a_d \in R^\times$. Then every element of $R[[t]]$ is congruent to a degree $\leq d-1$ polynomial in $R[t]$.

The significance of primitive elements lies in the following result:

Corollary 2.4.4. *Suppose there is a perfectoid algebra R over a perfectoid field K such that $L = K^\flat$ and $A = R^\flat$. Then the theta map $\theta : W(A^\circ) \to R^\circ$ associated with $\sharp : A^\circ \to R^\circ$ is surjective and its kernel is generated by a primitive element. Moreover, this primitive element can be taken to be the image of a primitive element of $W(\mathcal{O}_L)$ generating the kernel of $\theta : W(\mathcal{O}_L) \to \mathcal{O}_K$. In particular, θ induces a canonical $\mathcal{O}_K$-algebra isomorphism*

$$W(A^\circ) \otimes_{W(\mathcal{O}_L), \theta} \mathcal{O}_K \xrightarrow{\sim} R^\circ.$$

Proof. Surjectivity follows from the isomorphism $A^\circ/p^\flat \cong R^\circ/p$, where $p^\flat$ is any element of $L = K^\flat$ such that $|p^\flat|^\flat = |p|$. We claim that the kernel of $\theta : W(A^\circ) \to R^\circ$ contains a primitive element in $W(\mathcal{O}_L)$. If K is of characteristic p, then $R = A$ and this kernel contains the primitive element p. Assume K is of characteristic 0. Then $(p^\flat)^\sharp/p \in \mathcal{O}_K^\times$, so by the surjectivity of $\theta : W(\mathcal{O}_L) \to \mathcal{O}_K$ we find $u = \sum_{n \geq 0} [a_n] p^n \in W(\mathcal{O}_L)$ such that $\theta(u) = (p^\flat)^\sharp/p$. It immediately follows that $a_0^\sharp \in \mathcal{O}_K^\times$, or equivalently that $a_0 \in \mathcal{O}_L^\times$. In particular, $u \in W(\mathcal{O}_L)^\times$. Let $z = [p^\flat] + pu$. Then z is primitive in $W(\mathcal{O}_L)$ and killed by $\theta : W(\mathcal{O}_L) \to \mathcal{O}_K$. Its image in $W(A^\circ)$ is primitive and killed by $\theta : W(A^\circ) \to R^\circ$.

Once we know that the kernel of $\theta : W(A^\circ) \to R^\circ$ contains a primitive element, it must be generated by that element, since otherwise this kernel contains a non-zero stable element by Lemma 2.4.2, which is impossible by the definition of θ. □

The last isomorphism in the corollary tells us how to recover the $\mathcal{O}_K$-algebra R° from the tilt A°. For now this is only at the algebraic level, and we need to further understand how to recover the topology on R° from the topology on A°. We can rewrite the isomorphism as

$$W(A^\circ)/z \xrightarrow{\sim} R^\circ.$$

Thus we will study how to put a norm on $W(A^\circ)/z$.

We return to the general setting of a perfectoid algebra $(A, \|\cdot\|)$ over a perfectoid field L of characteristic p, with $\|\cdot\|$ power multiplicative. Fix a primitive element $z = [\bar{z}] + pu \in W(A^\circ)$. Let α be a real number such that

$$\|\bar{z}\| < \alpha \leq 1.$$

Let $Q_\alpha : W(A^\circ)/z \to [0,1]$ be the quotient semi-norm induced by the norm N_α on $W(A^\circ)$ (see Proposition 2.3.8), namely,

$$Q_\alpha(x \bmod z) := \inf_{x \in W(A^\circ) \text{ lifting } (x \bmod z)} N_\alpha(x).$$

Proposition 2.4.5. *The following statements hold.*

1. *The value of Q_α at $(x \bmod z)$ is equal to $N_\alpha(x)$ for any stable $x \in W(A^\circ)$ lifting $(x \bmod z)$ (which exists by Lemma 2.4.2). In particular, Q_α is independent of α.*

2. *Q_α is a power multiplicative norm, and the topology on $W(A^\circ)/z$ defined by Q_α is complete.*

Proof. (1) The second statement follows from the first, since for x stable we have $N_\alpha(x) = \|\bar{x}\|$ and this is independent of α. To prove the first statement, it suffices to show that for any stable $x \in W(A^\circ)$ and any $y \in W(A^\circ), y \equiv x \bmod z$, we have $N_\alpha(y) \geq N_\alpha(x)$. Suppose not. Let $d := x - y$. We claim that $N_\alpha(d) = \|\bar{d}\|$. Indeed, write $d = \sum [d_n]p^n, x = \sum [x_n]p^n, y = \sum [y_n]p^n$. Then $[d_n]$ is the coefficient in front of p^n for $\sum_{k=0}^n [x_k]p^k - \sum_{k=0}^n [y_k]p^k$, so

$$\|d_n\| \leq N_1(\sum_{k=0}^n [x_k]p^k - \sum_{k=0}^n [y_k]p^k) \leq \max_{0 \leq k \leq n} (N_1([x_k]p^k), N_1([y_k]p^k)) = \max_{0 \leq k \leq n} (\|x_k\|, \|y_k\|).$$

But $\|x_k\| \leq \|x_0\|$ by the stability of x, and $\|y_k\| \leq \alpha^{-k} N_\alpha(y) \leq \alpha^{-n} N_\alpha(x) = \alpha^{-n}\|x_0\|$. Hence $\alpha^n \|d_n\| \leq \|x_0\|$ for all n, and so $N_\alpha(d) \leq \|x_0\|$. By our assumption we have $\|y_0\| \leq N_\alpha(y) < N_\alpha(x) = \|x_0\|$, so $\|d_0\| = \|x_0 - y_0\| = \|x_0\|$. Hence $N_\alpha(d) \leq \|d_0\|$, which is equivalent to that $N_\alpha(d) = \|d_0\|$.

Now write $d = ze$ with $e \in W(A^\circ)$. Then $d = [\bar{z}]e + pue$. We have

$$N_\alpha(pue) = N_\alpha(pe) = \alpha N_\alpha(e) > \|\bar{z}\| N_\alpha(e) \geq N_\alpha([\bar{z}]e).$$

Hence

$$N_\alpha(d) = N_\alpha(pue) = \alpha N_\alpha(e).$$

But by the claim

$$N_\alpha(d) = \|d_0\| = \|\bar{z}\bar{e}\| \leq \|\bar{z}\|\|\bar{e}\| \leq \|\bar{z}\| N_\alpha(e) < \alpha N_\alpha(e),$$

a contradiction.

(2) Since N_α is a power multiplicative norm, to show that Q_α is a power multiplicative norm, in view of part 1 it suffices to show that for any non-zero stable element $x \in W(A^\circ)$ we have $N_\alpha(x) \neq 0$ and x^n is stable for all $n \geq 1$. The first is because $x \neq 0$, and for the second we have $N_1(x^n) = N_1(x)^n = \|\bar{x}\|^n = \|\bar{x}^n\| = \|\overline{x^n}\|$ by the power multiplicativity of N_1 and $\|\cdot\|$, which shows that x^n is stable. To show that the topology defined on $W(A^\circ)/z$ defined by Q_α is complete, we may assume $\alpha < 1$ by the last statement of part 1. We already know that $W(A^\circ)/z$ is Hausdorff since Q_α is a norm. It remains to show that the topology defined by N_α on $W(A^\circ)$ is complete, and this was already noted in the proof of Lemma 2.4.2. ∎

The next proposition completes the proof of the tilting equivalence (Theorem 2.2.7).

Proposition 2.4.6. *Assume that L is the tilt of a perfectoid field K, and let $z \in W(\mathcal{O}_L)$ be primitive and generate the kernel of $\theta : W(\mathcal{O}_L) \to \mathcal{O}_K$. Write $|\cdot|$ for the absolute value on K and $|\cdot|^\flat$ for the absolute value on L, related by $\sharp : L \to K$. Let $(A, \|\cdot\|)$ be a perfectoid L-algebra, with $\|\cdot\|$ power multiplicative and L-homogeneous.*

1. *For $\|\bar{z}\| < \alpha \leq 1$, the norm Q_α (which is independent of α) on $W(A^\circ)/z$ is $\mathcal{O}_K$-homogeneous (i.e., $Q_\alpha(\lambda x) = |\lambda| Q_\alpha(x), \forall \lambda \in \mathcal{O}_K, x \in W(A^\circ)/z$). Here we view $W(A^\circ)/z$ as an $\mathcal{O}_K$-algebra via $W(\mathcal{O}_L)/z \cong \mathcal{O}_K$.*

2. *Extend Q_α to a K-homogeneous norm on $R := (W(A^\circ)/z) \otimes_{\mathcal{O}_K} K$. Then (R, Q_α) is a perfectoid K-algebra, $R^\circ = W(A^\circ)/z$, and $R^\flat \cong A$.*

3. *If $A = S^\flat$ for a perfectoid K-algebra S, then R constructed from A as in part 2 is isomorphic to S.*

Proof. (1) Let $\lambda \in \mathcal{O}_K$ and $x \in W(A^\circ)/z$. Let $\mu \in W(\mathcal{O}_L)$ be a stable lift of λ, and $y \in W(A^\circ)$ a stable lift of x. Then it is easy to see that $\mu y \in W(A^\circ)$ is stable. Hence

$$Q_\alpha(\lambda x) = N_\alpha(\mu y) = \|\overline{\mu y}\| = \|\bar{\mu} \cdot \bar{y}\| = |\bar{\mu}|^\flat \|\bar{y}\| = |\bar{\mu}|^\flat Q_\alpha(x).$$

But if we write $\mu = \sum [\mu_n] p^n$, then $|\lambda| = |\sum \mu_n^\sharp p^n| = |\mu_0^\sharp| = |\bar{\mu}|^\flat$, since $|\sum_{n\geq 1} \mu_n^\sharp p^n| \leq |p||\mu_0^\sharp| < |\mu_0^\sharp|$ by the stability of μ.

(2) By Proposition 2.4.5, Q_α is complete and power multiplicative. Hence (R, Q_α) is a complete Banach K-algebra. Since Q_α is power multiplicative, R° is the closed unit ball in R. To check that it is $W(A^\circ)/z$, it suffices to show that for any stable $\mu \in W(\mathcal{O}_L)$ and stable $y \in W(A^\circ)$ with $|\bar{\mu}|^\flat \geq \|\bar{y}\|$, we have μ divides y in $W(A^\circ)$. Write $y = \sum_{n\geq 0} [y_n] p^n$. For each $n \geq 0$, we have $\|y_n\| \leq \|y_0\| \leq |\bar{\mu}|^\flat$. Since $A^\circ = \{t \in A \mid \|t\| \leq 1\}$ (since $\|\cdot\|$ is power multiplicative), this implies that $\bar{\mu}$ divides y_n in A°, and so $[\bar{\mu}]$ divides $[y_n]$ in $W(A^\circ)$. But μ can be written as $\mu = [\bar{\mu}]u$ for some $u \in W(\mathcal{O}_L)^\times$ by stability, so μ divides $[y_n]$ in $W(A^\circ)$. Say $[y_n] = \mu e_n$ for $e_n \in W(A^\circ)$. Then $e := \sum e_n p^n$ converges in $W(A^\circ)$, and we have $\mu e = y$, as desired. Finally, it remains to show that Frob is surjective on $R^\circ/p = W(A^\circ)/(z, p)$. But this is isomorphic to $(W(A^\circ)/p)/\bar{z} = A^\circ/\bar{z}$, and Frob is surjective on it by the perfectoidness of A. We leave it as an exercise to check that $R^\flat \cong A$.

(3) In Corollary 2.4.4 we already have an $\mathcal{O}_K$-algebra isomorphism $R^\circ \xrightarrow{\sim} S^\circ$ induced by θ. It suffices to check that this isomorphism is an isometry, provided that the norm Q_α on R arises from the power multiplicative norm on $A = S^\flat$ that comes from the power multiplicative norm on S via $\sharp$. This is an easy exercise. $\square$

By the proposition, the tilting functor

$$\{\text{perfectoid } K\text{-algebras}\} \to \{\text{perfectoid } K^\flat\text{-algebras}\}, \quad R \mapsto R^\flat.$$

is an equivalence, with a quasi-inverse given by $A \mapsto (W(A^\circ)/z) \otimes_{\mathcal{O}_K} K = W(A^\circ) \otimes_{W(\mathcal{O}_K^\flat), \theta} K$ equipped with norm Q_α.

Remark 2.4.7. Proposition 2.4.6 itself does not answer the following question: Given a perfectoid field L of characteristic p, is there always a perfectoid field K of characteristic 0 such that $K^\flat = L$? Moreover, how do we characterize the primitive elements $z \in W(\mathcal{O}_L)$ generating the kernel of $\theta_K : W(\mathcal{O}_L) \to \mathcal{O}_K$ for such K? In other words, how do we untilt L to characteristic 0? (Since for any characteristic p perfectoid field K we have $K^\flat = K$, the untilt problem is only interesting when we require the untilt is of characteristic 0.) It turns out that every primitive $z \in W(\mathcal{O}_L)$ generates $\ker \theta_K$ for an untilt K of L, and the set of primitive elements of $W(\mathcal{O}_L)$ modulo units completely classifies untilts of L. The uninteresting untilt L corresponds to $z = p$, and any primitive $z = [\bar{z}] + pu$ with $\bar{z} \neq 0$ corresponds to a characteristic 0 untilt. To prove these statements, the key step is to check that for any primitive z, the normed ring $(W(\mathcal{O}_L)/z, Q_\alpha)$ is the ring of integers in a perfectoid field. This is proved similarly as part 2 of Proposition 2.4.6.

In fact, it is more natural to study tilting and untilting of *perfectoid rings* (which are certain uniform analytic Huber rings), without specifying perfectoid base fields. Then tilting is an equivalence of categories from the category of all perfectoid rings of residue characteristic p to the category of pairs (A, z) where A is a perfectoid ring of characteristic p and z is a primitive element of $W(A^\circ)$ well defined up to units. This point of view is taken in [Ked17], and the proof is essentially the same as what we presented above.

2.5 Tilting at the level of adic spaces

Definition 2.5.1. Let K be a perfectoid field. By a *perfectoid pair* over K, we mean a Huber pair (R, R^+) such that R is a perfectoid K-algebra.

Given a perfectoid algebra R over K, the set of all possible rings of integral elements R^+ is identified with the set of integrally closed subrings of R/ϖ, where $\varpi \in K$ satisfies $|p| \leq |\varpi| < 1$; the point is that ϖ must lie in R^+ since a power of it lies in R^+ and R^+ is integrally closed. Take $\varpi^\flat \in K^\flat$ such that $|\varpi^\flat|^\flat = |\varpi|$. Then $R/\varpi \cong R^\flat/\varpi^\flat$. Hence the set of rings of integral elements in R is naturally identified with the set of rings of integral elements in the tilt $R^\flat$. Hence the tilting equivalence automatically upgrades to an equivalence

$$\{\text{perfectoid pairs over } K\} \to \{\text{perfectoid pairs over } K^\flat\}.$$

Remark 2.5.2. Let (R, R^+) be a perfectoid pair over K with tilt (A, A^+) over $K^\flat$. As noted above, $\varpi \in R^+$. Then A^+ can be described in terms of R^+ via $A^+ \cong \varprojlim_{\mathrm{Frob}} R^+/\varpi$. However, when we attempt to use the recipe similar to Proposition 2.4.6 to describe R^+ in terms of A^+ via $R^+ \cong W(A^+)/z$, we need to be slightly careful: The problem is that A^+ need not be an $\mathcal{O}_{K^\flat}$-algebra, so the chosen primitive element $z \in W(\mathcal{O}_{K^\flat})$ (such that $W(\mathcal{O}_{K^\flat})/z \cong \mathcal{O}_K$) may not have a well-defined image in $W(A^+)$. Nevertheless, it turns out that we can always multiply z by a unit in $W(\mathcal{O}_{K^\flat})$ (which may depend on A^+) to arrange that $z \in W(\mathcal{O}_{K^\flat} \cap A^+)$. Then the formula $R^+ \cong W(A^+)/z$ is valid.

This ambiguity in z goes away if we fix a perfectoid pair (R, R^+) over K with tilt (A, A^+) over $K^\flat$, and only consider perfectoid pairs under (R, R^+) on the K-side and perfectoid pairs under (A, A^+) on the $K^\flat$-side. (This will be the case when studying rational localizations of a perfectoid pair.) We can fix a primitive element $z \in W(A^+)$ such that $R^+ \cong W(A^+)/z$. Then for every perfectoid pair under (A, A^+), we can use the same z to obtain its untilt.

As noted in Remark 2.4.7, here the more natural point of view is again to formulate the tilting equivalence for perfectoid pairs without specifying perfectoid base fields, and on the tilted side to remember a primitive element up to units as part of the datum. See [Ked17, Theorem 2.3.9].

Lemma 2.5.3. *Let (A, A^+) be a Huber pair where A is a uniform Banach algebra over a non-archimedean field K. Let $U\left(\frac{f_1,\dots,f_n}{g}\right)$ be a rational subset of $\mathrm{Spa}(A, A^+)$ with $\sum f_i A = A$. Then there exists a pseudo-uniformizer $\epsilon \in K$ with the following property: For any $f_1', \dots, f_n', g' \in A$ satisfying*

$$\max(v(f_i'), v(\epsilon)) = \max(v(f_i), v(\epsilon)) \quad \text{and} \quad \max(v(g'), v(\epsilon)) = \max(v(g), v(\epsilon))$$

for all $v \in \mathrm{Spa}(A, A^+)$, we have $\sum_i f_i' A = A$ and $U\left(\frac{f_1',\dots,f_n'}{g'}\right) = U\left(\frac{f_1,\dots,f_n}{g}\right)$.

Proof. Let $a_1, \dots, a_n \in A$ be such that $\sum_i a_i f_i = 1$, and let ϵ_0 be a pseudo-uniformizer in K such that $\epsilon_0 a_i \in A^+$ for all i. Then for every $v \in \mathrm{Spa}(A, A^+)$ we have $v(a_i) \le v(\epsilon_0)^{-1}$ for all i, and hence

$$1 = v(\sum_i a_i f_i) \le v(\epsilon_0)^{-1} \max_i v(f_i).$$

Now take $\epsilon = \epsilon_0^2$. Then

$$\max_i v(f_i) > v(\epsilon), \ \forall v \in \mathrm{Spa}(A, A^+).$$

Now let f_i' and g' be as in the lemma, with respect to our chosen ϵ. Then

$$\max_i v(f_i') = \max_i v(f_i) > v(\epsilon), \ \forall v \in \mathrm{Spa}(A, A^+).$$

If $\sum f_i' A$ is a proper ideal, then by a general fact about Banach algebras, there exists $v \in \mathrm{Spa}(A, A^+)$ such that $v(f_i') = 0$ for all i, which contradicts with the above inequality. Hence $\sum_i f_i' A = A$.

Let $v \in U\left(\frac{f_1,\dots,f_n}{g}\right)$. Then $v(g) \ge \max_i v(f_i) > v(\epsilon)$, and so by the hypothesis on g' we have $v(g') = v(g) > 0$. Moreover we have $\max_i v(f_i') = \max_i v(f_i) \le v(g) = v(g')$, so we have $v \in U\left(\frac{f_1',\dots,f_n'}{g'}\right)$.

Conversely, let $v \in U\left(\frac{f_1',\dots,f_n'}{g'}\right)$. Then $v(g') \ge \max_i v(f_i') > v(\epsilon)$, and so by the hypothesis on g' we have $v(g') = v(g)$. Then $v(g) > 0$ and $v(g) = v(g') \ge \max_i v(f_i') = \max_i v(f_i)$, so $v \in U\left(\frac{f_1,\dots,f_n}{g}\right)$. ∎

Theorem 2.5.4. *Let (R, R^+) be a perfectoid pair over K with tilt (A, A^+) over $K^\flat$. Write $X = \mathrm{Spa}(R, R^+), X^\flat = \mathrm{Spa}(A, A^+)$.*

1. *The map $X \to X^\flat$ sending a valuation v to $v \circ \sharp$ (where $\sharp : A = R^\flat \to R$) is a homeomorphism, and preserves rational subsets.*

2. *If a rational subset $U \subset X$ corresponds to a rational subset $U^\flat \subset X^\flat$, then the Huber pairs $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ and $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$ are perfectoid over K and $K^\flat$ respectively, and the latter is the tilt of the former.*

Sketch of proof. For simplicity, assume that $R^+ = R^\circ, A^+ = A^\circ$ and fix a primitive element $z \in W(\mathcal{O}_K^\flat)$ generating the kernel of $\theta : W(\mathcal{O}_K^\flat) \to \mathcal{O}_K$. (Without this assumption, we shall just work with a primitive $z \in W(A^+)$ such that $R^+ \cong W(A^+)/z$, cf. Remark 2.5.2. The proof is the same.)

Step 1. The map $X \to X^\flat$ is well defined and continuous. More precisely, the pull-back of a rational subset is rational.

Well definedness is proved similarly as Proposition 2.1.10 (3). Let $U = U(f_1, \dots, f_n/g)$ be a rational subset of $X^\flat$. By Lemma 1.5.9, there is a pseudo-uniformizer $t \in K^\flat$ such that $v(g) > v(t)$ for all $v \in U$. We may then add t to the list of f_i's without changing U. Then the inverse image of U in X is $U(f_1^\sharp, \dots, f_n^\sharp/g^\sharp)$, and this is rational since $f_n^\sharp$ is a unit in R (as it lies in $K^\times$). Since rational subsets form a basis of the topology, it follows that $X \to X^\flat$ is continuous.

Step 2. Every rational subset of X is of the form $U(f_1^\sharp, \dots, f_n^\sharp/g^\sharp)$ as in Step 1. In particular, taking the inverse image induces a bijection $\{\text{rational subsets of } X^\flat\} \xrightarrow{\sim} \{\text{rational subsets of } X\}$.

Let $U(\phi_1, \dots, \phi_n/\gamma)$ be a rational subset of X. By Lemma 2.5.3, it suffices to show that for every pseudo-uniformizer $\epsilon \in K$ we can find $f_1, \dots, f_n, g$ such that

$$\max(v(f_i^\sharp), v(\epsilon)) = \max(v(\phi_i), v(\epsilon)) \quad \text{and} \quad \max(v(g^\sharp), v(\epsilon)) = \max(v(\gamma), v(\epsilon)) \quad \forall v \in X.$$

Indeed, as before we may assume that ϕ_n is a pseudo-uniformizer in K. There exists a pseudo-uniformizer $t \in K^\flat$ such that $t^\sharp$ divides ϕ_n in R^+. Thus we may assume that $\phi_n = t^\sharp$. Let ϵ be as in Lemma 2.5.3 with respect to $U(\phi_1, \dots, \phi_n/\gamma)$, and find f_i, g as above, with $f_n = t$. Then $U(\phi_1, \dots, \phi_n/\gamma) = U(f_1^\sharp, \dots, f_n^\sharp/g^\sharp)$ by Lemma 2.5.3, and this is the inverse image of the rational subset $U(f_1, \dots, f_n/g)$ of $X^\flat$ (which is rational since $f_n = t$).

We now show the existence of g with respect to ϵ, and the rest are the same. By simultaneously multiplying γ and ϵ by a pseudo-uniformizer in K, we may and shall assume that $\gamma \in R^+$.

Recall the Euclidean division from the proof of Lemma 2.4.2:

$$x = Q(x)z + R(x), \ \forall x \in W(A^+).$$

For every $x = \sum_{n \geq 0} [x_n]p^n \in W(A^+)$ and $w \in X^\flat$, the same argument as in that proof[3] shows that at least one of the following two possibilities happen: (We have slightly changed the division into two cases, so the current two cases are not mutually exclusive.)

(i) We have $w(\bar{z})w(x_n) \leq w(x_0)$ for all $n \geq 0$, with strict inequality for $n = 1$. In this case, $R(x)$ is w-stable in the sense that $R(x) = \sum [x'_n]p^n$ with $w(x'_0) \geq w(x'_n)$ for all n.

(ii) For some n we have $w(\bar{z})w(x_n) \geq w(x_0)$. In this case, every upper bound of $\{w(\bar{z})w(x_n)\}_n$ is an upper bound of $\{w(x'_n)\}_n$ where $R(x) = \sum [x'_n]p^n$.

Note that if x satisfies (i) with respect to w, then so does $R(x)$, since $w(\bar{z}) < 1$. Now let $x^{(0)} \in W(A^+)$ be an arbitrary lift of $\gamma \in R^+ \cong W(A^+)/z$, and inductively define $x^{(k+1)} = R(x^{(k)})$. Choose $l \in \mathbb{Z}_{\geq 1}$ such that $(|\bar{z}|^\flat)^l < |\epsilon|$. We claim that $g := \overline{x^{(l)}} \in W(A^+)/p \cong A^+$ satisfies the desired condition

$$\max(v(g^\sharp), v(\epsilon)) = \max(v(\gamma), v(\epsilon)), \ \forall v \in X.$$

Write $x^{(l)} = [g] + \sum_{n \geq 1} [a_n]p^n$. Let $v \in X$ with image $v^\flat = v \circ \sharp \in X^\flat$. If one of $x^{(k)}, k \leq l-1$ satisfies (i) with respect to $v^\flat$, then $x^{(l)}$ is $v^\flat$-stable. Then

$$\gamma = \theta(x^{(0)}) = \theta(x^{(l)}) = g^\sharp + \sum_{n \geq 1} a_n^\sharp p^n,$$

[3] We would like to replace $\|\cdot\|$ on A° as in that proof by w, and replace the norm $N_1 : \sum [x_n]p^n \mapsto \sup_n \|x_n\|$ on $W(A^\circ)$ by $\sum [x_n]p^n \mapsto \sup_n w(x_n)$. Since the target of w is an abstract totally ordered abelian group, the sup does not make sense on the nose. However all logical statements involving sup can be easily translated to statements only involving upper bounds. The only properties of N_1 used in that proof which are still needed in the current proof are:
(i) Strong triangle inequality. This generalizes to the following fact which is still a consequence of Lemma 2.3.9: If $\sum [x_n]p^n + \sum [y_n]p^n = \sum [t_n]p^n$, then for any upper bound ξ of $\{w(x_n)\}$ and any upper bound η of $\{w(y_n)\}$, $\max(\xi, \eta)$ is an upper bound of $\{w(t_n)\}$.
(ii) $N_1(xy) \leq N_1(x)$ (as we needed $N_1(u^{-1} \sum_{n \geq 0} [x_{n+1}\bar{z}]p^n) \leq N_1(\sum_{n \geq 0} [x_{n+1}\bar{z}]p^n)$). This generalizes to: If $(\sum [x_n]p^n)(\sum [y_n]p^n) = \sum [t_n]p^n$, and if ξ is an upper bound of $\{w(x_n)\}$, then ξ is an upper bound of $\{w(t_n)\}$. To show this, by similar method as the proof of Lemma 2.3.9, we know that z_n is a bi-homogeneous degree $(1,1)$ polynomial over $\mathbb{F}_p$ in the variables $x_k^{p^{-i}}, y_k^{p^{-i}}$ for $0 \leq k \leq n$. The desired statement follows.

and

$$v(g^\sharp) = v^\flat(g) > v(\sum_{n\geq 1} a_n^\sharp p^n)$$

since each $v(a_n^\sharp) = v^\flat(a_n) \leq v^\flat(g)$ and $v(p) < 1$. Hence $v(\gamma) = v(g^\sharp)$ in this case.

Otherwise, each of $x^{(0)}, \ldots, x^{(l-1)}$ satisfies (ii) with respect to $v^\flat$. Write $x^{(0)} = \sum [x_n] p^n$. Then every upper bound of $\{v^\flat(\bar{z})^l v^\flat(x_n)\}_n$ is an upper bound of $\{v^\flat(g), v^\flat(a_n), n \geq 1\}$. Since $v(\epsilon)$ is such an upper bound, we get $v(g^\sharp) = v^\flat(g) \leq v(\epsilon)$, as well as

$$v(\gamma) = v(g^\sharp + \sum_{n\geq 1} a_n^\sharp p^n) \leq v(\epsilon).$$

Step 3. Let $U^\flat$ be a rational subset of $X^\flat$ with inverse image $U \subset X$. We show that $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is a perfectoid pair with tilt $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$.

We reduce to the following two cases:

$$\begin{cases} U^\flat = U(f, 1/1), \\ U = U(f^\sharp, 1/1); \end{cases} \qquad \text{or} \qquad \begin{cases} U^\flat = U(1/g), \\ U = U(1/g^\sharp). \end{cases}$$

We only explain the proof in the first case, as the second case is similar. Write ϕ for $f^\sharp$, and write $\phi^{p^{-n}}$ for $(f^{p^{-n}})^\sharp$. Thus $\phi^{p^{-n}}$ is a p^n-th root of ϕ. Write $(B, B^+) = (\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$ and $(S, S^+) = (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$. Then we have

$$B = A\langle T\rangle/(T - f)$$

where $A\langle T\rangle = \{\sum_{n\geq 0} a_n T^n \mid a_n \in A, a_n \to 0\}$ is equipped with the Gauss norm $\|\sum_{n\geq 0} a_n T^n\| = \sup_n \|a_n\|_A$, and B is equipped with the quotient norm. (Note that $(T - f)$ is a closed ideal, so the quotient is still a Banach K-algebra.) Similarly

$$S = R\langle T\rangle/(T - \phi).$$

Our first goal is to write B and S as quotients of certain perfectoid algebras. Introduce the "perfectoid Tate algebra"

$$A' = A\langle T^{p^{-\infty}}\rangle := (A\langle T\rangle[T^{p^{-\infty}}])^\wedge.$$

Here every element of $A\langle T\rangle[T^{p^{-\infty}}]$ is of the form

$$\sum_{n\geq 0} a_n T^{n/p^N}$$

for some fixed $N \in \mathbb{Z}_{\geq 1}$ and $a_n \in A, a_n \to 0$. Define its Gauss norm to be $\sup_n \|a_n\|_A$. The completion in the definition of $A\langle T^{p^{-\infty}}\rangle$ is with respect to this norm. Similarly define $R' = R\langle T^{p^{-\infty}}\rangle$. It is easy to check that R' is a perfectoid K-algebra and its tilt is A'.

We claim that there is a topological isomorphism $B \cong A'/\overline{(T^{p^{-h}} - f^{p^{-h}})}_{h\geq 0}$, where the right hand side denotes the quotient of A' by the closure of the ideal generated by $\{T^{p^{-h}} - f^{p^{-h}} \mid h \geq 0\}$ equipped with the quotient topology. Indeed, for each $h \geq 0$, define

$$B_h = A\langle T\rangle[T^{p^{-h}}]/(T^{p^{-h}} - f^{p^{-h}}).$$

Then there is a natural A-algebra isomorphism

$$B_h \xrightarrow{\sim} B_{h+1}.$$

The point is that for every term of a power series in B_{h+1}, of the form

$$a \cdot T^{k/p^h + j/p^{h+1}}, \; k \in \mathbb{Z}_{\geq 0}, j \in \{1, 2, \ldots, p-1\},$$

we can use the relation $T^{1/p^{h+1}} = f^{1/p^{h+1}}$ to rewrite it as

$$af^{j/p^{h+1}} \cdot T^{k/p^h}.$$

One easily checks that if we apply this procedure to every term, then we obtain a power series in B_h (i.e., the new coefficients still tend to 0). Moreover, under this isomorphism the natural Gauss norms on the two sides compare as follows:

$$\|\cdot\|_{B_{h+1}} \leq \|\cdot\|_{B_h} \leq \max(1, \|f\|_A)^{p^{-h}} \cdot \|\cdot\|_{B_{h+1}}.$$

In particular, we have $B = B_0 \xrightarrow{\sim} B_h$ and

$$\|\cdot\|_{B_h} \leq \|\cdot\|_B \leq \max(1, \|f\|_A)^{1+p^{-1}+\cdots+p^{-h+1}} \|\cdot\|_{B_h}.$$

Since $\sum_k p^{-k} < \infty$, the claim follows.

By the claim, one checks that B is perfectoid. Indeed, we already know that it is a Banach $K^\flat$-algebra, so we only need to check that it is perfect by Lemma 2.2.5, which can be done using the claim. (Alternatively, there is even a much softer argument showing this, using the universal property of $B = \mathcal{O}_{X^\flat}(U^\flat)$ and the fact that any rational subset of $X^\flat$ is invariant under $w \mapsto w \circ (x \mapsto x^p)$.)

Similarly, we have a topological isomorphism

$$S \cong R'/(T^{p^{-h}} - \phi^{p^{-h}})^-_{h \geq 0}.$$

However, from this presentation we are unable to directly conclude that S is perfectoid. In the remaining part of the proof we show that S is the untilt of B. This will show that S is perfectoid, and it will also easily follow that (S, S^+) tilts to (B, B^+). By the functoriality of untilting, the surjection $A' \to B$ induces a surjection

$$\xi : (\text{the untilt of } A' \text{ over } K) = R' \to (\text{the untilt of } B \text{ over } K),$$

and $\ker \xi$ is the closure of the ideal of R' generated by $(T^{p^{-h}} - f^{p^{-h}})^{\sharp_{A'}}$ for $h \geq 0$, where $\sharp_{A'}$ is the usual multiplicative map $A' \to (\text{the untilt of } A' \text{ over } K) = R'$. It remains to show that

$$\ker \xi = (T^{p^{-h}} - \phi^{p^{-h}})^-_{h \geq 0}.$$

For $\supset$, note that ξ maps $T^{p^{-h}}$ and $\phi^{p^{-h}}$ to $(T^{p^{-h}})^{\sharp_B}$ and $(f^{p^{-h}})^{\sharp_B}$ respectively, but $T^{p^{-h}} = f^{p^{-h}}$ in B. For $\subset$, it suffices to show that in $W(A')$, each $[T^{p^{-h}} - f^{p^{-h}}]$ is contained in the closure (for the p-adic topology) of the ideal generated by $[T^{p^{-k}}] - [f^{p^{-k}}]$ for $k \geq 0$. This follows from the following general observation: For any $x, y \in W(A')$, since $[x - y] \equiv z_k^{p^k} \bmod p^{k+1}$ with $z_k = [x^{p^{-k}}] - [y^{p^{-k}}]$, we know that $[x - y]$ lies in the closure of the ideal generated by z_k for $k \geq 0$.

Step 4. The map $X \to X^\flat$ is a homeomorphism.

The spaces X and $X^\flat$ are T_0, so for every pair of distinct points there is a rational subset containing one and not the other. Hence by Step 3 we know that $X \to X^\flat$ is injective. It suffices to show that it is surjective. Then it is a homeomorphism since it induces a bijection between the sets of rational subsets.

Let $x \in X^\flat$. Let k_x be the residue field $\mathcal{O}_{X,x}$/maximal ideal, and $k_x^+ \subset k_x$ the image of $\mathcal{O}^+_{X,x}$. Then (k_x, k_x^+) is a Huber pair under the topology defined by v_x (the canonical valuation on $\mathcal{O}_{X,x}$, which descends to k_x). It is a general fact that the topology on k_x^+ is the same as the ϖ-adic topology, and moreover the completion is identified with the ϖ-adic completion of $\mathcal{O}^+_{X,x}$. Combining this fact with Step 3, we know that the completion of (k_x, k_x^+) is a perfectoid pair over $K^\flat$ (and under (A, A^+)). Its untilt over K is a perfectoid pair (l, l^+) under (R, R^+) where l is a perfectoid field. Let y be the image of the generic point under $\mathrm{Spa}(l, l^+) \to X$. One checks that y is a preimage of x.

□

Corollary 2.5.5. *Every perfectoid pair is stably uniform, and so the conclusions of Theorem 1.7.2 apply. Thus it is sheafy, the augmented Čech complex of any rational cover is exact, and Tate acyclicity holds.*

Definition 2.5.6. Let K be a perfectoid field. A *perfectoid space over* K is an adic space which is locally isomorphic to $\mathrm{Spa}(R, R^+)$ for perfectoid pairs (R, R^+) over K.

2.6 Almost mathematics and almost acyclicity

Let K be a perfectoid field and let $\varpi \in K$ be such that $|p| \leq |\varpi| < 1$.

Definition 2.6.1. An $\mathcal{O}_K$-module M is *almost zero*, if $\mathfrak{m}_K M = 0$. An $\mathcal{O}_K$-module morphism is an *almost isomorphism*, if its kernel and cokernel are both almost zero.

Remark 2.6.2. Note that $\sup_{x\in\mathfrak{m}_K} |x| = 1$. Thus for an almost zero $\mathcal{O}_K$-module M, there exists $x \in \mathfrak{m}_K$ whose absolute value is arbitrarily close to 1 (so x is "almost a unit") and $xM = 0$.

Lemma 2.6.3. *The class of almost zero $\mathcal{O}_K$-modules is closed under sub, quotient, and extensions.*

Proof. For extensions, use that for every $x \in \mathfrak{m}_K$ there exist $x_1, x_2 \in \mathfrak{m}_K$ such that $x_1x_2|x$ in $\mathcal{O}_K$. □

By the lemma and the general theory of Serre quotient category, we can form the quotient category

$$\mathcal{O}_K^a \text{ - mod} := \mathcal{O}_K \text{ - mod}/(\text{almost zero objects}).$$

This is an abelian category equipped with a functor $(\cdot)^a : \mathcal{O}_K \text{ - mod} \to \mathcal{O}_K^a \text{ - mod}$ which sends every almost isomorphism to an isomorphism and is universal for this property. We call this category the *category of almost $\mathcal{O}_K$-modules*. It has a symmetric monoidal structure $\otimes$ such that the functor $(\cdot)^a$ is monoidal. This allows us to consider commutative algebras over $\mathcal{O}_K^a$ (namely, commutative algebra objects in $\mathcal{O}_K^a$ - mod), as well as modules over them. A lot of the usual concepts in commutative algebra have analogues in this setting.

Remark 2.6.4. Concretely, we can construct the category $\mathcal{O}_K^a$ - mod as follows: Its objects are $\mathcal{O}_K$-modules. For an $\mathcal{O}_K$-module M, we write M^a for the corresponding object in $\mathcal{O}_K^a$ - mod. Morphisms are defined by:

$$\mathrm{Hom}_{\mathcal{O}_K^a}(M^a, N^a) := \mathrm{Hom}_{\mathcal{O}_K}(\mathfrak{m}_K \otimes_{\mathcal{O}_K} M, N),$$

and compositions of morphisms are defined using the canonical isomorphism $\mathfrak{m}_K = \mathfrak{m}_K^2 \cong \mathfrak{m}_K \otimes_{\mathcal{O}_K} \mathfrak{m}_K$. Namely, for $f : \mathfrak{m}_K \otimes_{\mathcal{O}_K} M \to N$ and $g : \mathfrak{m}_K \otimes_{\mathcal{O}_K} N \to U$, define their composition to be

$$g \circ f : \mathfrak{m}_K \otimes_{\mathcal{O}_K} M \cong \mathfrak{m}_K \otimes_{\mathcal{O}_K} \mathfrak{m}_K \otimes_{\mathcal{O}_K} M \xrightarrow{\mathrm{id}_{\mathfrak{m}_K} \otimes f} \mathfrak{m}_K \otimes_{\mathcal{O}_K} N \xrightarrow{g} U.$$

Define the functor $(\cdot)^a$ by sending every $f : M \to N$ to the morphism $f^a : M^a \to N^a$ corresponding to $\mathfrak{m}_K \otimes_{\mathcal{O}_K} M \to N, x \otimes m \mapsto xf(m)$. One checks that the pair $(\mathcal{O}_K^a \text{ - mod}, (\cdot)^a)$ indeed satisfies the universal property.

Similarly, we define the category $\mathcal{O}_K^a/\varpi$ - mod of "almost $\mathcal{O}_K/\varpi$-modules".

Definition 2.6.5. By a *perfectoid algebra over $\mathcal{O}_K^a$*, we mean a commutative $\mathcal{O}_K^a$-algebra A which is

- ϖ-adically complete: the natural map $A \to \varprojlim_n A/\varpi^n$ is an isomorphism (of $\mathcal{O}_K^a$-algebras).
- flat: the functor $A \otimes_{\mathcal{O}_K^a} (\cdot) : \mathcal{O}_K^a \text{ - mod} \to \mathcal{O}_K^a \text{ - mod}$ is exact.
- the map $\mathrm{Frob} : A/\varpi^{1/p} \to A/\varpi$ is an isomorphism (of $\mathcal{O}_K^a$-algebras). Here $\varpi^{1/p}$ denotes any element of $\mathcal{O}_K$ whose absolute value is $|\varpi|^{1/p}$.

Similarly, we define perfectoid $\mathcal{O}_K^a/\varpi$-algebras. We denote the categories of them by $\mathcal{O}_K^a$ - Perf and $\mathcal{O}_K^a/\varpi$ - Perf.

Theorem 2.6.6. *We have a sequence of equivalences of categories*

$$K \text{ - Perf} \underset{(1)}{\xrightarrow{\sim}} \mathcal{O}_K^a \text{ - Perf} \underset{(2)}{\xrightarrow{\sim}} \mathcal{O}_K^a/\varpi \text{ - Perf} \underset{(3)}{\xrightarrow{\sim}} \mathcal{O}_{K^\flat}^a/\varpi^\flat \text{ - Perf} \underset{(2)^{-1}}{\xrightarrow{\sim}} \mathcal{O}_{K^\flat}^a \text{ - Perf} \underset{(1)^{-1}}{\xrightarrow{\sim}} K^\flat \text{ - Perf}.$$

Moreover, the composition of all these equivalences is the tilting equivalence $R \mapsto R^\flat$.

Sketch of proof. The functor (1) is given by $R \mapsto R^\circ$, and its quasi-inverse is given by $A \mapsto A \otimes_{\mathcal{O}_K} K$ (or $A \mapsto A \otimes_{\mathcal{O}_{K^\flat}} K^\flat$). The equivalence (3) arises from the canonical isomorphism $\mathcal{O}_K/\varpi \cong \mathcal{O}_{K^\flat}/\varpi^\flat$ (where $\varpi^\flat \in K^\flat$ is such that $|\varpi^\flat|^\flat = |\varpi|$). The functor (2) is given by reduction mod ϖ. The proof that it is an equivalence will be sketched later using cotangent complexes. □

Theorem 2.6.7. *Let (R, R^+) be a perfectoid pair over a perfectoid field K, and let $X = \mathrm{Spa}(R, R^+)$. For any rational cover $\{U_i\}$ of X, the augmented Čech complex*

$$0 \to \mathcal{O}_X(X)^{oa} \to \prod_i \mathcal{O}_X(U_i)^{oa} \to \prod_{i,j} \mathcal{O}_X(U_i \cap U_j)^{oa} \to \cdots$$

is exact (in $\mathcal{O}_K^a$ - mod). In particular, since the above complex is isomorphic to the complex of $\mathcal{O}_K^a$-modules associated with the the augmented Čech complex for $\mathcal{O}_X^+$, we know that $H^i(X, \mathcal{O}_X^+)$ is almost zero for all $i \geq 1$.

Sketch of proof. **Step 0.** Reducing to a statement about norms.

We first reduce to showing that for a rational cover with two opens $\{U_1, U_2\}$, the complex

$$0 \to \mathcal{O}_X(X)^{oa} \to \mathcal{O}_X(U_1)^{oa} \times \mathcal{O}_X(U_2)^{oa} \to \mathcal{O}_X(U_1 \cap U_2)^{oa} \to 0$$

is exact. We then reduce to showing the following statement: The complex of K-Banach spaces

$$(\dagger) \qquad 0 \to \mathcal{O}_X(X) \xrightarrow{f_1} \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2) \xrightarrow{f_2} \mathcal{O}_X(U_1 \cap U_2) \to 0$$

is *almost optimal*, in the sense that it is exact, and that the quotient norm on the image of each map is equal to the subspace norm inherited from the target of the map. Here, every term is a uniform Banach K-algebra, and we equip it with the unique power-multiplicative norm (which is required to be K-homogeneous). Here uniqueness follows from the more general observation that any continuous K-algebra map between Banach K-algebras equipped with power multiplicative norms $\phi : A_1 \to A_2$ must be norm-decreasing in that sense that $\|\phi(x)\|_{A_2} \leq \|x\|_{A_1}$, $\forall x \in A_1$. This is because continuity of ϕ and the K-homogeneity of the norms imply that there exists a constant $C > 1$ such that $\|\phi(x)\|_{A_2} \leq C\|x\|_{A_1}$. One then applies this to x^n and use power-multiplicativity to get $\|\phi(x)\|_{A_2} \leq C^{1/n}\|x\|_{A_1}$.

Now we already know that $(\dagger)$ is exact by Corollary 2.5.5. The map f_1 is a continuous K-algebra map, so it is norm-decreasing by the above observation. The norm on $\mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$ sends (x, y) to $\max(\|x\|_{\mathcal{O}_X(U_1)}, \|y\|_{\mathcal{O}_X(U_2)})$, and f_2 sends (x, y) to $g_1(x) - g_2(y)$ where g_i is the natural continuous K-algebra map $\mathcal{O}_X(U_i) \to \mathcal{O}_X(U_1 \cap U_2)$. Hence f_2 is also norm-decreasing by the strong triangle inequality and the fact that g_i are norm-decreasing. Thus the condition that $(\dagger)$ is almost optimal is equivalent to the condition that for each $i = 1, 2$, each x in the image of f_i, and each $C > 1$, there exists a preimage of x along f_i whose norm is less than $C\|x\|$. As an exercise (using the non-discreteness of $|K^\times|$), this condition is also equivalent to asking that every element of the image of f_i of norm less than 1 has a preimage of norm less than 1.

Step 1. Reducing to characteristic p.

By Theorem 2.5.4, the tilt of $(\dagger)$ is the similar complex for a rational cover of $X^\flat$ with two opens

$$0 \to \mathcal{O}_{X^\flat}(X^\flat) \xrightarrow{f_1^\flat} \mathcal{O}_{X^\flat}(U_1^\flat) \times \mathcal{O}_{X^\flat}(U_2^\flat) \xrightarrow{f_2^\flat} \mathcal{O}_{X^\flat}(U_1^\flat \cap U_2^\flat) \to 0.$$

(Here $f_1^\flat$ is literally the tilt of f_1, and $f_2^\flat$ denotes the map $(x, y) \mapsto g_1^\flat(x) - g_2^\flat(y)$ where $g_i^\flat : \mathcal{O}_{X^\flat}(U_i^\flat) \to \mathcal{O}_{X^\flat}(U_1^\flat \cap U_2^\flat)$ is the tilt of g_i.) Supposing that this tilted complex is almost optimal, we show that $(\dagger)$ is almost optimal. We show that every element $t \in \mathrm{im}(f_2)$ of norm less than 1 has a preimage along f_2 of norm less than 1; the similar property for f_1 is proved in the same way (and slightly easier). By Proposition 2.4.5, t has a lift $\sum_{n \geq 0} [a_n] p^n \in W(\mathcal{O}_{X^\flat}(U_1^\flat \cap U_2^\flat)^\circ)$ along

$$\theta : W(\mathcal{O}_{X^\flat}(U_1^\flat \cap U_2^\flat)^\circ) \to \mathcal{O}_X(U_1 \cap U_2)^\circ$$

such that every $a_n \in \mathcal{O}_{X^\flat}(U_1^\flat \cap U_2^\flat)$ has norm less than 1. Then by our assumption, each a_n has a lift $(b_n, c_n) \in \mathcal{O}_{X^\flat}(U_1^\flat) \times \mathcal{O}_{X^\flat}(U_2^\flat)$ along $f_2^\flat$ which has norm less than 1, i.e., both $b_n \in \mathcal{O}_{X^\flat}(U_1^\flat)$ and $c_n \in \mathcal{O}_{X^\flat}(U_2^\flat)$ have norm less than 1, and $g_1^\flat(b_n) - g_2^\flat(c_n) = a_n$. Set

$$x = \theta_1(\sum_{n \geq 0} [b_n] p^n) \in \mathcal{O}_X(U_1)^\circ, \quad y = \theta_2(\sum_{n \geq 0} [c_n] p^n) \in \mathcal{O}_X(U_2)^\circ,$$

where θ_i is the θ-map

$$W(\mathcal{O}_{X^\flat}(U_i^\flat)^\circ) \to \mathcal{O}_X(U_i)^\circ.$$

By our proof of the tilting equivalence (see Proposition 2.4.6), we have $\|x\| \leq \sup_n \|b_n\|\alpha^n < 1$ (where we choose $\alpha \in (0,1)$ arbitrarily) and similarly $\|y\| < 1$. Then (x, y) is the desired lift of t along f_2.

The above reduction method follows [KL15, Prop. 8.3.2, 3.6.9]. Alternatively, Scholze [Sch12] uses Theorem 2.5.4 and Theorem 2.6.6 to reduce to characteristic p. The point is that for a complex of $\mathcal{O}_K^a$-modules whose terms are perfectoid $\mathcal{O}_K^a$-algebras, it being exact is equivalent to its reduction mod ϖ as a complex of $\mathcal{O}_K^a/\varpi$-modules, whose terms are perfectoid $\mathcal{O}_K^a/\varpi$-algebras, being exact.

Step 2. Proof in the case of characteristic p.

We now show that (†) is almost optimal assuming that K has characteristic p. Let $C > 1$. We show that every $x \in \mathrm{im}(f_i)$ has a lift $\tilde{x}$ along f_i such that $\|\tilde{x}\| < C\|x\|$. By the Banach open mapping theorem, there is a constant $D > 1$ such that every $x \in \mathrm{im}(f_i)$ has a lift $\tilde{x}$ along f_i satisfying $\|\tilde{x}\| < D\|x\|$. Applying this to x^{p^n}, we find a lift y of x^{p^n} satisfying $\|y\| < D\|x^{p^n}\|$. But then $y^{p^{-n}}$ is a lift of x, and it satisfies $\|y^{p^{-n}}\| < D^{p^{-n}}\|x\|$. Choosing n sufficiently large such that $D^{p^{-n}} < C$, we finish the proof. ☐

Remark 2.6.8. In [Sch12], the proof of Theorem 2.6.7 in the case of characteristic p consists in further reduction to the p-finite case and invoking the classical Tate acyclicity in the topologically finite type case. This is because at the time of that paper, Theorem 1.7.2 had not been discovered, so *a priori* one cannot use Tate acyclicity of $\mathcal{O}_X$ in the proof. Rather, Scholze proves this as a consequence of Theorem 2.6.7.

2.7 The cotangent complex

In the following, a ring always refers to a commutative ring with identity.

In classical commutative algebra and algebraic geometry, one encounters the following question: If $A \to B \to C$ are ring maps, then we have an exact sequence of C-modules (sometimes called the "first fundamental exact sequence")

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

How do we "complete" this exact sequence to a long exact sequence? The answer is provided by the cotangent complex. For any ring map $A \to B$, the cotangent complex $L_{B/A}$ is a complex of B-modules $(\cdots \to M_1 \to M_0)$ in homological degrees ≥ 0 that is well defined up to isomorphism in the derived category. For $A \to B \to C$ as above, we have an exact triangle

$$C \otimes_B^{\mathbb{L}} L_{B/A} \to L_{C/A} \to L_{C/B} \xrightarrow{+1}$$

In particular, the associated long exact sequence of homology completes the previous exact sequence:

$$\cdots \to H_2(L_{C/B}) \to H_1(C \otimes_B^{\mathbb{L}} L_{B/A}) \to H_1(L_{C/A}) \to H_1(L_{C/B}) \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

This long exact sequence entails another classical exact sequence when $B \to C$ is surjective with kernel I. In this case $\Omega_{C/B} = 0$ and $H_1(L_{C/B}) = I/I^2$ (as $C = B/I$-module). Hence we have the exact sequence

$$I/I^2 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to 0,$$

which is classically known as the "second fundamental exact sequence".

Fact 2.7.1. *In parts 1 to 3 below, assume that A is noetherian and $A \to B$ is of finite type.*

1. *If $A \to B$ is étale, then $L_{B/A} = 0$.*
2. *If $A \to B$ is smooth, then $L_{B/A}$ is isomorphic to $\Omega_{B/A}$ at degree 0, and $\Omega_{B/A}$ is finite projective over B.*
3. *If $A \to B$ is locally complete intersection, then $L_{B/A}$ is isomorphic to $(P_1 \to P_0)$ where P_i are finite projective over B.*
4. *Let $A \to B$ and $A \to A'$ be ring maps, and let $B' = B \otimes_A A'$. Assume that either one of $A \to B$ and $A \to A'$ is flat. Then we have*

$$A' \otimes_A^{\mathbb{L}} L_{B/A} \cong L_{B'/A'}.$$

5. *Let $A \to B$ be a ring map over characteristic p that is* relatively perfect *in the sense that the A-algebra map $B \otimes_{A,\mathrm{Frob}} A \to B, b \otimes a \mapsto b^p \otimes a$ is an isomorphism. Then $L_{B/A} = 0$.*

Theorem 2.7.2 (Invariance of "étale like" algebras under nilpotent thickenings of the base)**.** *For any ring A, let $\mathscr{C}(A)$ denote the category of flat A-algebras B such that $L_{B/A} = 0$. (These are the "étale like" A-algebras.) Then for any surjection $A' \to A$ with nilpotent kernel, the functor $(\cdot) \otimes_{A'} A : \mathscr{C}(A') \to \mathscr{C}(A)$ is an equivalence.*

Proof. This is a special case of Illusie's deformation theory. □

Lemma 2.7.3. *Let K be a non-archimedean field, and ϖ a pseudo-uniformizer. Let $n \in \mathbb{Z}_{\geq 1}$. Write A_n for $\mathcal{O}_K/\varpi^n$. Let B_n be a flat A_n-algebra such that $B_n \otimes_{A_n} A_1$ is relatively perfect over A_1. Then $L_{B_n/A_n} = 0$.*

Proof. We prove by induction on n. If $n = 1$, this is part 5 of Fact 2.7.1. Assume $n \geq 2$. Consider the exact sequence of A_n-modules

$$0 \to A_1 \xrightarrow{\times \varpi^{n-1}} A_n \to A_{n-1} \to 0.$$

Applying $(\cdot) \otimes^{\mathbb{L}}_{A_n} L_{B_n/A_n}$ to it and using part 4 of Fact 2.7.1, we get an exact triangle

$$L_{B_1/A_1} \to L_{B_n/A_n} \to L_{B_{n-1}/A_{n-1}} \xrightarrow{+1},$$

where $B_i := B_n \otimes_{A_n} A_i$ for $i \in \{1, n-1\}$. By induction hypothesis, $L_{B_i/A_i} = 0$ for $i \in \{1, n-1\}$, so $L_{B_n/A_n} = 0$. □

Example 2.7.4. Let R be a perfect $\mathbb{F}_p$-algebra. Then R is flat and relatively perfect over $\mathbb{F}_p$. By Theorem 2.7.2 and Lemma 2.7.3, for any $n \geq 1$ there is a unique (up to unique isomorphism) flat $\mathbb{Z}/p^n$-algebra lifting R. Moreover this algebra has trivial cotangent complex over $\mathbb{Z}/p^n$. We know that the ring of truncated Witt vectors $W_n(R) = W(R)/p^n$ is such a $\mathbb{Z}/p^n$-algebra. It then easily follows that $W(R) = \varprojlim_n W_n(R)$ is the unique flat and p-adically complete $\mathbb{Z}_p$-algebra lifting R.

We can now prove the equivalence (2) in Theorem 2.2.7.

Theorem 2.7.5. *Reduction mod ϖ induces an equivalence of categories $\mathcal{O}_K^a$ - Perf $\xrightarrow{\sim} \mathcal{O}_K^a/\varpi$ - Perf.*

Sketch of proof. (See [Bha17, Thm. 6.2.5].) Write $\mathscr{C}_n$ for the category of flat $\mathcal{O}_K/\varpi^n$-algebras B such that B/ϖ is relatively perfect over $\mathcal{O}_K/\varpi$. Write $\mathscr{C}$ for the category of flat and ϖ-adically complete $\mathcal{O}_K$-algebras B such that B/ϖ is is relatively perfect over $\mathcal{O}_K/\varpi$. By Theorem 2.7.2 and Lemma 2.7.3, the $\mathscr{C}_n$'s are all naturally equivalent to each other. By taking the inverse limit it is also easy to see that they are all equivalent to $\mathscr{C}$. (Starting with any $B_n \in \mathscr{C}_n$ for a fixed n, we get the corresponding objects $B_m \in \mathscr{C}_m$ for all m, and they form an inverse system $(\cdots \to B_2 \to B_1)$. The inverse limit $\varprojlim_m B_m$ is then the corresponding object of $\mathscr{C}$.)

We now construct the inverse functor $\mathcal{O}_K^a/\varpi$ - Perf $\to$ $\mathcal{O}_K^a$ - Perf as follows. Start with $\bar{A} \in \mathcal{O}_K^a/\varpi$ - Perf. The functor $(\cdot)^a$ from $\mathcal{O}_K$-algebras (resp. $\mathcal{O}_K/\varpi$-algebras) to $\mathcal{O}_K^a$-algebras (resp. $\mathcal{O}_K^a/\varpi$-algebras) has a left adjoint denoted by $(\cdot)_{!!}$. One checks that $\bar{A}_{!!} \in \mathscr{C}_1$. Let $\tilde{A} \in \mathscr{C}$ be the image of $\bar{A}_{!!}$ under the equivalence $\mathscr{C}_1 \simeq \mathscr{C}$. Let $A = \tilde{A}^a$. This is an $\mathcal{O}_K^a$-algebra, and one checks that it is perfectoid. We define the desired inverse functor by sending $\bar{A}$ to A.

To check that $A/\varpi \cong \bar{A}$, we have

$$A/\varpi = \tilde{A}^a/\varpi \cong (\tilde{A}/\varpi)^a \cong (\bar{A}_{!!})^a \cong \bar{A},$$

where the last isomorphism uses a general property of $(\cdot)_{!!}$.

It remains to check that if $\bar{A} = B/\varpi$ for a perfectoid $\mathcal{O}_K^a$-algebra B, then A constructed above is naturally isomorphic to B. In this case, one checks that $B_{!!} \in \mathscr{C}$. The image of $B_{!!}$ under $\mathscr{C} \simeq \mathscr{C}_1$ is $B_{!!}/\varpi \cong (B/\varpi)_{!!} = \bar{A}_{!!}$ (as $(\cdot)_{!!}$ commutes with colimits). Hence $B_{!!} \cong \tilde{A}$, and it follows that $B \cong (B_{!!})^a \cong \tilde{A}^a = A$. □

References

[Bha17] B Bhatt. Lecture notes for a class on perfectoid spaces, 2017. available at https://www.math.ias.edu/~bhatt/teaching/mat679w17/lectures.pdf.

[Bou64] Nicolas Bourbaki. *Algèbre commutative. Chapitre 6: Valuations*. Number 1308 in Éléments de mathématique, Fasc. XXX. Actualités Scientifiques et Industrielles, Hermann, Paris, 1964.

[Hub93] Roland Huber. Continuous valuations. *Math. Z.*, 212(3):455–477, 1993.

[Ked13] Kiran S Kedlaya. Nonarchimedean geometry of witt vectors. *Nagoya Mathematical Journal*, 209:111–165, 2013.

[Ked15] Kiran S Kedlaya. New methods for (φ, γ)-modules. *Research in the Mathematical Sciences*, 2(1):20, 2015.

[Ked17] Kiran S Kedlaya. Sheaves, stacks, and shtukas. *Perfectoid Spaces: Lectures from the 2017 Arizona Winter School*, 242, 2017.

[KL15] KS Kedlaya and R Liu. Relative p-adic hodge theory: foundations, astérisque 371. *Soc. Math. France, Paris*, 2015.

[Sch12] Peter Scholze. Perfectoid spaces. *Publications mathématiques de l'IHÉS*, 116(1):245–313, 2012.

[Ser13] Jean-Pierre Serre. *Local fields*. Springer Science & Business Media, 2013.