REPRESENTATIONS OF P-ADIC GROUPS

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Abstract.

Contents

A description of the course

The topic of the course is representation theory of p-adic groups. This is the study of typically infinite dimensional representations of certain "p-adic Lie groups". More specifically, we will work towards the statement and proof of **Bernstein's decomposition theorem** for the category of smooth representations of a p-adic reductive group, after laying foundations in the general theory of representations of locally profintie groups and recalling the basic structure theory of p-adic groups. If time allows, we will discuss the Langlands classification of representations and/or the Local Langlands Correspondence.

Prerequisites. Familiarity with the representation theory of finite groups will be helpful to build some intuition, although not strictly necessary. No prior exposure to representation theory beyond that is required. Knowledge about the general theory of linear algebraic groups over a field will be helpful, as we will go through that pretty fast. Basic familiarity with the p-adic numbers will be assumed.

References. We will loosely follow Renard's book [Ren10]. Another source is Alan Roche's lecture notes (=Chapter 1 of the Ottawa Lectures [CN09]). Also useful is Ngô's lecture notes [Ngô16]. For a guide to the literature see [Ren10].

1. Motivations

Lect.1, Jan 25

We start by recalling the field of p-adic numbers $\mathbb{Q}_p$. Let p be a prime. On $\mathbb{Q}$, we have the p-adic absolute value $|\cdot|_p$ defined as follows: If $x = 0$, then $|x|_p = 0$. If $x \neq 0$, then write $x = p^n y$ with $n \in \mathbb{Z}$ and $y \in \mathbb{Q}^\times$ such that y has a reduced fraction in which p does not appear in the numerator or denominator. Then $|x|_p = p^{-n}$. We define $\mathbb{Q}_p$ to be the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$; this is analogous to $\mathbb{R}$ as the completion of $\mathbb{Q}$ with respect to the usual absolute value $|\cdot|_\infty$.

Over $\mathbb{R}$, we have the theory of Lie groups, which are smooth manifolds (locally isomorphic to $\mathbb{R}^n$) equipped with a group structure. Over $\mathbb{Q}_p$, one might also consider "p-adic Lie groups" as "smooth p-adic manifolds" with a group structure. For our purpose, we do not need to go into the question of how to make sense of these. Instead, we simply consider p-adic algebraic groups, i.e., subgroups of $\mathrm{GL}_n(\mathbb{Q}_p) = \{\text{invertible } n \times n \text{ matrices over } \mathbb{Q}_p\}$ defined by systems of polynomial equations over $\mathbb{Q}_p$ in the n^2 coordinates.[1] We shall also impose two more conditions: reductive, and connected (which are conditions for the underlying algebraic group over $\mathbb{Q}_p$).

Example 1.1. We have the following examples of connected reductive p-adic algebraic groups:

- $\mathrm{GL}_n(\mathbb{Q}_p)$, $\mathrm{SL}_n(\mathbb{Q}_p) = \{g \in \mathrm{GL}_n(\mathbb{Q}_p) \mid \det(g) = 1\}$.
- Fix a degree d extension $E/\mathbb{Q}_p$. Fix a $\mathbb{Q}_p$-basis of E. Then we have an injective ring homomorphism $E \hookrightarrow M_{d\times d}(\mathbb{Q}_p)$, sending $x \in E$ to the $\mathbb{Q}_p$-linear endomorphism of $E \cong \mathbb{Q}_p^d$ given by $y \mapsto xy$. The resulting map $\mathrm{GL}_n(E) \hookrightarrow M_{nd\times nd}(\mathbb{Q}_p)$ induces an injective group homomorphism $\mathrm{GL}_n(E) \hookrightarrow \mathrm{GL}_{nd}(\mathbb{Q}_p)$, and this is a subgroup defined by polynomial equations over $\mathbb{Q}_p$.

Exercise 1.2. Find these equations, and justify the other claims made above.

[1] The correct definition is more general, in that we allow polynomial equations in n^2+1 variables: The first n^2 variables a_{ij} stand for the coordinates, and the last variable b satisfies the equation that $\det(a_{ij})b = 1$. In other words, we allow polynomial equations in the n^2 coordinates and $\det^{-1}$.

Similarly, $\mathrm{SL}_n(E) \hookrightarrow \mathrm{SL}_{nd}(\mathbb{Q}_p)$.

- Fix $J \in \mathrm{GL}_n(\mathbb{Q}_p)$ anti-symmetric, i.e., $J = -J^t$. (Thus n is even.) We have the symplectic group $\mathrm{Sp}(J) = \{g \in \mathrm{GL}_n(\mathbb{Q}_p) \mid g^t J g = J\}$.
- Fix $J \in \mathrm{GL}_n(\mathbb{Q}_p)$ symmetric, i.e., $J = J^t$. We have the special orthogonal group $\mathrm{SO}(J) = \{g \in \mathrm{SL}_n(\mathbb{Q}_p) \mid g^t J g = J\}$.[2]
- Fix a quadratic extension $E/\mathbb{Q}_p$, and fix $J \in \mathrm{GL}_n(E)$ hermitian, i.e., $J^t = \bar{J}$, where $\overline{(\cdot)}$ denotes the non-trivial element of $\mathrm{Gal}(E/\mathbb{Q}_p)$. We have the unitary group $\mathrm{U}(J) = \{g \in \mathrm{GL}_n(E) \mid \bar{g}^t J g = J\}$. Using the method in the second example above, one shows that $\mathrm{U}(J)$ is a subgroup of $\mathrm{GL}_{2n}(\mathbb{Q}_p)$ defined by polynomial equations over $\mathbb{Q}_p$.

  Exercise 1.3. Show this.

We will also replace $\mathbb{Q}_p$ by a more general non-archimedean local field F, and consider connected reductive algebraic groups $G \subset \mathrm{GL}_n(F)$ defined by polynomials over F. We endow G with the subspace topology inherited from $\mathrm{GL}_n(F) \subset F^{n^2}$. Here F^{n^2} has the product topology of the natural topology of F as a local field. For instance, if $F = \mathbb{Q}_p$, the topology is defined by $|\cdot|_p$. Then G is Hausdorff, locally compact, and $1 \in G$ has a neighborhood basis consisting of compact open subgroups of G. For instance, $1 \in \mathrm{GL}_n(\mathbb{Q}_p)$ has a neighborhood basis $1 + p^k M_{n\times n}(\mathbb{Z}_p), k \geq 1$, which are compact open subgroups. Moreover, G is *totally disconnected*, meaning that each connected component in G is a singleton.[3] Totally disconnected spaces are very different from the "everyday topological spaces" such as manifolds: In a manifold, every point has a neighborhood homeomorphic to an Euclidean space, which is connected and infinite. On a general totally disconnected space, there is no interesting way of doing analysis (differentiation, etc.), and this fact actually simplifies the theory considerably.

Definition 1.4. A *representation of* G is a vector space V over $\mathbb{C}$ (typically infinite dimensional) together with a homomorphism $\pi : G \to \mathrm{Aut}(V)$. We often write $g \cdot v$ for $\pi(g) \cdot v, g \in G, v \in V$. We call (V, π) *smooth*, if for all $v \in V$, the stabilizer $\mathrm{Stab}_v G = \{g \in G \mid g \cdot v = v\}$ is open. Equivalently, the action map $G \times V \to V, (g, v) \mapsto g \cdot v$ is continuous, for the discrete topology on V.

Lect.2, Jan 27

Exercise 1.5. Show the equivalence. Also show that the smooth condition is equivalent to the condition that for each $v \in V$, the map $G \to V, g \mapsto g \cdot v$ is locally constant.

We are interested in the category $\mathcal{M}(G)$ of all smooth representations of G. Here a morphism $(V, \pi) \to (W, \rho)$ is by definition a linear map $\phi : V \to W$ that is G-equivariant, i.e., $\phi(g \cdot v) = g \cdot \phi(v)$ for all $g \in G, v \in V$. (We also call such a map a *G-map* or a *G-linear map*.)

Why do we care about smooth representations of G?

(1) p-adic groups are important, so we should study their representations. Smooth representations turn out to form a very natural class.
(2) Smooth representations of G arise as local components of automorphic representations. Consider a reductive group $\mathbb{G}$ over a global field, say $\mathbb{Q}$. Then the theory

[2] This satisfies the connected condition, whereas the orthogonal group $\mathrm{O}(J) = \{g \in \mathrm{GL}_n(\mathbb{Q}_p) \mid g^t J g = J\}$ does not.

[3] This does not contradict with our assumption that G is a connected algebraic group, as the latter refers to the Zariski topology of the underlying algebraic variety, which is very different from the topology coming from F.

of automorphic representations of $\mathbb{G}$ roughly amounts to the study of functions on $\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles over $\mathbb{Q}$. This theory is a vast generalization of the theory of modular forms, and it has utter importance in number theory and the Langlands program. It turns out that each automorphic representation is determined by the data of a smooth representation of the p-adic group $\mathbb{G}(\mathbb{Q}_p)$, one for each prime p, together with a certain representation of the real Lie group $\mathbb{G}(\mathbb{R})$.

(3) Roughly speaking, the Local Langlands Correspondence (LLC) predicts a relationship between $\mathcal{M}(G)$ and the category of (certain) ${}^LG$-valued representations $\mathrm{WD}_F \to {}^LG$. Here WD_F is the Weil–Deligne group of F, a variant of $\mathrm{Gal}(\overline{F}/F)$, and ${}^LG$ is the Langlands dual group of G. When $G = \mathrm{GL}_n(F)$, ${}^LG = \mathrm{GL}_n(\mathbb{C})$. For $G = \mathrm{GL}_1(F)$, the LLC is essentially local class field theory for F. Thus $\mathcal{M}(G)$ secretly carries arithmetic information about F. For p-adic groups G other than GL_n, the LLC is a topic of active current research.

The structure of $\mathcal{M}(G)$ is much more complicated than representations of a finite group. Recall that every representation of a finite group Γ (over $\mathbb{C}$) is a union of finite dimensional representations, so in particular irreducible representations (i.e., those that do not have non-trivial sub-representations) must be finite dimensional. The category of finite dimensional representations of Γ is a semi-simple category, i.e., every finite dimensional representation is the direct sum of irreducible representations. Moreover, the number of isomorphism classes of irreducible representations is finite, and equal to the number of conjugacy classes in Γ. In contrast, the irreducible representations in $\mathcal{M}(G)$ are in general not finite dimensional, and the number of isomorphism classes of them is also infinite. For this reason, we would lose too much if we only consider finite dimensional smooth representations, and that is why in our definition of $\mathcal{M}(G)$ we allow infinite dimensionality. The category $\mathcal{M}(G)$ is not semi-simple; there exist smooth representations which are not direct sums of irreducible ones. One goal of our course is to *decompose the category* $\mathcal{M}(G)$ into a direct sum of certain subcategories, and moreover say something about the structures of these subcategories. Concretely, this means for every $(V,\pi) \in \mathcal{M}(G)$ we decompose it into a direct sum in a canonical way such that each summand belongs to one of these prescribed subcategories, and moreover we require that the decomposition be respected by all morphisms in $\mathcal{M}(G)$. This is the content of the *Bernstein decomposition theorem.*

2. Td spaces and groups

Recall profinite topological spaces. Let $(I,\le)$ be a directed set (i.e., $\le$ is a reflexive and transitive relation on I, and for any $i,j \in I$ there exists $k \in I, i \le k, j \le k$), and let $(X_i)_{i\in I}$ be a projective system of finite sets. Thus each X_i is a finite set, and for each pair $i \le j \in I$, we have a transition map $\phi_{i,j} : X_j \to X_i$ satisfying $\phi_{i,j} \circ \phi_{j,k} = \phi_{i,k} : X_k \to X_i$ for $i \le j \le k$. (Similarly, one defines the notion of a projective system in an arbitrary category.)

The inverse limit $\varprojlim_{i\in I} X_i$ is the subset of $\prod_{i\in I} X_i$ consisting of $(x_i)_i$ such that $\phi_{i,j}(x_j) = x_i$ for all $i,j \in I$ with $i \le j$. We endow it with the subspace topology inherited from the product topology on $\prod_{i\in I} X_i$ (each X_i having the discrete topology). Concretely, a basis of the topology of $\prod_{i\in I} X_i$ is given as follows. Fix a finite subset $I_0 \subset I$, and for each $i \in I_0$ fix $Y_i \subset X_i$. Let

$$U = \prod_{i\in I-I_0} X_i \times \prod_{i\in I_0} Y_i.$$

For varying choices of I_0 and $(Y_i)_{i\in I_0}$, the U's form a basis of the topology. Lect.3, Jan 30

Exercise 2.1. The set U as above is also closed in $\prod_{i\in I} X_i$.

Definition 2.2. A topological space X is *profinite* if it is homeomorphic to $\varprojlim_{i\in I} X_i$ for a projective system of finite sets as above.

Lemma 2.3. *A topological space X is profinite if and only if it is Hausdorff, compact, and totally disconnected, i.e., every connected component is a singleton.*

Proof. "Only if": Write $X = \varprojlim_{i\in I} X_i$. Hausdorff is easy to check. For compactness, note that X is a closed subspace of $\prod_{i\in I} X_i$. The latter is compact by Tychonoff's theorem (product of compact spaces is compact), and so X is compact. For totally disconnectedness, use that the image of a connected subset of X in each X_i must be connected, and therefore a singleton.

"If": Let I be the set of maps $f : X \to \mathbb{Z}$ such that $\mathrm{im}(f)$ is finite and each $f^{-1}(n)$ is open. Informally, I is the set of ways of decomposing X into a finite disjoint union of open subsets. For $f, g \in I$, define $f \leq g$ if there exists a (necessarily unique) map $\phi_{f,g} : \mathrm{im}(g) \to \mathrm{im}(f)$ such that $f = \phi_{f,g} \circ g$ (i.e., the partition corresponding to g refines that to f). Then $(\mathrm{im}(f))_{f\in I}$ is a projective system of fintie sets, and we have a natural continuous map $\Phi : X \to \varprojlim_{f\in I} \mathrm{im}(f), x \mapsto (f(x))_f$. We claim Φ is a homeomorphism. Since the source is compact and the target is Hausdorff, it suffices to check that Φ is a bijection.

To show Φ is surjective, let $\underline{n} = (n_f)_f \in \varprojlim_f \mathrm{im}(f)$. We need to find a preimage of $\underline{n}$. For each $f \in I$, let $C_f = f^{-1}(n_f)$. Then C_f is clopen[4], and we have $C_g \subset C_f$ whenever $f \leq g$. Note that $\Phi^{-1}(\underline{n}) = \bigcap_f C_f$. If this were empty, then by compactness of X we know that a finite sub-intersection is empty. Finding a common upper bound of the indices, we get some $f \in I$ such that C_f is empty, a contradiction. This shows that Φ is surjective.

To show Φ is injective, by Lemma 2.4 below, we have $\{x\}$ is the intersection of all clopens containing x, for any $x \in X$. If $\Phi(x) = \Phi(y)$, then for any clopen U containing x it also contains y, since the characteristic function 1_U of U is an element of I. Taking the intersections of all such clopens we conclude $x = y$. □

Lemma 2.4. *Let X be a Hausdorff compact topological space. Then for any $x \in X$, the connected component of X containing x is the intersection of all clopens containing x.*

Proof. Fix $x \in X$, and let $(Z_i)_{i\in I}$ be all clopens containing x. Let $S = \bigcap_i Z_i$. Clearly the connected component containing x is contained in S, so it remains to show that S is connected. Suppose not. Since S is closed, we have $S = A \sqcup B$ for two disjoint non-empty closed subsets A, B of X. Since X is compact, so are A, B. Since X is Hausdorff, the compact subsets A and B are separated by open neighborhoods, i.e., there are disjoint opens U and V in X such that $A \subset U, B \subset V$. Since $X - U \cup V$ is compact and $(X - U \cup V) \cap S = \emptyset$, we know that $X - U \cup V$ is disjoint from a finite sub-intersection of the Z_i's, which is itself a Z_i. Namely, $Z_i \subset U \cup V$, and so $Z_i = (Z_i \cap U) \sqcup (Z_i \cap V)$. It then follows that both $Z_i \cap U$ and $Z_i \cap V$ are clopens. Suppose without loss of generality $x \in A$. Then $Z_i \cap U$ is one of the Z_j's, and hence $S \subset Z_i \cap U \subset U$, and so $B = \emptyset$, a contradiction. □

Corollary 2.5. *A closed subset Z of a profinite space X is profinite. Moreover, X has a basis of topology consisting of compact open profinite subsets.*

Proof. We know Z is still Hausdorff and compact. Clearly Z is totally disconnected as X is. Hence Z is profinite by Lemma 2.3. For the second assertion, we have already seen in

[4] "Clopen" stands for "closed and open".

Exercise 2.1 that X has a basis of topology consisting of clopens. But by the first assertion every clopen is compact and profinite. □

Proposition 2.6. *Let X be a Hausdorff topological space. TFAE:*

(1) *The topology has a basis consisting of open compact sets.*
(2) *X is locally compact and totally disconnected.*
(3) *Each point has an open neighborhood that is a profinite space.*

Proof. (1) ⇒ (2). Locally compact is clear. To see totally disconnected, let $x, y \in X$ be distinct. Then x and y are separated by open compact neighborhoods, which are in particular clopens. Hence x, y cannot be in the same connected component.

(2) ⇒ (3). Let $x \in X$. Since X is locally compact, there exist open $U \subset X$ and compact $K \subset X$ such that $x \in U \subset K$. Now K is Hausdorff compact totally disconnected, so profinite by Lemma 2.3. Therefore by Corollary 2.5 , the open set U in K must be a union of opens U_i in K which are profinite. The point x must lie in such a U_i. But U_i is open in K and contained in $U \subset K$, so it is open in U, which implies that it is open in X. Hence we have found an open neighborhood of x in X which is profinite.

(3) ⇒ (1). This follows from Corollary 2.5.

□

Definition 2.7. A Hausdorff topological space satisfying the above equivalent conditions is called *a td space*, or *a locally profinite space*.

Remark 2.8. A topological space is profinite if and only if it is td and compact.

Lect.4, Feb 1

Definition 2.9. A topological group is called *a td group*, or *a locally profinite group*, if its underlying topological space is td.

Remark 2.10. A topological group is td if and only if it is Hausdorff and 1 has an open neighborhood that is a profinite space. This is sufficient because we can translate the neighborhood U of 1 to a neighborhood gU of g for any g in the group, and gU is homeomorphic to g.

Fact 2.11. *Let G be a td group. Then $1 \in G$ has a neighborhood basis consisting of open compact subgroups.*

In practice, the td groups we are interested in (the p-adic groups) will be easily seen to satisfy this fact.

Exercise 2.12. Prove Fact 2.11 as follows. Let K be an arbitrary compact open neighborhood of 1. It suffices to construct a compact open subgroup H of G contained in K. Do this in the following steps:

(1) For every $x \in K$ there is an open neighborhood V_x of 1 such that $xV_x^2 \subset K$.
(2) There is an open neighborhood V of 1 such that $KV \subset K$. In particular, $V \subset K$ and $V^2 \subset K$.
(3) We may take V in (2) to satisfy $V^{-1} = V$.
(4) Let H be the subgroup of G generated by V. Then H is an open subgroup of G, and $H \subset K$.
(5) Any open subgroup of a topological group is closed. Hence H is closed, and therefore compact as it is contained in K.

Example 2.13. Let p be a prime. Then $\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ is a profinite group (under addition). It is identified (as a topological group/ring) with the subgroup/subring $\{x \in \mathbb{Q}_p \mid |x|_p < 1\}$ of $\mathbb{Q}_p$, which is open in $\mathbb{Q}_p$. Thus $\mathbb{Q}_p$ is a td group. The subgroups $p^n\mathbb{Z}_p, n \geq 1$, are open and compact, and form a neighborhood basis of 0.

Similarly, the group $\mathbb{Z}_p^\times$ of multiplicative units in the ring $\mathbb{Z}_p$ is profinite and has presentation $\mathbb{Z}_p^\times = \varprojlim_n (\mathbb{Z}/p^n\mathbb{Z})^\times$. It is an open subgroup of $\mathbb{Q}_p^\times$. Hence $\mathbb{Q}_p^\times$ is td. The subgroups $1 + p^n\mathbb{Z}_p, n \geq 1$, are open and compact, and they form a neighborhood basis of 1.

Example 2.14. For any prime p, $\mathrm{GL}_n(\mathbb{Q}_p)$ is a td group, and $\mathrm{GL}_n(\mathbb{Z}_p) \subset \mathrm{GL}_n(\mathbb{Q}_p)$ is an open, profinite subgroup. In fact, $\mathrm{GL}_n(\mathbb{Z}_p) \cong \varprojlim_k \mathrm{GL}_n(\mathbb{Z}/p^k\mathbb{Z})$ as topological groups. It is open in $\mathrm{GL}_n(\mathbb{Q}_p)$ because it is defined by the (non-strict) inequalities $|a_{ij}|_p \leq 1, |\det(a_{ij})|_p = 1$. We have compact open subgroups $1 + p^k M_n(\mathbb{Z}_p)$ of $\mathrm{GL}_n(\mathbb{Z}_p)$ for $k \geq 1$, and they form a neighborhood basis of 1.

Example 2.15. More generally, let F be a non-archimedean local field. Recall that this means F is a complete topological field with respect to a discrete non-archimedean absolute value $|\cdot|_F$ (i.e., an absolute value satisfying the strong triangle inequality $|a+b| \leq \max(|a|, |b|)$ and such that $|F^\times|$ is a discrete subgroup of $\mathbb{R}^\times$) such that the topology is locally compact. The locally compact condition is equivalent to the condition that the ring of integers $\mathcal{O}_F = \{x \in F \mid |x| \leq 1\}$, which is an open subring and a DVR, should have finite residue field. Any generator π of the unique maximal ideal of $\mathcal{O}_F$ is called a uniformizer. We have $\mathcal{O}_F \cong \varprojlim_k \mathcal{O}_F/\pi^k$ is a profinite group, with each $\mathcal{O}_F/\pi^k$ being a finite ring. Also $\mathcal{O}_F^\times \cong \varprojlim_k (\mathcal{O}_F/\pi^k)^\times$ is profinite. The groups $F, F^\times$ are td, containing $\mathcal{O}_F, \mathcal{O}_F^\times$ as open subgroups respectively. As before, $\mathrm{GL}_n(\mathcal{O}_F) \cong \varprojlim_k \mathrm{GL}_n(\mathcal{O}_F/\pi^k)$ is profinite, and open in the td group $\mathrm{GL}_n(F)$. We have compact open subgroups $1 + \pi^k M_n(\mathcal{O}_F)$, forming a neighborhood basis of 1.

Clearly if G is a closed subgroup of $\mathrm{GL}_n(F)$, then G is also td, and $G \cap \mathrm{GL}_n(\mathcal{O}_F)$ is an open profinite subgroup of G. The compact open subgroups $G \cap (1 + \pi^k M_n(\mathcal{O}_F))$ of G form a neighborhood basis of 1. This applies in particular to the case where $G \subset \mathrm{GL}_n(F)$ is defined by polynomial equations.

3. Functions and distributions

Let X be a td space. Define[5]

- $C^\infty(X)$ = the $\mathbb{C}$-vector space of locally constant functions $X \to \mathbb{C}$.
- $C_c^\infty(X)$ = the subspace of $C^\infty(X)$ consisting of compactly supported functions. We define the support of $f \in C_c^\infty(X)$ to be the smallest (i.e., intersection of all) compact set $K \subset X$ such that $f = 0$ outside K.
- $C^{-\infty}(X)$ = the linear dual of $C_c^\infty(X)$.
- $C_c^{-\infty}(X)$ = the $\mathbb{C}$-vector space of linear maps $\alpha : C^\infty(X) \to \mathbb{C}$ satisfying the following compactly supported condition: there exists a compact subset $K \subset X$ such that $\langle \alpha, f \rangle = 0$ for all $f \in C^\infty(X)$ such that $f|_K \equiv 0$. We define the support of α to be the smallest such K, i.e., the intersection of all such K.

The dual of the inclusion $C_c^\infty(X) \hookrightarrow C^\infty(X)$ is a (surjective) map $C^\infty(X)^* \to C^{-\infty}(X)$. Composing with the inclusion $C_c^{-\infty}(X) \hookrightarrow C^\infty(X)^*$, we obtain a map

$$C_c^{-\infty}(X) \longrightarrow C^{-\infty}(X).$$

[5]Our notations $C_c^\infty, C^{-\infty}, C_c^{-\infty}$ correspond to the symbols $\mathcal{D}, \mathcal{D}', \mathcal{E}'$ in [Ren10].

Lemma 3.1. *This map is injective.*

Proof. Let α be in the kernel. Since the topology of X has a basis consisting of compact open sets, the compact support of α is contained in a compact open set $K \subset X$.For any $f \in C^\infty(X)$, define $f^K : X \to \mathbb{C}$ to be the same as f on $X - K$ and 0 on K. Then $f^K \in C^\infty(X)$ since both $X - K$ and K are open. Moreover $f - f^K \in C_c^\infty(X)$. Since $\alpha|_{C_c^\infty(X)} = 0$, we have $\langle \alpha, f\rangle = \langle \alpha, f^K\rangle$, which is zero since f^K vanishes on K. Hence $\alpha = 0$. □

We shall think of the injection $C_c^{-\infty}(X) \to C^{-\infty}(X)$ as inclusion, think of $C^{-\infty}(X)$ as *the space of all distributions* on X, and $C_c^{-\infty}(X)$ as *the space of all compactly supported distributions* on X.[6] Note that $C_c^{-\infty}(X)$ is also identified with the set of $\alpha \in C^{-\infty}(X)$ for which there exists a compact set K such that $\langle \alpha, f\rangle = 0$ for all $f \in C_c^\infty(X)$ such that $f|_K \equiv 0$. Lect.5, Feb 3

Example 3.2. For any compact open subset $K \subset X$, its characteristic function 1_K is in $C_c^\infty(X)$. It is locally constant because both K and $X - K$ are open.

Exercise 3.3. Every function in $C_c^\infty(X)$ is a finite linear combination of characteristic functions of disjoint compact open subsets of X.

Example 3.4. Fix $x \in X$. We have the Dirac distribution $\delta_x : C^\infty(X) \to \mathbb{C}, f \mapsto f(x)$. We have $\delta_x \in C_c^{-\infty}(X)$, and its support is $\{x\}$.

Example 3.5. Let X be an infinite set equipped with discrete topology. Then X is td. We have an identification

$$C^\infty(X) \xrightarrow{\sim} \prod_{x\in X} \mathbb{C}, \quad f \mapsto (f(x))_x$$

under which

$$C_c^\infty(X) \xrightarrow{\sim} \bigoplus_{x\in X} \mathbb{C},$$

and an identification

$$C^{-\infty}(X) \xrightarrow{\sim} \prod_{x\in X} \mathbb{C}, \quad \alpha \mapsto (\langle \alpha, 1_{\{x\}}\rangle)_x,$$

under which

$$C_c^{-\infty}(X) \xrightarrow{\sim} \bigoplus_{x\in X} \mathbb{C}.$$

For instance, consider the functional $\alpha : C_c^\infty(X) \to \mathbb{C}, f \mapsto \sum_{x\in X} f(x)$. The sum is always finite, hence well defined. Then this α corresponds to $(1, 1, \cdots)$ in the third identification above, and $\alpha \in C^{-\infty}(X) - C_c^{-\infty}(X)$.

We think of the natural pairing $C_c^{-\infty}(X) \times C^\infty(X) \to \mathbb{C}, (\alpha, f) \mapsto \langle \alpha, f\rangle$ as "integrating f against α", and sometimes even write[7]

$$\int_{x\in X} f(x)\alpha(x), \text{ or simply } \int_X f\alpha$$

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[6] We have pairings $C^{-\infty}(X) \otimes C_c^\infty(X) \to \mathbb{C}$ and $C_c^{-\infty}(X) \otimes C^\infty(X) \to \mathbb{C}$. The way to remember is that ∞ and $-\infty$ both appear, and c appears once.

[7] Here the notation $\alpha(x)$ is formal, as the "value" of α at $x \in X$ does not make sense. However, the physicists would think that $\alpha(x)$ is an actual function. For instance, the Dirac distribution δ_0 over $\mathbb{R}$ supported at 0 would be thought of as an actual function on $\mathbb{R}$ such that $\delta_0(x) = 0$ for $x \neq 0$, $\delta_0(0) = +\infty$, and $\int_{\mathbb{R}} \delta_0(x)dx = 1$. Then by formal manipulation one shows that for any $f \in C^\infty(\mathbb{R})$, indeed $\int_{\mathbb{R}} f(x)\delta_0(x)dx = f(0)$. Thus integration against the "function" $\delta_0(x)$ (or rather $\delta_0(x)dx$), plays the role of the Dirac distribution $f \mapsto f(0)$.

for $\langle \alpha, f\rangle$. As a generalization, we would like to integrate vector-valued functions f against $\alpha \in C_c^{-\infty}(G)$. Thus let V be a $\mathbb{C}$-vector space (maybe infinite dimensional), and let $C^\infty(X, V)$ denote the space of locally constant functions $X \to V$. Let $\alpha \in C_c^{-\infty}(X)$ and $f \in C^\infty(X, V)$. As in the above proof, α is supported in a compact open $K \subset X$. Since f is locally constant, there is a partition of K into finitely many open subsets $U_1, \cdots, U_n$ such that f takes constant value $v_i \in V$ on each U_i. Define

$$\int_X f\alpha = \langle \alpha, f\rangle := \sum_i \langle \alpha, 1_{U_i}\rangle v_i \in V.$$

This definition is motivated by the following formal computation:

$$\begin{aligned}\int_{x\in X} f(x)\alpha(x) = \int_{x\in K} f(x)\alpha(x) &= \sum_i \int_{x\in U_i} f(x)\alpha(x)\\ &= \sum_i \left(\int_{x\in U_i}\alpha(x)\right)\cdot v_i = \sum_i\left(\int_{x\in X} 1_{U_i}(x)\alpha(x)\right)\cdot v_i.\end{aligned}$$

The definition of $\int_G f\alpha$ is independent of choices and gives a bilinear pairing $C_c^{-\infty}(X) \otimes C^\infty(X, V) \to V$. When V is finite dimensional and identified with $\mathbb{C}^n$, we can identify $f \in C^\infty(G, V)$ with a tuple $(f_1, \cdots, f_n) \in C^\infty(G)^n$. Then $\int f\alpha = (\int f_1\alpha, \cdots, \int f_n\alpha) \in V$.

Exercise 3.6. Verify these claims.

Example 3.7. Let $x_0 \in X$ and let δ_{x_0} be the Dirac distribution. For any $f \in C^\infty(X, V)$, we have $\langle \delta_{x_0}, f\rangle = \int f(x)\delta_{x_0}(x) = f(x_0)$.

4. Distributions on a td group

Let G be a td group. We have two representations l, r of G on $C^\infty(G)$, called *left translation and right translation*:

$$(l(g)f)(x) = f(g^{-1}x), \qquad (r(g)f)(x) = f(xg).$$

(The g^{-1} in the first formula is to make sure l is a left action.) Since G is a topological group, left or translation of a function indeed preserves the property of being locally constant. Moreover, it preserves the property of compactly supported. Thus we have sub-representations $(C_c^\infty(G), l) \subset (C^\infty(G), l)$, $(C_c^\infty(G), r) \subset (C^\infty(G), r)$. Define left/right translation on $C^{-\infty}(G)$ by

$$\langle l(g)\alpha, f\rangle = \langle \alpha, l(g^{-1})f\rangle, \quad \langle r(g)\alpha, f\rangle = \langle \alpha, r(g^{-1})f\rangle, \qquad \forall \alpha \in C^{-\infty}(G), f \in C_c^\infty(G).$$

(Again the negative powers are introduced to make left actions.) Similarly, define left/right translation on $C_c^{-\infty}(G)$. Then again $C_c^{-\infty}(G)$ is a sub-representation of $C^{-\infty}(G)$ when we consider the left/right translation.

Lemma 4.1. *Let G be a td group. Every element of $C_c^\infty(G)$ is right invariant by a compact open subgroup $K \subset G$, i.e., fixed by $r(K)$. In particular $(C_c^\infty(G), r)$ is a smooth representation. Same for "right" replaced by "left".*

Proof. Let $f \in C_c^\infty(G)$. By Exercise 3.3, we may assume that $f = 1_Y$ for a compact open subset $Y \subset G$. For each $y \in Y$, $y^{-1}Y$ is a neighborhood of 1, and hence contains a compact open subgroup $K_y \subset G$. That is, $yK_y \subset Y$. Since Y is compact and each yK_y is open, there exist finitely many $y_1, \cdots, y_n \in Y$ such that $Y = \bigcup_{i=1}^n y_iK_{y_i}$. Let $K = \bigcap_{i=1}^n K_{y_i}$. Then K is a compact open subgroup, and Y is right invariant by K. Hence $f = 1_Y$ is right invariant by K. ∎

Exercise 4.2. Let K be a compact open subgroup of G. Then $C_c^\infty(G)^{r(K)}$ has basis $\{1_{gK} \mid gK \in G/K\}$. In particular this is identified with the space of finitely supported functions on G/K.

Theorem 4.3. *Let G be a td group. Then $C^{-\infty}(G)^{l(G)}$ is one-dimensional. Moreover, one can choose a basis μ to be* positive *in the sense that $\langle \mu, f\rangle > 0$ for every non-zero $f \in C_c^\infty(G)$ such that $f \geq 0$.*

Remark 4.4. Such a positive μ will be called a *Haar distribution*, or more precisely a *left Haar distribution*; it is unique up to scaling by $\mathbb{R}_{>0}$. It is an incarnation of a *(left) Haar measure*, which is a left G-invariant Borel measure $\mathrm{vol}(\cdot)$ on G, unique up to $\mathbb{R}_{>0}$, for any locally compact Hausdorff topological group G. Namely, our μ is given by integration against a Haar measure. Conversely, given μ, we can recover the measures of compact open sets K by $\mathrm{vol}(K) = \langle \mu, 1_K\rangle$. If we replace "left" by "right", we obtain the notion of a right Haar distribution, which is again unique up to scaling by $\mathbb{R}_{>0}$. When the adjective "left" or "right" is omitted, "left" is always understood.

Lect.6, Feb 6

Proof of Theorem 4.3. Fix a compact open subgroup K_0, and let S be the set of all compact open subgroups of K_0. Then S is a neighborhood basis of 1. By Lemma 4.1, we have

$$C_c^\infty(G) = \bigcup_{K\in S} C_c^\infty(G)^{r(K)}.$$

Hence

$$C^{-\infty}(G) = \varprojlim_{K\in S} (C_c^\infty(G)^{r(K)})^*,$$

and

$$C^{-\infty}(G)^{l(G)} = \varprojlim_{K\in S} \{\text{left } G\text{-inv elts of } (C_c^\infty(G)^{r(K)})^*\}.$$

Denote by V_K the space of left G-invariant elements of $(C_c^\infty(G)^{r(K)})^*$, i.e., $\mu \in (C_c^\infty(G)^{r(K)})^*$ such that

$$\langle \mu, l(g)f\rangle = \langle \mu, f\rangle, \quad \forall f \in C_c^\infty(G)^{r(K)}, g \in G.$$

(Note that $C_c^\infty(G)^{r(K)}$ is stable under $l(g)$, so the above makes sense.) Let us analyze V_K. By Exercise 4.2, $C_c^\infty(G)^{r(K)}$ can be identified with the space of finitely supported functions on G/K. From this one sees that its dual space is naturally identified with $\prod_{G/K} \mathbb{C}$. Since left multiplication by G is transitive on G/K, we see that V_K is one-dimensional, with a canonical basis $\alpha_K = (1, 1, \cdots)$ defined by

$$\langle \alpha_K, f\rangle = \sum_{y\in G/K} f(y).$$

Exercise 4.5. Verify this.

Moreover, if $K, L \in S$ with $K \subset L$, then the transition map $V_K \to V_L$ sends α_K to $[L : K]\alpha_L$. Thus to give a left G-invariant element of $C^{-\infty}(G)$, is the same as giving a number c_K for each $K \in S$, satisfying $c_L = [L : K]c_K$ whenever $K \subset L$. Clearly this is possible by taking $c_K := c/[K_0 : K]$ for a constant c, and unique up to scaling (i.e., the choice of the constant c). If we choose $c > 0$, then the resulting μ will be positive. □

By the proof of Theorem 4.3, we know that

$$\int_G f(g)\mu(g)$$

can be computed as follows: Find a sufficiently small compact open subgroup K such that f is right K-invariant. Then

$$\int_G f(g)\mu(g) = \sum_{y\in G/K} f(y)\,\mathrm{vol}_\mu(K),$$

where $\mathrm{vol}_\mu(K) := \langle \mu, 1_K\rangle$. Moreover, we know that $\mathrm{vol}_\mu(\cdot)$ satisfies that for any compact open subgroups K, L with $K \subset L$, we have

$$\mathrm{vol}_\mu(L) = [L : K]\,\mathrm{vol}_\mu(K).$$

Finally, the left G-invariance of μ clearly follows from the fact that the expression $\sum_{y\in G/K} f(y)$ is invariant under left translation on f.

Now $C^{-\infty}(G)^{l(G)} \subset C^{-\infty}(G)$ is stable under right translation, so it is a one-dimensional sub-representation of $(C^{-\infty}(G), r)$. Therefore $r(G)$ must act on it by a homomorphism $\Delta_G : G \to \mathbb{C}^\times$. That is, we have

$$r(g)\mu = \Delta_G(g)\mu$$

for all $g \in G$ and $\mu \in C^{-\infty}(G)^{l(G)}$.

Definition 4.6. We call Δ_G the *modulus character of* G. If it is trivial, we call G *unimodular.*

Remark 4.7. G is unimodular if and only if a left Haar distribution is the same as a right Haar distribution.

Remark 4.8. Let μ be a Haar distribution. For $h \in G$, the number $\Delta_G(h)$ is characterized by the equation

$$\int_G f(gh^{-1})\mu(g) = \Delta_G(h)\int_G f(g)\mu(g), \qquad \forall f \in C_c^\infty(G).$$

If K is a compact open subgroup, we get $\Delta_G(K) = \{1\}$ by taking $f = 1_K$ and $h \in K$ in the above. Thus, as a function on G, Δ_G is left and right invariant by any compact open subgroup K. We also conclude that any compact td group (i.e. profinite group) is unimodular.

Lect.7, Feb 8

Remark 4.9. We have seen that $C^{-\infty}(G)$ contains the one-dimensional subspace $C^{-\infty}(G)^{l(G)}$. Also recall that $C^{-\infty}(G)$ contains $C_c^{-\infty}(G)$, the space of compactly supported distributions. These two subspaces are in general disjoint (unless G is compact), since the conditions "left G-invariant" and "compactly supported" are not compatible.

5. The Hecke algebra

Let G be a td group.

Definition 5.1. Let $\mathcal{H}(G) = \bigcup_K C_c^{-\infty}(G)^{l(K)} \subset C_c^{-\infty}(G)$, where K runs over compact open subgroups. We call $\mathcal{H}(G)$ the *Hecke algebra of* G. (Right now $\mathcal{H}(G)$ is just a $\mathbb{C}$-vector space, but later we will define a product on it.)

Clearly $\mathcal{H}(G)$ is stable under right translation by G. It is also stable under left translation by G, since the conjugate of a compact open subgroup is still a compact open subgroup.

For any $f_0 \in C_c^\infty(G)$ and $\alpha \in C^{-\infty}(G)$, we write $f_0\alpha$ for the functional $C^\infty(G) \to \mathbb{C}$, $f \mapsto \langle \alpha, ff_0\rangle$. This makes sense since $ff_0 \in C_c^\infty(G)$. Moreover, $f_0\alpha$ is compactly supported since α is, i.e., $f_0\alpha \in C_c^{-\infty}(G)$. Actually, it is easy to see that all elements of $C_c^{-\infty}(G)$ are of this form.

Proposition 5.2. *Let μ be a Haar distribution. Then $\mathcal{H}(G) = \{f_0\mu \mid f_0 \in C_c^\infty(G)\}$.*

Proof. The containment "$\supset$" is clear since each $f_0 \in C_c^\infty(G)$ is left invariant under some K, and μ is left invariant under G. We now prove "$\subset$". Let $\alpha \in \mathcal{H}(G)$, and suppose it is left invariant by a compact open subgroup K. There exist finitely many cosets $Kg_1, \cdots, Kg_n$ whose union contains the support of α. Then

$$\alpha = \sum_i 1_{Kg_i}\alpha.$$

Note that each $1_{Kg_i}\alpha$ is still left K-invariant, and is supported inside Kg_i. Thus up to replacing α by $1_{Kg_i}\alpha$, we may assume that α is left K-invariant and supported in Kg_1 for some $g_1 \in G$. The space $\{f_0\mu \mid f_0 \in C_c^\infty(G)\}$ is stable under right translation by G (since any right translate of μ is a scalar multiple of μ), so we may replace α by a right translate of it. Thus we may assume that α is left K-invariant and supported in K. Since K has unique Haar distribution up to scalar, we know that there exists $c \in \mathbb{C}$ such that

$$\langle \alpha, f\rangle = c\langle \mu, f\rangle$$

for all $f \in C_c^\infty(G)$ supported in K. But this means $\alpha = c1_K\mu$. □

Corollary 5.3. *We have $\mathcal{H}(G) = \bigcup_K C_c^{-\infty}(G)^{r(K)} = \bigcup_K C_c^{-\infty}(G)^{l(K),r(K)}$ where K runs over compact open subgroups.*

Proof. The second equality follows from the first, and for the first we only need to prove the direction "$\subset$" by symmetry. By Proposition 5.2, it suffices to show that for any $f_0 \in C_c^\infty(G)$, the element $\alpha = f_0\mu$ is right invariant by some compact open subgroup. Now f_0 is right invariant by some compact open subgroup K, and μ is also right invariant by K, since $\Delta_G|_K = 1$. □

Corollary 5.4. *Fix a Haar distribution μ. We have an isomorphism $C_c^\infty(G) \xrightarrow{\sim} \mathcal{H}(G), f_0 \mapsto f_0\mu$. This is an isomorphism between the left translation representations.*

Proof. We only need to prove injectivity. Suppose $f_0 \in C_c^\infty(G)$ is such that $f_0\mu = 0$. Let $\bar{f}_0$ be the complex conjugate of f_0. Then

$$0 = \langle f_0\mu, \bar{f}_0\rangle = \langle \mu, f_0\bar{f}_0\rangle,$$

which implies $f_0\bar{f}_0 = 0$ by the positivity of μ and the fact that $f_0\bar{f}_0 \geq 0$. Hence $f_0 = 0$. □

Remark 5.5. By symmetry, if we pick a right Haar measure μ', we also have an isomorphism $C_c^\infty(G) \xrightarrow{\sim} \mathcal{H}(G), f_0 \mapsto f_0\mu'$, between the right translation representations.

6. Smooth representations and convolution

Let G be a td group. A representation (V, π) of G is called *smooth*, if for every $v \in V$, $\mathrm{Stab}_v\, G$ is an open subgroup of G. Equivalently, $V = \bigcup_K V^K$ where K runs through compact open subgroups of G.

Exercise 6.1. Show the equivalence. (Use that a subgroup of G is open if and only if it contains an open subgroup of G.)

Example 6.2. We have already seen four examples of smooth representations: $(C_c^\infty(G), l)$, $(C_c^\infty(G), r)$, $(\mathcal{H}(G), l)$, $(\mathcal{H}(G), r)$. The first is isomorphic to the third by multiplying with a Haar distribution. Similarly, the second is isomorphic to the fourth, by multiplying with a right Haar distribution. Actually the first two are also isomorphic to each other under $f \mapsto \check{f}$, where $\check{f}(g) = f(g^{-1})$. Note that $(C^\infty(G), l \text{ or } r)$ is in general not smooth.

Exercise 6.3. Let $G = \mathrm{SL}_2(\mathbb{Q}_p)$. Show that every finite dimensional smooth representation (V, π) of G is trivial, i.e., $\pi(g) = \mathrm{id}_V, \forall g \in G$, in the following steps. (This exercise also works with slight modification for $\mathbb{Q}_p$ replaced by any non-archimedean local field.)

(1) The subgroup $K := \ker(\pi : G \to \mathrm{GL}(V))$ is open.

(2) Suppose H is a normal subgroup of G containing $\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ and for all $x \in \mathbb{Q}_p$. Then H also contains $\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}$ for all x, and G is generated by H.

(3) Show that $K = G$ by using (2) and noting that
$$\begin{pmatrix} p^k & \\ & p^{-k} \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} p^{-k} & \\ & p^k \end{pmatrix} = \begin{pmatrix} 1 & p^{2k}x \\ & 1 \end{pmatrix}.$$

We now introduce a construction of vital importance. Let (V, π) be a smooth representation. For each $v \in V$, the function $F_v : G \to V, g \mapsto g \cdot v$ is locally constant (since it is right invariant by some compact open subgroup K fixing v). Sending v to F_v defines a linear map $V \to C^\infty(G, V)$.

Definition 6.4. Define a linear map $C_c^{-\infty}(G) \to \mathrm{End}(V), \alpha \mapsto \pi(\alpha)$ by
$$\pi(\alpha) \cdot v := \langle \alpha, F_v \rangle \in V, \quad \forall v \in V,$$
where $\langle , \rangle$ is the pairing $C_c^{-\infty}(G) \otimes C^\infty(G, V) \to V$. In the integral notation, we have
$$\pi(\alpha) \cdot v = \int_G F_v(g)\alpha(g) = \int_G (\pi(g)v)\alpha(g).$$

Lect.8, Feb 10

To do the above "integral", we can find a sufficiently small compact open subgroup K such that F_v is right K-invariant, and find finitely many cosets $g_1K, \cdots, g_nK$ covering $\mathrm{supp}(\alpha)$. Then the integral is by definition equal to
$$\sum_{i=1}^n \langle \alpha, 1_{g_iK} \rangle F_v(g_i) = \sum_{i=1}^n \langle \alpha, 1_{g_iK} \rangle \pi(g_i)v.$$
Note that F_v is right K-invariant precisely when $v \in V^K$. Hence we have the following formula:
$$\pi(\alpha) \cdot v = \sum_{gK \in G/K} \langle \alpha, 1_{gK} \rangle \pi(g)v, \qquad \forall v \in V^K, \tag{6.1}$$
where it is understood that only finitely many summands are non-zero.

Example 6.5. Consider the Dirac distribution $\delta_{g_0} \in C_c^{-\infty}(G)$ at a fixed $g_0 \in G$. Then in the summation (6.1), only the term indexed by $gK = g_0K$ survives, and that term is equal to $\pi(g_0)v$. Hence we have $\pi(\delta_{g_0}) = \pi(g_0)$. In this sense the action of $C_c^\infty(G)$ on V extends the original action of G on V.

Example 6.6. Let K be a compact open subgroup, and μ a Haar distribution. Let
$$\alpha = \mathrm{vol}_\mu(K)^{-1} 1_K \cdot \mu \in \mathcal{H}(G) \subset C_c^{-\infty}(G).$$
Note that the definition of α depends only on K, not on the choice of μ. Let $v \in V$, and let U be a compact open subgroup contained in K such that $v \in V^U$. Then by (6.1) (applied to U) we have
$$\pi(\alpha)v = \sum_{gU \in G/U} \langle \alpha, 1_{gU} \rangle \pi(g)v = \sum_{gU \in K/U} \mathrm{vol}_\mu(K)^{-1} \mathrm{vol}_\mu(U) \pi(g)v = [K : U]^{-1} \sum_{gU \in K/U} \pi(g)v.$$

This is the average of the finite K-orbit of v.

Example 6.7. Let K be a compact open subgroup, μ a Haar distribution, and $g_0 \in G$. Let $\alpha = \mathrm{vol}_\mu(K)^{-1} 1_{g_0 K} \cdot \mu \in \mathcal{H}(G)$. Let $v \in V^K$. Then

$$\pi(\alpha)v = \mathrm{vol}_\mu(K)^{-1} \sum_{gK \in G/K} \langle \mu, 1_{g_0 K} 1_{gK} \rangle \pi(g) v = \pi(g_0)v.$$

Definition 6.8. Take $(V, \pi) = (C_c^\infty(G), l)$ or $(\mathcal{H}(G), l)$. For $\alpha \in C_c^{-\infty}(G)$ and $v \in V$, we denote $l(\alpha) \cdot v$ by $\alpha * v$, and call it the *convolution product*, which is $\mathbb{C}$-bilinear. Note that if we fix a Haar distribution μ and write elements of $\mathcal{H}(G)$ as $f_0 \mu$ with $f_0 \in C_c^\infty(G)$, then $\alpha * (f_0 \mu) = (\alpha * f_0)\mu$.

Concretely, for $\alpha \in C_c^{-\infty}(G)$ and $f \in C_c^\infty(G)$, the value of the function $\alpha * f \in C_c^\infty(G)$ at $h \in G$ is

$$\left(\int_{g \in G} (l(g)f)\alpha(g) \right)(h) = \int_{g \in G} f(g^{-1}h)\alpha(g).$$

(The first integral is valued in $C_c^\infty(G)$, and we evaluate it at h.) In the special case where $\alpha = f_0 \mu \in \mathcal{H}(G)$, we have

$$(\alpha * f)(h) = \int_G f_0(g) f(g^{-1}h) d\mu(g).$$

This integral is also called the convolution of the functions f_0 and f (with respect of μ).

Proposition 6.9. *Let (V, π) be a smooth representation of G. Let $\alpha \in C_c^{-\infty}(G)$ and $\beta \in \mathcal{H}(G)$. Then for every $v \in V$, $\pi(\alpha * \beta)v = \pi(\alpha)\pi(\beta)v$.*

Corollary 6.10. *The convolution product on $\mathcal{H}(G)$ is associative.*

Proof. Apply the proposition to $(V, \pi) = (C_c^\infty(G), l)$. □

Thus we have an associative ring $(\mathcal{H}(G), *)$. (We know $*$ is $\mathbb{C}$-bilinear, so in particular we have the distributive laws.) It is in general non-commutative, and non-unital (i.e., not having a multiplicative identity), as we will soon see later.

Corollary 6.11. *Let (V, π) be a smooth representation. Then the $\mathbb{C}$-linear map $(\mathcal{H}(G), *) \to (\mathrm{End}(V), \circ), \alpha \mapsto \pi(\alpha)$ is a ring homomorphism, in the sense of non-unital rings.*

Proof. Immediate from the proposition. □

Proof of Proposition 6.9. Fix a Haar distribution μ, and write $\beta = f_0 \mu$ with $f_0 \in C_c^\infty(G)$. Let K be a compact open subgroup fixing v. Up to shrinking K, we may assume that f_0 is right K-invariant, and thus reduce, by linearity, to the case $f_0 = 1_{g_0 K}$ for some $g_0 \in G$. Let $U = g_0 K g_0^{-1}$, which is also a compact open subgroup of G. Then f_0 is left U-invariant. Hence by (6.1) we have

$$\alpha * f_0 = \sum_{gU \in G/U} \langle \alpha, 1_{gU} \rangle l(g) f_0 = \sum_{gU \in G/U} \langle \alpha, 1_{gU} \rangle 1_{g g_0 K},$$

and

$$\alpha * \beta = \sum_{gU \in G/U} \langle \alpha, 1_{gU} \rangle 1_{g g_0 K} \cdot \mu.$$

Since $v \in V^K$, by Example 6.7 we have

$$\pi(\alpha * \beta)v = \sum_{gU \in G/U} \langle \alpha, 1_{gU} \rangle \, \mathrm{vol}_\mu(K) \pi(g g_0) v.$$

On the other hand, since $v \in V^K$, by Example 6.7 we have

$$\pi(\beta)v = \mathrm{vol}_\mu(K)\pi(g_0)v,$$

and this is fixed by $U = g_0 K g_0^{-1}$. Hence by (6.1) we have

$$\pi(\alpha)\pi(\beta)v = \mathrm{vol}_\mu(K) \sum_{gU \in G/U} \langle \alpha, 1_{gU} \rangle \pi(g)\pi(g_0)v.$$

The proof is complete since $\pi(gg_0) = \pi(g)\pi(g_0)$. □

Lect.9, Feb 13

For any compact open subgroup $K \subset G$, define $e_K \in \mathcal{H}(G)$ by

$$e_K = 1_K \,\mathrm{vol}_\mu(K)^{-1}\mu,$$

where μ is a Haar distribution. This definition is independent of μ.

Proposition 6.12. *The following statements hold:*

1. *Let (V, π) be a smooth representation. Then $\pi(e_K)V = V^K$. Moreover $v \in V$ is in V^K if and only if $\pi(e_K)v = v$.*
2. $e_K * e_K = e_K$.

Proof. For (1), we have already noted in Example 6.6 that for any $v \in V$, $\pi(e_K)v$ is the average over the finite K-orbit of v. It is then elementary to check the statements in (1). (2) follows from (1) since e_K is fixed by $l(K)$. □

Remark 6.13. By (1), we know that if $v \in \pi(e_K)V$ then $v = \pi(e_K)v = v$. This also formally follows from (2) and the fact that $\pi : \mathcal{H}(G) \to \mathrm{End}(V)$ is a ring homomorphism.

Recall that $\mathcal{H}(G) = \bigcup_K C_c^{-\infty}(G)^{l(K), r(K)}$, so its definition is "unbiased" towards "left" or "right". The following exercise makes precise the symmetry between "left" and "right".

Exercise 6.14. Consider the involution $\alpha \mapsto \alpha'$ on $C^{-\infty}(G)$ given by pull-back along the homeomorphism $G \to G, g \mapsto g^{-1}$. Concretely, $\langle \alpha', f \rangle = \langle \alpha, \check{f} \rangle$, where $\check{f}(g) = f(g^{-1})$, for all $f \in C_c^\infty(G)$.

1. For any $g \in G$ and $\alpha \in C^{-\infty}(G)$, $(l(g)\alpha)' = r(g)\alpha'$.
2. The subspaces $\mathcal{H}(G) \subset C_c^{-\infty}(G) \subset C_c^\infty(G)$ are both stable under the involution $\alpha \mapsto \alpha'$.
3. For each compact open subgroup $K \subset G$, the element $e_K \in \mathcal{H}(G)$ satisfies $e'_K = e_K$. (Hint: beware that in general if μ is a Haar distribution on G then $\mu \neq \mu'$; however use that K is unimodular.)

Lemma 6.15. *For $\alpha, \beta \in \mathcal{H}(G)$, we have $\alpha * \beta = r(\beta')\alpha$.*

Proof. Fix a Haar distribution μ, and write $\alpha = f_1\mu, \beta = f_2\mu$, for $f_1, f_2 \in C_c^\infty(G)$. Find a compact open subgroup K such that f_1 is right K-invariant, and f_2 is left K-invariant. By bilinearity, we may assume that $f_1 = 1_{g_1K}, f_2 = 1_{Kg_2}$ for $g_1, g_2 \in G$. Write v for $\mathrm{vol}_\mu(K)$. Since β is fixed by $l(K)$, we have

$$\alpha * \beta = l(\alpha)\beta = \sum_{gK \in G/K} l(g)\beta \cdot \langle \alpha, 1_{gK} \rangle = v l(g_1)\beta = v 1_{g_1 K g_2}\mu.$$

Since α is fixed by $r(K)$, we have

$$r(\beta')\alpha = \sum_{gK \in G/K} r(g)\alpha \cdot \langle \beta', 1_{gK} \rangle = \sum_{gK \in G/K} r(g)\alpha \cdot \langle \beta, 1_{Kg^{-1}} \rangle = r(g_2^{-1})\alpha \langle \mu, 1_{Kg_2} \rangle.$$

Now

$$r(g_2^{-1})\alpha = 1_{g_1Kg_2} \cdot r(g_2^{-1})\mu = \Delta_G(g_2^{-1})1_{g_1Kg_2}\mu,$$

and

$$\langle \mu, 1_{Kg_2}\rangle = \langle \mu, r(g_2^{-1})1_K\rangle = \langle r(g_2)\mu, 1_K\rangle = \Delta_G(g_2)v.$$

Hence $r(\beta')\alpha = v1_{g_1Kg_2}\mu = \alpha * \beta$. □

Lect.10, Feb 15

Corollary 6.16. *Let $\alpha \in \mathcal{H}(G)$ and K be a compact open subgroup. Then α is left (resp. right) K-invariant if and only if $e_K * \alpha = \alpha$ (resp. $\alpha * e_K = \alpha$), if and only if $\alpha \in e_K * \mathcal{H}(G)$ (resp. $\alpha \in \mathcal{H}(G) * e_K$).*

Proof. The version with "left" follows from Proposition 6.12 applied to $(V, \pi) = (\mathcal{H}(G), l)$. The version with "right" follows from Proposition 6.12 applied to $(V, \pi) = (\mathcal{H}(G), r)$, Exercise 6.14 (3), and Lemma 6.15. □

Corollary 6.17. *For every compact open subgroup $K \subset G$, we have*

$$e_K * \mathcal{H}(G) * e_K = \{\alpha \in \mathcal{H}(G) \mid \alpha = e_K * \alpha * e_K\} = \mathcal{H}(G)^{l(K),r(K)}.$$

Proof. The first equality follows from the fact that $e_K * e_K = e_K$. The second equality follows from Corollary 6.16. □

Corollary 6.18. *We have*

$$\mathcal{H}(G) = \bigcup_K e_K * \mathcal{H}(G) = \bigcup_K \mathcal{H}(G) * e_K = \bigcup_K e_K * \mathcal{H}(G) * e_K,$$

where the unions are over compact open subgroups K.

Exercise 6.19. For a compact open subgroup K, the subset $e_K * \mathcal{H}(G) * e_K \subset \mathcal{H}(G)$ is a subring of $\mathcal{H}(G)$, and it has the multiplicative unity e_K.

Example 6.20. Let G be a finite, discrete group. There is a canonical choice of Haar distribution μ, namely that normalized by $\mathrm{vol}_\mu(\{1\}) = 1$. We have $C^\infty(G) = C_c^\infty(G) \cong \mathcal{H}(G) = C_c^{-\infty}(G) = C^{-\infty}(G)$, and for $f \in C^\infty(G), \langle \mu, f\rangle = \sum_{g\in G} f(g)$. We can identify $\mathcal{H}(G)$ with the group ring $\mathbb{C}[G]$, where $[g] \in \mathbb{C}[G]$ corresponds to the Dirac distribution $\delta_g = 1_{\{g\}}\mu$. Then $*$ on $\mathcal{H}(G)$ corresponds to the usual product on $\mathbb{C}[G]$.

Example 6.21. If G is discrete, the multiplicative unity of $(\mathcal{H}(G), *)$ is e_K, for $K = \{1\}$. This is nothing but the Dirac distribution δ_1 at $1 \in G$.

Proposition 6.22. *Let G be a non-discrete td group. Then $\mathcal{H}(G)$ does not have a multiplicative unity.*

Proof. Suppose e is the multiplicative unity. Let K be a compact open subgroup such that e is left invariant by K. Then $e \in e_K * \mathcal{H}(G)$, and so $e_K * e = e = e_K$. But if K' is any compact open subgroup of K, then the same argument shows $e = e_{K'}$. Hence $K = K'$, which implies that G has a smallest compact open subgroup. This is only possible when G is discrete. □

7. Smooth representations as Hecke modules

Let G be a td group.

Corollary 7.1. *Let (V, π) be a smooth representation of G. We have $V = \bigcup_K \pi(e_K)V$, where K runs over compact open subgroups of G.*

Proof. We have $\pi(e_K)V = V^K$. □

Definition 7.2. A *(left) $\mathcal{H}(G)$-module* is an abelian group V together with a ring homomorphism $\pi : \mathcal{H}(G) \to \mathrm{End}(V)$ (i.e., $\pi(\alpha \pm \beta) = \pi(\alpha) \pm \pi(\beta), \pi(\alpha * \beta) = \pi(\alpha) \circ \pi(\beta)$ for all $\alpha, \beta \in \mathcal{H}(G)$). It is called *non-degenerate*, if $V = \bigcup_K \pi(e_K)V$ where K runs over compact open subgroups of G. Let $\mathcal{M}(\mathcal{H}(G))$ be the category of non-degenerate $\mathcal{H}(G)$-modules, with the obvious definition of morphisms. Given an $\mathcal{H}(G)$-module (V, π), we often write $\alpha \cdot v$ for $\pi(\alpha) \cdot v$ for $\alpha \in \mathcal{H}(G), v \in V$.

Remark 7.3. Since $\mathcal{H}(G) = \bigcup_K e_K * \mathcal{H}(G)$, an $\mathcal{H}(G)$-module V is non-degenerate if and only if $V = \pi(\mathcal{H}(G)) \cdot V$. Thus the notion of non-degenerate modules is intrinsic to the ring $\mathcal{H}(G)$ itself, and does not depend on the relationship between $\mathcal{H}(G)$ and the group G.

Exercise 7.4. Recall that $\mathcal{H}(G)$ has multiplicative unity if and only if G is discrete, and in this case the multiplicative unity is $e = \delta_1$. Suppose this is the case. Then a $\mathcal{H}(G)$-module V is non-degenerate if and only if e acts by 1.

Exercise 7.5. Let V be a non-degenerate $\mathcal{H}(G)$-module. Then there is a unique structure of $\mathbb{C}$-vector space on V such that

$$\lambda(\alpha \cdot v) = (\lambda\alpha) \cdot v, \quad \forall \alpha \in \mathcal{H}(G), v \in V, \lambda \in \mathbb{C}.$$

Moreover, the map $\pi : \mathcal{H}(G) \to \mathrm{End}(V)$ factors through a $\mathbb{C}$-linear map $\mathcal{H}(G) \to \mathrm{End}_{\mathbb{C}}(V)$. Every morphism in $\mathcal{M}(\mathcal{H}(G))$ is automatically $\mathbb{C}$-linear.

Example 7.6. Let $\mathcal{H}(G)$ act on $\mathbb{Z}$ by zero. Then $\mathbb{Z}$ is a degenerate $\mathcal{H}(G)$-module, and there is no $\mathbb{C}$-vector space structure.

Starting with a smooth G-representation (V, π), we have constructed a non-degenerate $\mathcal{H}(G)$-module structure on V, namely

$$\pi(\alpha)v = \int_{g \in G} \pi(g)v\alpha(g), \qquad \forall \alpha \in H(G), v \in V.$$

(It is non-degenerate by Corollary 7.1.) Now if (V, π) is non-degenerate $\mathcal{H}(G)$-module, we can also construct a smooth G-representation on V, as an inverse to the previous construction. To wit, let $v \in V$ and $g \in G$. We define $\pi(g) \cdot v$ as follows. Let K be a compact open subgroup such that $v \in \pi(e_K)V$. Define

$$\pi(g)v := \pi(l(g)e_K)v = \pi(1_{gK} \, \mathrm{vol}_\mu(K)^{-1}\mu) \cdot v.$$

Exercise 7.7. Check that this way we indeed obtain a smooth representation of G on V. Also check that for a fixed $\mathbb{C}$-vector space V, the two constructions described above give inverse bijections between the set of smooth G-representation structures on V and the set of non-degenerate $\mathcal{H}(G)$-module structures on V.

Lect.11, Feb 17

Lemma 7.8. *Let $(V_1, \pi_1), (V_2, \pi_2)$ be smooth G-representations. A linear map $\phi : V_1 \to V_2$ is a morphism of G-representations if and only if it is a morphism of $\mathcal{H}(G)$-modules.*

Proof. The "only if" direction is easy to verify. We check the "if" part. Thus we assume ϕ satisfies $\phi \circ \pi_1(\alpha) = \pi_2(\alpha) \circ \phi$ for all $\alpha \in \mathcal{H}(G)$. Let $v \in V_1$ and $g \in G$ be arbitrary. We need to check

$$\phi(\pi_1(g)v) = \pi_2(g)\phi(v).$$

Let K be a compact open subgroup fixing both v and $\phi(v)$. Let $\alpha = l(g)e_K$. Then

$$\phi(\pi_1(g)v) = \phi(\pi_1(\alpha)v) = \pi_2(\alpha)\phi(v) = \pi_2(g)\phi(v).$$

□

In summary, a smooth G-representation is "the same thing" as a non-degenerate $\mathcal{H}(G)$-module, and a morphism between smooth G-representations is "the same thing" as a morphism between non-degenerate $\mathcal{H}(G)$-modules. Abstractly, this implies we have an equivalence of categories

$$F : \mathcal{M}(G) \to \mathcal{M}(\mathcal{H}(G)),$$

inducing the identity functor on the underlying vector spaces. Note that this is a very special kind of equivalence, as F has a literal inverse: a functor $F^{-1} : \mathcal{M}(\mathcal{H}(G)) \to \mathcal{M}(G)$ such that $F \circ F^{-1}$ and $F^{-1} \circ F$ are the identity functors on the nose (not just isomorphic to the identity functors).

Clearly $\mathcal{M}(G)$ and $\mathcal{M}(\mathcal{H}(G))$ are both abelian categories (that is, the kernel and cokernel of a morphism are naturally smooth G-representations or non-degenerate $\mathcal{H}(G)$-modules), and F is an equivalence of abelian categories.

Example 7.9. In $\mathcal{M}(G)$ we have $(\mathcal{H}(G), l)$. The corresponding object in $\mathcal{M}(\mathcal{H}(G))$ is $\mathcal{H}(G)$ as a left $\mathcal{H}(G)$-module by left multiplication.

8. The categorical center

Let $\mathcal{C}$ be an abelian category. We define its center to be $Z(\mathcal{C}) = \mathrm{End}(\mathrm{id}_{\mathcal{C}})$, the endomorphism ring of the identity functor $\mathcal{C} \to \mathcal{C}$. Concretely, an element $F \in Z(\mathcal{C})$ is a family of endomorphisms $F_M \in \mathrm{End}(M)$ for all $M \in \mathcal{C}$, such that for any morphism $\phi : M_1 \to M_2$ in $\mathcal{C}$ we have $\phi \circ F_{M_1} = F_{M_2} \circ \phi$. To add or multiply two elements $F, G \in Z(\mathcal{C})$, we add or multiply there components $F_M, G_M \in \mathrm{End}(M)$. Here multiplication in $\mathrm{End}(M)$ is composition.

Example 8.1. Let A be a (non-commutative) unital ring, and $\mathcal{C}$ the category of left A-modules such that $1 \in A$ acts as identity. There is natural ring isomorphism between $Z(\mathcal{C})$ and $Z(A)$, the center of A. Given $z \in Z(A)$, we define $F_M(m) = zm$ for all $M \in \mathcal{C}$ and $m \in M$. Then $F_M \in \mathrm{End}(M)$ because left multiplication by z commutes with left multiplication by A. Moreover, the F_M's are compatible with morphisms. Hence we obtain an element $F = (F_M)_M \in Z(\mathcal{C})$. We define $\Phi : Z(A) \to Z(\mathcal{C})$ sending z to F. Conversely, suppose $F = (F_M)_M \in Z(\mathcal{C})$. Consider the left A-module A. Let $z = F_A(1)$. Note that $F_A : A \to A$ commutes with left multiplication by A (since it is an A-module endomorphism), and commutes with right multiplication by A (since right multiplication by any fixed $a \in A$ is an endomorphism of the left A-module A). Hence for all $a \in A$ we have $F_A(a) = F_A(a \cdot 1) = az = F_A(1 \cdot a) = za$, and so $z \in Z(A)$. We define $\Psi : Z(\mathcal{C}) \to Z(A)$ sending F to z. Clearly $\Psi \circ \Phi = \mathrm{id}$. We show $\Phi \circ \Psi = \mathrm{id}$. Let $F \in Z(\mathcal{C})$ and $z = \Psi(F)$. Now for arbitrary $M \in \mathcal{C}$ and $m \in M$, we have a morphism $\phi : A \to M, a \mapsto am$. Then $F_M(m) = F_M(\phi(1)) = \phi(F_A(1)) = \phi(z) = zm$. Hence $F = \Phi(z)$.

Now let G be a td group. Write $\mathcal{H}$ for $\mathcal{H}(G)$, and we simply write $\alpha \cdot \beta$ or $\alpha\beta$ for the convolution product $\alpha * \beta$. For each compact open subgroup K, we write $\mathcal{H}_K$ for $e_K\mathcal{H}e_K$. Recall that this is a subring of $\mathcal{H}(G)$, and it is unital with unity e_K. The centers $Z(\mathcal{H}_K)$

form a projective system as follows: If $K' \subset K$, then $\mathcal{H}_K \subset \mathcal{H}_{K'}$, and we define the transition map $Z(\mathcal{H}_{K'}) \to Z(\mathcal{H}_K), z \mapsto e_K z e_K = z e_K^2 = z e_K$. This is well defined, and is a unital ring homomorphism (since $zw \mapsto zwe_K = ze_K w = ze_K e_K w = ze_K we_K$, and $e_{K'} \mapsto e_{K'} e_K = e_K$). When $K'' \subset K' \subset K$, the transition map $Z(\mathcal{H}_{K''}) \to Z(\mathcal{H}_K)$ is indeed the composition of the transition maps $Z(\mathcal{H}_{K''}) \to Z(\mathcal{H}_{K'})$ and $Z(\mathcal{H}_{K'}) \to Z(\mathcal{H}_K)$ since $e_{K'} e_K = e_K$.

Definition 8.2. The *Bernstein center* of G is the ring $\mathcal{Z}_G = \varprojlim_K Z(\mathcal{H}_K)$. This is a commutative unital ring.

Theorem 8.3. *We have* $Z(\mathcal{M}(G)) = Z(\mathcal{M}(\mathcal{H})) \cong \mathcal{Z}_G$.

Proof. Let $z \in \mathcal{Z}_G$. Write z_K for the component of z in $Z(\mathcal{H}_K)$. We define $F \in Z(\mathcal{M}(\mathcal{H}))$ associated with z as follows. Let $V \in \mathcal{M}(\mathcal{H})$ and $v \in V$. Then $v \in e_K V$ for some compact open subgroup K. Define $F_V(v) = z_K v$.

We check this is independent of the choice of K. For this, it suffices to show $z_K v = z_{K'} v$ for $K' \subset K$ and $v \in e_K V$. We have $z_K = z_{K'} e_K$, so

$$z_K v = z_{K'} e_K v = z_{K'} v,$$

since $v = e_K v$.

We now check that F_V is an $\mathcal{H}$-module endomorphism of V. Let $\alpha \in \mathcal{H}(G)$ and $v \in V$. Find K such that $\alpha \in \mathcal{H}_K$ and $v \in e_K V$. Then $\alpha v \in e_K V$, and so $F_V(\alpha v) = z_K \alpha v = \alpha z_K v = \alpha F_V(v)$.

It is easy to see that the family $F = (F_V)_V$ is compatible with morphisms in $\mathcal{M}(\mathcal{H})$, and we omit the proof. Thus starting with $z \in \mathcal{Z}_G$, we have constructed $F \in Z(\mathcal{M}(\mathcal{H}))$. We denote this map by $\Phi : \mathcal{Z}_G \to Z(\mathcal{M}(\mathcal{H}))$.

Lect.12, Feb 20

Conversely, let $F = (F_V)_V \in Z(\mathcal{M}(\mathcal{H}))$. Take $V = \mathcal{H}$ as a left $\mathcal{H}$-module by left multiplication (which is non-degenerate), and write F_0 for the corresponding F_V. For every compact open subgroup K, let $z_K = F_0(e_K)$. Note that for any $\alpha, \beta, \gamma \in \mathcal{H}(G)$, we have $F_0(\alpha\beta\gamma) = \alpha F_0(\beta)\gamma$, where α comes out since F_0 is an endomorphism of the left $\mathcal{H}$-module $\mathcal{H}$, and β comes out since right multiplication by β is an endomorphism of the left $\mathcal{H}$-module $\mathcal{H}$. Hence

$$z_K = F_0(e_K e_K e_K) = e_K z_K e_K,$$

which implies $z_K \in \mathcal{H}_K$, and moreover for all $\alpha \in \mathcal{H}_K$ we have

$$\alpha z_K = \alpha z_K e_K = F(\alpha e_K e_K) = F(\alpha) = F(e_K e_K \alpha) = e_K z_K \alpha = z_K \alpha,$$

which implies $z_K \in Z(\mathcal{H}_K)$.

If $K' \subset K$, then

$$z_K = F_0(e_K) = F_0(e_{K'} e_K e_{K'}) = e_{K'} z_K e_{K'}.$$

Hence $(z_K)_K \in \mathcal{Z}_G$.

We define $\Psi : Z(\mathcal{M}(\mathcal{H})) \to \mathcal{Z}_G$ sending F to (z_K) as above. It is easy to see that $\Psi \circ \Phi = \mathrm{id}$. We now show $\Phi \circ \Psi = \mathrm{id}$. Suppose $\Psi(F) = z$.

For general $V \in \mathcal{M}(\mathcal{H})$, and for $v \in V$, consider the morphism $\phi : \mathcal{H} \to V, \alpha \mapsto \alpha v$. Assume $v \in e_K V$. Then

$$F_V(v) = F_V(\phi(e_K)) = \phi(F_0(e_K)) = \phi(z_K) = z_K v.$$

Hence $F = \Phi(z)$. □

Remark 8.4. If G is discrete, then $\mathcal{H}$ is unital, and $\mathcal{Z}_G = Z(\mathcal{H})$. In this case Theorem 8.3 is a special case of Example 8.1. In fact, one can generalize Theorem 8.3 to the following purely algebraic setting: Let $\mathcal{H}$ be an associative (non-unital) ring satisfying $\mathcal{H} = \bigcup_{e\in I} e\mathcal{H}e$, where I is the set of idempotents of $\mathcal{H}$ (i.e., elements e satisfying $e^2 = e$). Assume that for any $e, f \in I$, there exists $g \in I$ such that $e, f \in g\mathcal{H}g$. Then one can define a projective system of commutative unital rings $Z(e\mathcal{H}e)$ indexed by $e \in I$ as before, and the inverse limit is identified with the center of the category of non-degenerate left $\mathcal{H}$-modules.

9. Contragredient and admissible

Let G be a td group. Given an arbitrary representation (V, π), define the *smooth part*:

$$V^\infty := \bigcup_{K \text{ c.o.s.}} V^K = \{v \in K \mid \operatorname{Stab}_v G \text{ is open}\}.$$

This is a sub-representation of (V, π), and is itself a smooth representation.

Example 9.1. $(\mathcal{H}(G), l)$ is the smooth part of $(C_c^{-\infty}(G), l)$.

Let (V, π) be a smooth representation. Then the linear dual V^* is naturally a G-representation via

$$\langle g \cdot f, v\rangle = \langle f, g^{-1} \cdot v\rangle, \quad \forall g \in G, f \in V^*, v \in V.$$

Definition 9.2. The *contragredient of* V is $(V^*)^\infty$. We denote it by $(V^\vee, \pi^\vee)$. This is a smooth G-representation.

Let (V, π) be a smooth representation. The natural map $\phi : V \to (V^\vee)^*, v \mapsto (f \mapsto f(v))$ is G-linear, and hence $\phi(V)$ lies in the smooth part of $(V^\vee)^*$, i.e. $(V^\vee)^\vee$. Therefore we have a natural G-linear map $\phi : V \to (V^\vee)^\vee$. If V is finite dimensional, then $V^\vee = V^*$, and ϕ is an isomorphism. In the infinite dimensional case, ϕ need not be an isomorphism.

Definition 9.3. A smooth representation V is called *admissible*, if V^K is finite dimensional for every (compact) open subgroup K.

Remark 9.4. It suffices to check the condition for all sufficiently small K.

Example 9.5. For $(V, \pi) = (C_c^\infty(G), l)$, for every compact open subgroup K, V^K has a basis $\{1_{Kg} \mid Kg \in K\backslash G\}$. Hence this representation is admissible if and only if $K\backslash G$ is finite for every K, if and only if G is compact.

Lemma 9.6. *Let (V, π) be a smooth representation, and K a compact open subgroup. The restriction map $r : (V^*)^K \to (V^K)^*$ is an isomorphism, and its inverse is given by $f \mapsto \tilde{f} = f \circ \pi(e_K)$. In particular, if (V, π) is admissible, then so is its contragredient.*

Proof. We define the inverse map. Recall that we have a linear map

$$\pi(e_K) : V \to \pi(e_K)V = V^K,$$

and $\pi(e_K)|_{V^K} = \mathrm{id}$. Hence for every $f \in (V^K)^*$, we can extend f to $\tilde{f} : V \to \mathbb{C}, v \mapsto f(\pi(e_K)v)$. For any $k \in K$ and $v \in V$,

$$\tilde{f}(\pi(k)v) = f(\pi(e_K)\pi(k)v) = f(\pi(e_K)v) = \tilde{f}(v)$$

(because $\pi(e_K)$ is taking average over K-orbits, and the K-orbit of $\pi(k)v$ is the same as that of v). Hence $\tilde{f} \in (V^*)^K$. We claim that $f \mapsto \tilde{f}$ is inverse to r. We already know $r \circ \tilde{\ } = \mathrm{id}$. To check $\tilde{\ } \circ r = \mathrm{id}$, let $F \in (V^*)^K$ and let $f = r(F)$. Then for every $v \in V$, $F(v) = F(\pi(e_K)v) = f(\pi(e_K)v) = \tilde{f}(v)$, where the first equality is because $\pi(e_K)$ is taking K-averages and F is K-invariant. $\square$

Theorem 9.7. *Let V be a smooth representation. Then V is admissible if and only if the natural map $\phi : V \to (V^\vee)^\vee$ is an isomorphism.*

Lect.13, Feb 22

Proof. For each compact open subgroup K, ϕ induces a map $\phi_K : V^K \to ((V^\vee)^\vee)^K$. By Lemma 9.6 (and the fact that $((\cdot)^*)^K = ((\cdot)^\vee)^K$), we know that $((V^\vee)^\vee)^K$ is naturally identified with $((V^K)^*)^*$. Under this identification, ϕ_K is the natural double dual map $V^K \to ((V^K)^*)^*$ for the vector space V^K. This is an isomorphism if and only if V^K is finite dimensional.[8] Now since V and $(V^\vee)^\vee$ are both smooth, ϕ is an isomorphism if and only if ϕ_K is an isomorphism for all K. □

10. Schur's Lemma and the separation lemma

Let G be a td group. A smooth representation (V, π) is called *irreducible*, if it does not have non-trivial sub-representations. It is called *finite-length*, if there is an integer $n \geq 0$ such that every chain of proper sub-representations $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_k = V$ satisfies $k \leq n$. The minimal such n is called the length of (V, π). Thus the length of a non-zero irreducible representation is 1.

Proposition 10.1. *Let $(V, \pi), (V', \pi')$ be two irreducible representations. Every non-zero element $\phi \in \mathrm{Hom}_G(V, V)$ is an isomorphism.*

Proof. Since $\mathrm{im}(\phi)$ is a non-zero sub-representation of V', $\mathrm{im}(\phi) = V'$. Also $\ker(\phi)$ is a proper sub-representation of V, so $\ker(\phi) = 0$. Hence ϕ is an isomorphism. □

When (V, π) and (V', π') are not isomorphic, the proposition implies that $\mathrm{Hom}_G(V, V') = 0$. When they are isomorphic, we can pick an isomorphism $\phi_0 : V \to V'$. Then we have a bijection $\mathrm{End}_G(V) = \mathrm{Hom}_G(V, V) \xrightarrow{\sim} \mathrm{Hom}_G(V, V'), \phi \mapsto \phi_0 \circ \phi$. The study of $\mathrm{Hom}_G(V, V')$ reduces to $\mathrm{End}_G(V)$. Note that $\mathrm{End}_G(V)$ is a unital $\mathbb{C}$-algebra, with multiplication given by composition.

Definition 10.2. We call a td group G *countable at infinity*, if for any compact open subgroup K, G/K is countable.

It suffices to check the condition for one K, since for compact open subgroups K and K', $K \cap K'$ is also a compact open subgroup, and both $[K : K \cap K']$ and $[K' : K \cap K']$ are finite.

Recall that a topological space is called *second countable*, if it admits a countable basis of opens.

Lemma 10.3. *Let G be a second countable td group. Then G is countable at infinity.*

Proof. Let K be a compact open subgroup. Let $\{U_i\}_{i \in \mathbb{Z}}$ be a basis of opens. Then each $gK \in G/K$ contains a U_i, so by the axiom of choice we get a map $\phi : G/K \to \mathbb{Z}$ such that gK contains $U_{\phi(gK)}$ for every $gK \in G/K$. This map is obviously injective. □

Example 10.4. Let F be a non-archimedean local field. Then every subspace of F^n is second countable, so in particular, every closed subgroup of $\mathrm{GL}_N(F)$ is a second countable td group.

To check this, it suffices to check that F is second countable. But $F \cong F_0^{[F:F_0]}$ where $F_0 = \mathbb{Q}_p$ or $\mathbb{F}_q((t))$, so we reduce to the case $F = F_0$. Since F is a metric space, it suffices to find a dense countable subset $X \subset F$. One then obtains a basis consisting of balls $B(x, r)$ for $x \in X, r \in \mathbb{Q}_{>0}$. We can take $X = \mathbb{Q}$ if $F = \mathbb{Q}_p$, and $X = \mathbb{F}_q(t)$ if $F = \mathbb{F}_q((t))$.

[8] For an infinite dimensional vector space W, the cardinality of W is strictly smaller than that of W^*. (This uses the axiom of choice.)

Theorem 10.5 (Schur's Lemma)**.** *Assume that G is countable at infinity. For any non-zero irreducible smooth representation (V, π), we know that $\dim V$ is at most countable, and we have $\mathrm{End}_G(V) = \mathbb{C}$.*

Proof. Let $v \in V - \{0\}$. Then v is fixed by a compact open subgroup K. The subrepresentation generated by v is non-zero, hence equal to V, and as a vector space it is generated by the G-orbit of v. But the G-orbit of v has cardinality at most that of G/K. Hence V is countably generated as a vector space.

Now suppose $E = \mathrm{End}_G(V)$ is strictly larger than $\mathbb{C}$, and let $\phi \in E - \mathbb{C}$. By Proposition 10.1, every non-zero element of the (unital) ring E is inveritble, i.e., E is a division algebra. We have the $\mathbb{C}$-algebra map $\gamma : \mathbb{C}[t] \to E, t \mapsto \phi$. Let $\ker \gamma = (f(t))$. If $f \neq 0$, then since $\mathbb{C}[t]/(f(t))$ is a division algebra, $\deg f = 1$. (If $\deg f \geq 2$, then $\mathbb{C}[t]/(f(t))$ has zero divisors.) This means $\phi \in \mathbb{C}$, a contradiction. Hence $\ker \gamma = 0$, i.e., γ is injective. Since E is a division algebra, γ induces an injective map $\mathbb{C}(t) \to E$.

Note that for each fixed $v \in V - \{0\}$, the map $\mathrm{ev}_v : E \to V, \rho \mapsto \rho(v)$ is injective. Indeed, if $\rho(v) = 0$, then $\ker \rho \neq 0$, and then since V is irreducible we have $\ker \rho = V$, from which $\rho = 0$. Since V has at most countable basis, the same holds for the $\mathbb{C}$-vector space E. But $\mathbb{C}(t)$ does not have countable basis, as $\{(t-a)^{-1}\}_{a\in\mathbb{C}}$ is an uncountable linearly independent set. A contradiction. □

Lect.14, Feb 24

Let G be a td group. There is a natural multiplicative map $Z(G) \to \mathcal{Z}_G$ sending z to $(F_V)_V$ where $F_V = \pi(z)$ for $(V, \pi) \in \mathcal{M}(G)$.

Proposition 10.6. *Let (V, π) be a smooth representation of a td group G such that $\mathrm{End}_G(V) = \mathbb{C}$. Then the following statements hold.*

1. *There is a unique character (i.e., group homomorphism) $\chi : Z(G) \to \mathbb{C}^\times$ such that $\pi(z)v = \chi(z)v$ for all $z \in Z(G), v \in V$. Moreover, χ is trivial on $Z(G) \cap K$ for some compact open subgroup $K \subset G$.*
2. *There is a unique character (i.e., unital ring homomorphism) $\tilde{\chi} : \mathcal{Z}_G \to \mathbb{C}$ such that $F_V(v) = \tilde{\chi}(F)v$ for all $F = (F_V)_V \in \mathcal{Z}_G, v \in V$. Moreover, $\tilde{\chi}$ factors through $\mathcal{Z}_G \to Z(\mathcal{H}_K)$ for some compact open subgroup K.*
3. *χ and $\tilde{\chi}$ are compatible with respect to $Z(G) \to \mathcal{Z}_G$.*

Proof. Obvious. (For the "moreover" parts in (2) and (3), take $v \in V - \{0\}$ and let K be such that $v \in V^K$. Then test $Z(G)$ or $\mathcal{Z}_G$ on v.) □

We call both χ and $\tilde{\chi}$ the *central character* of (V, π), by abuse of language. These exist when G is countable at infinity and (V, π) is irreducible.

Corollary 10.7. *Let G be an abelian, second countable td group. Then every irreducible smooth representation of G is one-dimensional.*

Proof. By Theorem 10.5 and Proposition 10.6, G must act on any irreducible smooth representation by a character $G \to \mathbb{C}^\times$. □

We now proceed to discuss the separation lemma. We first state a lemma in abstract ring theory.

Lemma 10.8. *Let R be an associative ring, not necessarily unital or commutative. Assume that R is a $\mathbb{C}$-algebra (i.e., R is a $\mathbb{C}$-vector space such that the multiplication map $R \times R \to R$ is $\mathbb{C}$-bilinear), and assume that $\dim_{\mathbb{C}} R$ is at most countable. Let $\phi \in R$ be a non-nilpotent element. Then there exists a simple left R-module M on which ϕ is non-zero.*

Theorem 10.9 (The separation lemma)**.** *Let G be a second countable td group. Then $\dim_{\mathbb{C}} \mathcal{H}(G)$ is at most countable, and for every non-zero $\alpha \in \mathcal{H}(G)$ there exists an irreducible $(V, \pi) \in \mathcal{M}(G)$ such that $\pi(\alpha) \neq 0$. (Thus every pair of distinct elements of $\mathcal{H}(G)$ can be separated by an irreducible smooth representation.)*

Proof. We prove the theorem using Lemma 10.8.

Since G is second countable, $1 \in G$ has a countable neighborhood basis. Since every neighborhood of 1 contains a compact open subgroup, there exists a countable decreasing sequence of compact open subgroups $K_1 \supset K_2 \supset \cdots$ forming a neighborhood basis of 1. Then $\mathcal{H}(G)$ is the increasing union $\bigcup_n \mathcal{H}(G)_{K_n}$. Each $\mathcal{H}(G)_{K_n}$ is countable-dimensional since it has a basis indexed by $K_n \backslash G / K_n$ and G/K_n is already countable by Lemma 10.3. In general, a countable increasing union $\bigcup_n W_n$ of countable-dimensional vector spaces $W_1 \subset W_2 \subset W_3 \subset \cdots$ is countable-dimensional.[9] This implies the first statement.

To show the second statement, it suffices to find $\alpha' \in \mathcal{H}(G)$ such that $\alpha * \alpha'$ is non-nilpotent and then apply Lemma 10.8. Indeed, note that a simple $\mathcal{H}(G)$-module V is necessarily non-degenerate, since $\mathcal{H}(G)V \subset V$ is a non-zero sub-module and must equal V. Thus the simple $\mathcal{H}(G)$-module provided by Lemma 10.8 will be an irreducible smooth G-representation V on which $\alpha\alpha'$ is non-zero. In particular α is non-zero on V.

We now show the existence of α'. Fix a Haar distribution μ and use this to identify $\mathcal{H}(G)$ with $C_c^\infty(G)$. Let α correspond to $f \neq 0$ in $C_c^\infty(G)$. Define $f^* \in C_c^\infty(G)$ by $f^*(g) = \overline{f(g^{-1})}$. Let $f_2 = f * f^*$. Then

$$f_2(1) = (f * f^*)(1) = \int_{g \in G} f(g) f^*(g^{-1}) \mu(g) = \int_{g \in G} |f(g)|^2 \mu(g) > 0,$$

and so $f_2 \neq 0$. Note that for general $F, H \in C_c^\infty(G)$ we have $(F * H)^* = H^* * F^*$. Hence

$$f_2^* = f^{**} * f^* = f * f^* = f_2.$$

Define $f_4 = f_2 * f_2^* = f_2 * f_2$. As before $f_4 \neq 0$, and $f_4^* = f_4$. Then define $f_8 = f_4 * f_4$, etc. Take $\alpha' = f^* = f^* \mu$. Then $(\alpha\alpha')^{2^{n-1}} = f_{2^n} \mu \neq 0$. Hence $\alpha\alpha'$ is non-nilpotent. □

Proof of Lemma 10.8. We first reduce to the case where R is unital. Define $\tilde{R} = R \oplus \mathbb{C}$, and define multiplication on $\tilde{R}$ by $(r, a)(r', a') = (rr' + a'r + ar', aa')$. (Here $a'r$ and ar' are defined by the scalar multiplication on R.) Then $\tilde{R}$ is a unital $\mathbb{C}$-algebra, with unity $(0, 1)$. The inclusion $R \hookrightarrow \tilde{R}$ is a ring homomorphism, so ϕ remains non-nilpotent in $\tilde{R}$. Note that if M is a simple unital $\tilde{R}$-module, then it is also a simple R-module. Indeed, if M is a unital $\tilde{R}$-module, then a subgroup of M is $\tilde{R}$-stable if and only if it is R-stable.

We can thus assume that R is unital, and we need to find a simple unital R-module M such that ϕ is non-zero on M. Lect.15, Feb 27

Since R is unital, we have $\mathbb{C} \subset R$. We claim there exists $a \in \mathbb{C}^\times$ such that $\phi - a$ is not invertible. Suppose not. Then $(\phi - a)^{-1}$ is an uncountable collection of elements, for $a \in \mathbb{C}^\times$. Hence they are linearly dependent, and so there exist $c_1, \cdots, c_n \in \mathbb{C}^\times$ and $a_1, \cdots, a_n \in \mathbb{C}^\times$ such that the a_i's are distinct and

$$\sum_i c_i (\phi - a_i)^{-1} = 0.$$

[9] Such a space can be written as a countable direct sum of conutable-dimensional subspaces, by splitting each $W_n \hookrightarrow W_{n+1}$ as $W_{n+1} = W_n \oplus Y_n$. Then use $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$.

Clearing the denominators we get $P(\phi) = 0$ with $P(X) = \sum_i c_i \prod_{j,j\neq i}(X - a_j) \in \mathbb{C}[X]$. Note

$$P(a_i) = c_i \prod_{j,j\neq i}(a_i - a_j) \neq 0,$$

so $P \neq 0$. Hence the ideal $I \subset \mathbb{C}[X]$ consisting of $f(X)$ such that $f(\phi) = 0$ is non-zero. Let $I = (f)$. Then f has a non-zero root $a \in \mathbb{C}^\times$ since otherwise $f(X) = X^n$ and ϕ is nilpotent. But $\phi - a$ is invertible, so $f(X)/(X - a)$ is also in I, a contradiction.

We have proved the claim. Now let $a \in \mathbb{C}^\times$ be such that $\phi - a$ is not invertible. Let $J \subset R$ be a maximal left ideal containing $\phi - a$ (which exists by Zorn's lemma, just as in the commutative case). Then $M = R/J$ is a simple, unital, left R-module. If ϕ acts as zero on M, then $\phi(1 + J) = 0 + J$, so $\phi \in J$. Since $\phi - a \in J$, we have $a \in J$, a contradiction since a is invertible. Hence ϕ is non-zero on M. □

The following result is the converse of Corollary 10.7.

Corollary 10.10. *Let G be a second countable td group. Suppose every irreducible smooth representation of G is one-dimensional. Then G is abelian.*

Proof. Let $\alpha, \beta \in \mathcal{H}(G)$. Then for every irreducible $(V, \pi) \in \mathcal{M}(G)$, we have $\pi(\alpha)\pi(\beta) = \pi(\beta)\pi(\alpha)$ since $\dim V = 1$. Then by Theorem 10.9, $\alpha\beta = \beta\alpha$ in $\mathcal{H}(G)$. Hence $\mathcal{H}(G)$ is commutative. Now let $g, h \in G$, and let K be a compact open subgroup. Then

$$(l(g)e_K) * (l(h)e_K) = l(g)(e_K * l(h)e_K) = l(g)(l(h)e_K * e_K) = l(gh)e_K.$$

Similarly this is equal to $l(hg)e_K$. Thus $ghK = hgK$. Since K is arbitrary and G is Hausdorff, we have $gh = hg$. □

Exercise 10.11. In the above proof we have shown that any td group whose Hecke algebra is commutative is necessarily abelian. The converse is also true. (Check that a Haar distribution on an abelian td group is invariant under $g \mapsto g^{-1}$.)

11. Fixed vectors under a compact open subgroup

Let G be a td group. Write $\mathcal{H}$ for $\mathcal{H}(G)$, and write $\mathcal{H}_K$ for $e_K\mathcal{H}e_K$, for every compact open subgroup $K \subset G$. In the following, by *$\mathcal{H}_K$-modules* we mean left unital $\mathcal{H}_K$-modules. Let $\mathcal{M}(\mathcal{H}_K)$ be the category of them. We have a functor $(V, \pi) \mapsto V^K$ from $\mathcal{M}(G)$ to $\mathcal{M}(\mathcal{H}_K)$. Indeed, the $\mathcal{H}$-action on V restricts to an $\mathcal{H}_K$-action on V^K since $V^K = \pi(e_K)V$.

Theorem 11.1. *A smooth representation $(V, \pi) \in \mathcal{M}(G)$ is irreducible if and only if for all compact open subgroups K, V^K is a simple $\mathcal{H}_K$-module. (We count 0 as a simple $\mathcal{H}_K$-module.)*

Proof. "Only if". If $V^K = 0$, there is nothing to prove. Suppose $0 \neq v \in V^K$. Since V is irreducible, it is a simple $\mathcal{H}$-module. Now $\mathcal{H}v$ is an $\mathcal{H}$-submodule of V, and it is moreover non-degenerate since $\mathcal{H} = \mathcal{H}\mathcal{H}$. Therefore $\mathcal{H}v$ is a non-zero (as it contains $e_Kv = v$) sub-representation of V, and hence it is equal to V. Then $V^K = e_KV = e_K\mathcal{H}v = e_K\mathcal{H}e_Kv = \mathcal{H}_Kv$. Hence the $\mathcal{H}_K$-module V^K is generated by v. Since $v \in V^K$ is arbitrary, V^K is simple.

"If". Suppose $W \subset V$ is a non-zero sub-representation. Then for each K, $W^K \subset V^K$ is an $\mathcal{H}_K$-submodule, and for sufficiently small K we have $W^K \neq 0$. For such K, we must then have $W^K = V^K$ by the simplicity of V^K. Taking the union over K we get $W = V$. □

Our next goal is the proof of the following theorem. From now on, we fix a compact open subgroup $K \subset G$.

Theorem 11.2. *The construction $V \mapsto V^K$ induces a bijection from the set of isomorphism classes of irreducible $V \in \mathcal{M}(G)$ with $V^K \neq 0$ (such representations are said to be K-*spherical*), to the set of isomorphism classes of non-zero simple $\mathcal{H}_K$-modules.*

We need several lemmas in preparation.

Lemma 11.3. *The functor $\mathcal{M}(G) \to \mathcal{M}(\mathcal{H}_K), V \mapsto V^K$ is exact, i.e., it takes exact sequences to exact sequences.*

Proof. Taking invariants under a group is always left exact. Hence we only need to prove that if $V \to W$ is a surjection of smooth G-representations, then the induced map $V^K \to W^K$ is surjective. Let $w \in W^K$, and let $v \in V$ be an arbitrary lift. Then $e_K v \in V^K$ is a lift of $w = e_K w$. □

Lect.16, Mar 1

Lemma 11.4. *For any $V \in \mathcal{M}(G)$, the set of sub-representations $W \subset V$ such that $W^K = 0$ has a unique maximal element.*

Proof. If V_1, V_2 are sub-representations of V such that $V_1^K = V_2^K = 0$, then applying the exact functor $(\cdot)^K$ to the exact sequences $0 \to V_1 \to V_1 + V_2 \to V_2/(V_1 \cap V_2) \to 0$ and $0 \to V_1 \cap V_2 \to V_2 \to V_2/(V_1 \cap V_2) \to 0$ we get $(V_1 + V_2)^K = 0$. Take the sum of all sub-representations W such that $W^K = 0$. □

Definition 11.5. For $V \in \mathcal{M}(G)$, let V^0 be the maximal sub-representation such that $(V^0)^K = 0$. Let $V_0 = V/V^0$.

Remark 11.6. Applying the exact functor $(\cdot)^K$ to the exact sequence $0 \to V^0 \to V \to V_0 \to 0$, we see that the natural map $V \to V_0$ induces $V^K \xrightarrow{\sim} V_0^K$.

Lemma 11.7. *For any $V \in \mathcal{M}(G)$, we have $(V_0)^0 = 0$.*

Proof. Let $W = (V_0)^0$, and let $\tilde{W}$ be the preimage of W in V. Then $W^K = 0$. By the exactness of $(\cdot)^K$ and the short exact sequence $0 \to V^0 \to \tilde{W} \to W \to 0$, we have $\tilde{W}^K = (V^0)^K = 0$. Hence $\tilde{W} \subset V^0$, and so $W = 0$. □

Proof of Theorem 11.2. We have already seen that this map is well defined. We now give the inverse construction. For any $W \in \mathcal{M}(\mathcal{H}_K)$, consider $\mathcal{H} \otimes_{\mathcal{H}_K} W$, which is a non-degenerate $\mathcal{H}$-module and hence an object of $\mathcal{M}(G) \cong \mathcal{M}(\mathcal{H})$. Define $\mathcal{F}(W) = (\mathcal{H} \otimes_{\mathcal{H}_K} W)_0 \in \mathcal{M}(G)$. We will check that $\mathcal{F}$ induces the inverse of the map in the theorem.

Step 1. For any $W \in \mathcal{M}(\mathcal{H}_K)$, we have $\mathcal{F}(W)^K \cong W$ as $\mathcal{H}_K$-modules. We have $\mathcal{F}(W)^K \cong (\mathcal{H} \otimes_{\mathcal{H}_K} W)^K = e_K \mathcal{H} \otimes_{\mathcal{H}_K} W$ and we have a $\mathcal{H}_K$-module map

$$W \to e_K \mathcal{H} \otimes_{\mathcal{H}_K} W, w \mapsto e_K \otimes w.$$

This map is surjective since every pure tensor $e_K \alpha \otimes w \in e_K \mathcal{H} \otimes_{\mathcal{H}_K} W$ is equal to

$$e_K \alpha \otimes e_K w = e_K \alpha e_K \otimes w = e_K e_K \alpha e_K \otimes w = e_K \otimes (e_K \alpha e_K) w.$$

It is injective since its composition with $e_K \mathcal{H} \otimes_{\mathcal{H}_K} W \to W, e_K \alpha \otimes w \mapsto (e_K \alpha e_K) w$ (which is well defined!) is the identity on W. Thus $\mathcal{F}(W)^K \cong W$ as $\mathcal{H}_K$-modules.

Step 2. For any non-zero simple $\mathcal{H}_K$-module W, $\mathcal{F}(W)$ is irreducible. We first show a weaker statement. We call a G-representation *good* if it is generated as a representation by its K-fixed vectors. We shall show that $\mathcal{F}(W)$ is good. We observe that $\mathcal{H} \otimes_{\mathcal{H}_K} W$ is good. Indeed, for any $\alpha \otimes w \in \mathcal{H} \otimes_{\mathcal{H}_K} W$, it is equal to $\alpha \otimes e_K w = \alpha e_K \otimes w$, and hence lies in the $\mathcal{H}$-submodule generated by $e_K \otimes w \in e_K \mathcal{H} \otimes_{\mathcal{H}_K} W = (\mathcal{H} \otimes_{\mathcal{H}_K} W)^K$. Now $(\cdot)^K$ is exact, from which it is easy to see that the quotient of a good representation is good. Hence $\mathcal{F}(W)$ is good.

We now show that $\mathcal{F}(W)$ is irreducible. Suppose it has a sub-representation U. If $U^K \neq 0$, then $U^K \subset \mathcal{F}(W)^K \cong W$ and so $U^K = \mathcal{F}(W)^K$ since W is a simple $\mathcal{H}_K$-module. Then $U = \mathcal{F}(W)$ since $\mathcal{F}(W)$ is good. Thus assume $U^K = 0$. By Lemma 11.7 we have $\mathcal{F}(W)^0 = 0$. Since U is contained in $\mathcal{F}(W)^0$, it is zero.

Step 3. If $V \in \mathcal{M}(G)$ is irreducible with $V^K \neq 0$, then $V \cong \mathcal{F}(V^K)$. We already know that V^K is a simple $\mathcal{H}_K$-module, and in the above we have already shown that $\mathcal{F}(V^K)$ is irreducible. Hence it suffices to construct one non-zero G-map between V and $\mathcal{F}(V^K)$. Define $\phi : \mathcal{H} \otimes_{\mathcal{H}_K} V^K \to V$ by $\alpha \otimes v \mapsto \alpha v$. Clearly ϕ is well defined and is a $\mathcal{H}$-module map. If we take $v \neq 0$ in V^K, then $\phi(e_K \otimes v) = v$, so $\phi \neq 0$. It remains to show that ϕ factors through $\mathcal{F}(V^K)$, i.e., it kills $U = (\mathcal{H} \otimes_{\mathcal{H}_K} V^K)^0$. Clearly $\phi(U) \subset V^0 \subset V$. Since V is irreducible and $V^K \neq 0$, we have $V^0 = 0$. Hence $\phi(U) = 0$ as desired. ☐

Lect.17, Mar 3

Exercise 11.8. Check the details of the proof of injectivity in Step 1.

Theorem 11.9 (Schur's lemma at finite level)**.** *Assume that $\mathcal{H}_K$ is countable-dimensional (which is true if G is countable at infinity). Then for every non-zero simple $\mathcal{H}_K$-module W, we have $\dim_{\mathbb{C}} W$ is at most countable, and $\mathrm{End}_{\mathcal{H}_K} W = \mathbb{C}$.*

Proof. The first statement follows from the fact that W is a quotient vector space of $\mathcal{H}_K$. The second statement is proved in exactly the same way as Theorem 10.5. ☐

Corollary 11.10. *Assume that $\mathcal{H}_K$ is countable-dimensional, and that it is commutative. Then the irreducible $V \in \mathcal{M}(G)$ with $V^K \neq 0$ are classified by characters (i.e. unital ring homomorphisms) $\chi : \mathcal{H}_K \to \mathbb{C}$, i.e., the $\mathbb{C}$-points of $\mathrm{Spec}\, \mathcal{H}_K$.*

Proof. In this case, all non-zero simple $\mathcal{H}_K$-modules are one-dimensional by Theorem 11.9, and their isomorphism classes are in bijection with characters $\mathcal{H}_K \to \mathbb{C}$. ☐

Example 11.11. We have seen that G is abelian if and only if $\mathcal{H} = \mathcal{H}(G)$ is commutative. However, $\mathcal{H}_K$ can be commutative without G being abelian. For example, let $G = \mathrm{GL}_n(F)$ and $K = \mathrm{GL}_n(\mathcal{O}_F)$, where F is a non-archimedean local field. Then $\mathcal{H}_K$ is commutative. In fact, it is isomorphic to $\mathbb{C}[t_1^{\pm}, \cdots, t_n^{\pm}]^{S_n}$ by the Satake isomorphism. Characters on $\mathcal{H}_K$ are thus classified by unordered n-tuples of elements of $\mathbb{C}^\times$. For a given irreducible $(V, \pi) \in \mathcal{M}(G)$ with $V^K \neq 0$, the corresponding n-tuple in $\mathbb{C}^\times$ is called *the Satake parameter* of (V, π).

Note that there exist plenty of irreducible $(V, \pi) \in \mathcal{M}(G)$ with $V^K = 0$ for $K = \mathrm{GL}_n(\mathcal{O}_F)$. For instance, one can always twist an irreducible (V, π) with $V^K \neq 0$ by the one-dimensional representation $G \xrightarrow{\det} F^\times \xrightarrow{\mu} \mathbb{C}^\times$, where μ is a ramified character (i.e., $\mu|_{\mathcal{O}_F^\times}$ is not constantly 1), and obtain an irreducible representation whose K-fixed part is 0.

12. Compact representations

Let G be a td group. We assume that it is unimodular.

Definition 12.1. A smooth representation (V, π) is called *compact*, if for every compact open subgroup $K \subset G$ and $v \in V$, the map $f_{K,v} : G \to V, g \mapsto \pi(e_K)\pi(g^{-1})v$ is compactly supported.

Let (V,π) be a smooth representation. Let $v \in V$ and $\lambda \in V^\vee$, i.e., λ is a linear functional on V invariant under some compact open subgroup. Associated to (v,λ) we have the *matrix coefficient* $\phi_{v,\lambda} : G \to \mathbb{C}, g \mapsto \langle \lambda, \pi(g^{-1})v\rangle$. Clearly $\phi_{v,\lambda}$ is bi-invariant under some compact open subgroup K, namely any K fixing both λ and v. In particular, $\phi_{v,\lambda} \in C^\infty(G)$.

Theorem 12.2. *Let $(V,\pi) \in \mathcal{M}(G)$. Among the following conditions, (1) is equivalent to (2), and they both imply (3).*

1. *(V,π) is compact.*
2. *For all $v \in V, \lambda \in V^\vee$, the function $\phi_{v,\lambda}$ is compactly supported.*
3. *For all compact open subgroup $K \subset G$ and $v \in V$, $\mathrm{im}(f_{K,v})$ spans a finite-dimensional subspace of V.*

Proof. (1) $\Rightarrow$ (2). Let K be a compact open subgroup fixing λ. Then

$$\phi_{v,\lambda}(g) = \langle \lambda, \pi(g^{-1})v\rangle = \langle \lambda, \pi(e_K)\pi(g^{-1})v\rangle = \langle \lambda, f_{K,v}(g)\rangle.$$

Hence the support of $\phi_{v,\lambda}$ is contained in that of $f_{K,v}$, which is compact.

(2) $\Rightarrow$ (3). If not, then there are infinitely many $g_1, g_2, \cdots \in G$ such that $f_{K,v}(g_i) = \pi(e_K)\pi(g_i^{-1})v$ are linearly independent. Note that these vectors are all in V^K. Hence we can extend them to a basis of V^K (by Zorn's lemma), and then define a functional $\lambda' : V^K \to \mathbb{C}$ taking all $f_{K,v}(g_i)$ to 1. Define λ as the composition $V \xrightarrow{\pi(e_K)} V^K \xrightarrow{\lambda'} \mathbb{C}$. Then λ is in $V^\vee$ since it is K-invariant. Moreover, λ takes every $f_{K,v}(g_i)$ to 1. Since λ is K-invariant, we have

$$\phi_{v,\lambda}(g_i) = \langle \lambda, f_{K,v}(g_i)\rangle = 1.$$

By assumption $\phi_{v,\lambda}$ is compactly supported, and by construction it is right K-invariant. Hence $g_1K, g_2K, \cdots$, are only finitely many right K-cosets. But then $f_{K,v}(g_i)$ are only finitely many distinct vectors, a contradiction.

(2) + (3) $\Rightarrow$ (1). For arbitrary K, v, we need to show $f_{K,v}$ is compactly supported. Let E be the span of $\mathrm{im}(f_{K,v})$, which is finite-dimensional by assumption. Since $E \subset V^K$, we can find finitely many linear functionals $\lambda_i' : V^K \to \mathbb{C}$ whose restrictions to E form a basis of E^*. Extend each λ_i' to $\lambda_i : V \to \mathbb{C}$ as before via $\pi(e_K) : V \to V^K$. Then λ_i are K-invariant and in particular in $V^\vee$, and we have $\mathrm{supp}(f_{K,v}) \subset \bigcup_i \mathrm{supp}(\phi_{v,\lambda_i})$. Indeed, if $f_{K,v}(g) \neq 0$, then there exists $1 \le i \le n$ such that

$$0 \neq \langle \lambda_i', f_{K,v}(g)\rangle = \langle \lambda_i, f_{K,v}(g)\rangle = \phi_{v,\lambda_i}(g).$$

□

Lect.18, Mar 6

We say a representation (V,π) is finitely generated, if there is a finite subset S of V such that no proper sub-representation of V contains S. Equivalently, V is spanned by $\bigcup_{v\in S} \pi(G)v$. If (V,π) is irreducible, then it is finitely generated and in fact generated by a single non-zero vector.

Corollary 12.3. *Let (V,π) be a compact, finitely generated representation. Then it is admissible.*

Proof. Let K be a compact open subgroup. We need to check that V^K is finite-dimensional. Let $v_1, \cdots, v_n \in V$ generate V as a G-representation, i.e., V is spanned by $\bigcup_i \pi(G)v_i$. Let E_i be the span of $\mathrm{im}(f_{K,v_i})$. Then $\sum_i E_i$ contains $\pi(e_K) \cdot \sum_i \pi(G) \cdot v_i = \pi(e_K)V = V^K$. By Theorem 12.2 E_i is finite-dimensional. Hence V^K is finite-dimensional. □

We can understand the matrix coefficient construction more conceptually as follows. Let $(V,\pi)\in\mathcal{M}(G)$. Then the matrix coefficient construction $(v,\lambda)\mapsto\phi_{v,\lambda}\in C^\infty(G)$ is bilinear. Thus it gives rise to a linear map $\mathrm{MC}=\mathrm{MC}_{V,\pi}:V\otimes V^\vee\to C^\infty(G)$. Now for two td groups G_1,G_2, and $V_1\in\mathcal{M}(G_1),V_2\in\mathcal{M}(G_2)$, $V_1\otimes V_2$ is naturally a smooth $G\times G$-representation, by $(g_1,g_2)\cdot v_1\otimes v_2=(g_1v_1)\otimes(g_2v_2)$. Thus $V\otimes V^\vee\in\mathcal{M}(G\times G)$. It can be easily checked that $\mathrm{MC}:V\otimes V^\vee\to C^\infty(G)$ is $G\times G$-equivariant, where the action on the target is

$$(g_1,g_2)\cdot f=l(g_1)r(g_2)f:g\mapsto f(g_1^{-1}gg_2).$$

Note that we have a natural $G\times G$-equivariant map $\Phi:V\otimes V^\vee\to\mathrm{End}_{\mathbb{C}}(V),v\otimes\lambda\mapsto(x\mapsto\lambda(x)v)$, where the action on the target is $(g_1,g_2)\phi=\pi(g_1)\circ\phi\circ\pi(g_2^{-1})$. Since $V\otimes V^\vee$ is a smooth $G\times G$-representation, Φ lands in the smooth part $\mathrm{End}_{\mathbb{C}}(V)^\infty$ of $\mathrm{End}_{\mathbb{C}}(V)$ (with respect to the $G\times G$-representation on $\mathrm{End}_{\mathbb{C}}(V)$).

Lemma 12.4. *Let $(V,\pi)\in\mathcal{M}(G)$ be admissible. Then $\Phi:V\otimes V^\vee\to\mathrm{End}_{\mathbb{C}}(V)^\infty$ is an isomorphism.*

Proof. Since the source and target are both smooth $G\times G$-representations, it suffices to check that for any compact open subgroup $K\subset G$, Φ induces an isomorphism $(V\otimes V^\vee)^{K\times K}\xrightarrow{\sim}\mathrm{End}_{\mathbb{C}}(V)^{K\times K}$. Injectivity is easy to see. We prove surjectivity by counting the dimensions. Using the projection $\pi(e_K):V\to V^K$, it is easy to check that the restriction induces an injective map $\mathrm{End}_{\mathbb{C}}(V)^{K\times K}\to\mathrm{End}_{\mathbb{C}}(V^K)$. Thus $\dim(\mathrm{End}_{\mathbb{C}}(V)^{K\times K})\le\dim(V^K)^2$ (which is finite). But $V^K\otimes(V^K)^*$ maps injectively into $(V\otimes V^\vee)^{K\times K}$ (recall that $(V^K)^*\cong(V^\vee)^K$). Hence $\dim(V^K)^2\le\dim(V\otimes V^\vee)^{K\times K}$. □

The upshot is, for a smooth admissible (V,π), we can think of the matrix coefficient construction as a $G\times G$-equivariant map

$$\mathrm{MC}_{V,\pi}:\mathrm{End}_{\mathbb{C}}(V)^\infty\cong V\otimes V^\vee\longrightarrow C^\infty(G).$$

Now let $(V,\pi)\in\mathcal{M}(G)$ be compact and finitely generated. Then it is admissible, and $\mathrm{MC}_{V,\pi}$ lands in $C_c^\infty(G)\subset C^\infty(G)$. From now on we assume that G is unimodular and fix a Haar distribution μ. Using μ we identify $C_c^\infty(G)$ with $\mathcal{H}(G)$. Since μ is bi-G-invariant, the $G\times G$-action on $C_c^\infty(G)$ considered before corresponds to the natural $G\times G$-action on $\mathcal{H}(G)$ by left and right translation. In this case, we have obtained a map of smooth $G\times G$-representations

$$\mathrm{MC}_{V,\pi}:\mathrm{End}_{\mathbb{C}}(V)^\infty\longrightarrow\mathcal{H}(G).$$

Note that for any $(W,\tau)\in\mathcal{M}(G)$, the map $\tau:\mathcal{H}(G)\to\mathrm{End}_{\mathbb{C}}(W),\alpha\mapsto\tau(\alpha)$ is $G\times G$-equivariant, and in particular lands in $\mathrm{End}_{\mathbb{C}}(W)^\infty$. (For this we do not need the assumption that G is unimodular, as long as we consider the natural $G\times G$-action on $\mathcal{H}(G)$.)

Theorem 12.5. *Assume G is unimodular and second countable, and let $(V,\pi)\in\mathcal{M}(G)$ be compact and irreducible (in particular finitely generated and admissible). Let $(W,\tau)\in\mathcal{M}(G)$ be irreducible. Then the composition*

$$\mathrm{End}_{\mathbb{C}}(V)^\infty\xrightarrow{\mathrm{MC}_{V,\pi}}\mathcal{H}(G)\xrightarrow{\tau}\mathrm{End}_{\mathbb{C}}(W)^\infty$$

is zero if (V,π) is not isomorphic to (W,τ), and is a non-zero scalar $c(V,\pi)$ depending proportionally on μ if $(W,\tau)=(V,\pi)$.

Proof. First assume V is not isomorphic to W. It suffices to prove that for each fixed $\lambda\in V^\vee$ and $w\in W$, the composition

$$V\xrightarrow{v\mapsto\phi_{v,\lambda}}\mathcal{H}(G)\xrightarrow{\alpha\mapsto\tau(\alpha)w}W$$

is zero. One checks that this map is G-equivariant, and so it must be zero since V and W are non-isomorphic irreducible G-representations.

Now assume $(W, \tau) = (V, \pi)$. By the exercise below, $\mathrm{End}(V)^\infty \cong V \otimes V^\vee$ is irreducible as $G \times G$-representation. By Schur's Lemma, the map in question must be a scalar. To show it is non-zero, first let α be a non-zero element of the image of $\mathrm{MC}_{V,\pi}$. (For instance we can pick $v \in V, \lambda \in V^\vee$ such that $\langle \lambda, v \rangle \neq 0$. Then $\phi_{v,\lambda}(1) \neq 0$.) If $\pi(\alpha) = 0$, then there must be an irreducible representation τ, not isomorphic to π, such that $\tau(\alpha) \neq 0$ by the Separation Lemma. This is a contradiction to what we have already shown. □

Exercise 12.6. Let G_1, G_2 be td groups, and V_i an irreducible smooth representation of G_i. Then $V_1 \otimes V_2$ is an irreducible smooth representation of $G_1 \times G_2$. If V_1 is irreducible and admissible, then $V_1^\vee$ is irreducible (and admissible) as a smooth G_1-representation. (Use $V_1 \cong V_1^{\vee\vee}$.)

Lect.19, Mar 8

Definition 12.7. The inverse of $c(V, \pi)$ is called the *formal degree* of (V, π), denoted by $d(V, \pi)$.[10]

Example 12.8. Let G be a second countable, compact td group. Then G is unimodular, and every smooth representation is compact. It is easy to see that every irreducible smooth representation is finite-dimensional. Its formal degree is its dimension, if we normalize the Haar distribution such that G has volume 1. Indeed, to verify the last statement we easily reduce to the case where G is finite, by modding out the kernel of $G \to \mathrm{GL}(V)$. Now assume that G is finite. Take $T = \mathrm{id} \in \mathrm{End}(V)$. Then $\mathrm{MC}(T)$ sends g to the trace of $\pi(g^{-1})$, i.e., it is the complex conjugate of the character χ of (V, π). We have

$$\pi(\mathrm{MC}(T)) = |G|^{-1} \sum_{g \in G} \bar{\chi}(g)\pi(g).$$

If we take its trace, we get $|G|^{-1} \sum_g \bar{\chi}(g)\chi(g)$, which is 1 by the Schur orthogonality relations satisfied by irreducible characters. Hence the original $\pi(\mathrm{MC}(T)) = c(V)\,\mathrm{id}_V$ must be $(\dim V)^{-1}\,\mathrm{id}_V$.

Proposition 12.9. *Let G be a td group for which Schur's lemma holds (e.g., if G is countable at infinity). If a compact irreducible representation of G exists, then the center $Z(G)$ of G is compact.*

Proof. Let (V, π) be such a representation. By Schur's lemma, $Z(G)$ acts on V by a character $\chi : Z(G) \to \mathbb{C}^\times$. For any $v \in V$ and $\lambda \in V^\vee$, we have

$$\phi_{v,\lambda}(g) = \langle \lambda, \pi(g^{-1})v \rangle = \chi(g)^{-1} \langle \lambda, v \rangle$$

for all $g \in Z(G)$. We can choose v, λ such that $\langle \lambda, v \rangle \neq 0$. Then $\phi_{v,\lambda}$ is non-vanishing on $Z(G)$. But $\phi_{v,\lambda}$ is compactly supported, so $Z(G)$ is contained in a compact subset of G. One easily checks that in general $Z(G)$ is a closed subgroup of G. Hence in our case it is compact. □

Remark 12.10. When we study representation theory of p-adic reductive groups, compact representations usually do not exist, since $Z(G)$ is usually non-compact. (For instance, for F a non-archimedean local field, the center of $\mathrm{GL}_n(F)$ is $F^\times$, which is non-compact.) However, there is a closely related class of representations, called supercuspidal representations. They will be the building blocks of all representations in a suitable sense.

[10] The inversion is overlooked in [Ren10, §IV.1].

13. Decomposition with respect to a fixed compact representation

From now on, let G be a second countable, unimodular td group. Fix a compact representation (W,τ) of G. (Assume it exists.) Our next goal is the following: For any $(V,\pi)\in\mathcal{M}(G)$, we would like to have a decomposition $V=V_1\oplus V_2$ into sub-representations, such that V_1 is a direct sum of copies of (W,τ), and V_2 is such that every irreducible subquotient of it is non-isomorphic to (W,τ). Here a *subquotient* of a representation U means a representation of the form U_1/U_2, where $0\subset U_2\subset U_1\subset U$ are sub-representations.

Theorem 13.1. *For any compact open subgroup $K\subset G$, there is a unique element $e_{K,\tau}\in\mathcal{H}(G)$ with the following property: For any irreducible $(V,\pi)\in\mathcal{M}(G)$, $\pi(e_{K,\tau})=\pi(e_K)$ if $\pi\cong\tau$, and $\pi(e_{K,\tau})=0$ otherwise.*

Proof. The uniqueness follows from the Separation Lemma. For the existence, we can take $e_{K,\tau}=c(W,\tau)^{-1}\mathrm{MC}_{W,\tau}(\tau(e_K))$. It satisfies the desired property since for any irreducible π, we have $\pi\circ\mathrm{MC}_{W,\tau}=c(W,\tau)$ when $\pi\cong\tau$ and $=0$ otherwise, by Theorem 12.5. □

Proposition 13.2. *For two compact open subgroups K,K' of G with $K'\subset K$, we have*

$$e_{K',\tau}*e_{K,\tau}=e_{K,\tau}*e_{K',\tau}=e_{K',\tau}*e_K=e_K*e_{K',\tau}=e_{K,\tau}.$$

In particular, $e_{K,\tau}$ is idempotent, and it lies in $\mathcal{H}(G)_K=e_K\mathcal{H}(G)e_K$.

Proof. Use the unique characterization of $e_{K,\tau}$. For instance, $e_K*e_{K',\tau}$ acts on $\pi=\tau$ via $\pi(e_K)\pi(e_{K'})=\pi(e_K)$, and acts on irreducible π not isomorphic to τ via $\pi(e_K)\circ 0=0$. □

Proposition 13.3. *For any compact open subgroup $K\subset G$ and any $g\in G$, we have $l(g)r(g)e_{K,\tau}=e_{gKg^{-1},\tau}$.*

Proof. In general, for any $(V,\pi)\in\mathcal{M}(G)$, the map $\pi:\mathcal{H}(G)\to\mathrm{End}_{\mathbb{C}}(V),\alpha\mapsto\pi(\alpha)$ is $G\times G$-equivariant. It follows that $\pi(l(g)r(g)\alpha)=\pi(g)\pi(\alpha)\pi(g)^{-1}$. Using this one checks that $l(g)r(g)e_{K,\tau}$ satisfies the characterizing properties of $e_{gKg^{-1},\tau}$. □

Lect.20, Mar 10

Let $(V,\pi)\in\mathcal{M}(G)$. Define $p_\tau:V\to V$ as follows. For $v\in V$, find a compact open subgroup $K\subset G$ fixing v. Then define $p_\tau(v):=\pi(e_{K,\tau})v$. (Here V is not necessarily irreducible, so it is not the case that $\pi(e_{K,\tau})$ is either 0 or $\pi(e_K)$.) This definition is independent of the choice of K. Indeed, for $K'\subset K$, we have

$$\pi(e_{K',\tau})v=\pi(e_{K',\tau})\pi(e_K)v=\pi(e_{K,\tau})v,$$

where the first equality is because v is fixed by K, and the second equality follows from Proposition 13.2.

Lemma 13.4. *The map $p_\tau:V\to V$ is idempotent.*

Proof. Let $v\in V$. Assume that v is fixed by a compact open subgroup K. Then $p_\tau(v)=\pi(e_{K,\tau}v)=\pi(e_K*e_{K,\tau})v\in\pi(e_K)V=V^K$. Hence $p_\tau^2v=\pi(e_{K,\tau})\pi(e_{K,\tau})v=\pi(e_{K,\tau}*e_{K,\tau})v=\pi(e_{K,\tau})v=p_\tau(v)$. □

Since p_τ is idempotent, we have a canonical decomposition

$$V\cong p_\tau V\oplus(1-p_\tau)V.$$

Lemma 13.5. *The map $p_\tau:V\to V$ is G-equivariant. In particular the above decomposition is a decomposition into sub-representations.*

Proof. Let $g \in G$. We need to check only check $\pi(g)p_\tau = p_\tau\pi(g)$. Let $x \in V^K$. Then $\pi(g)x \in V^{gKg^{-1}}$. We have

$$p_\tau\pi(g)x = \pi(e_{gKg^{-1},\tau})\pi(g)x = \pi(l(g)r(g)e_{K,\tau})\pi(g)x = \pi(g)\pi(e_{K,\tau})\pi(g^{-1})\pi(g)x = \pi(g)p_\tau(x).$$

□

It follows directly from the definition of p_τ, that if we have a morphism $\phi : V \to V'$ in $\mathcal{M}(G)$, then ϕ is compatible with $p_\tau \in \mathrm{End}_G(V)$ and $p_\tau \in \mathrm{End}_G(V')$. In other words, p_τ is in the Bernstein center $\mathcal{Z}_G = Z(\mathcal{M}(G))$. Recall that $\mathcal{Z}_G$ is canonically isomorphic to the inverse limit of the centers of $\mathcal{H}(G)_K$. The projection of p_τ in the center of $\mathcal{H}(G)_K$ is nothing but $e_{K,\tau}$.

Now let p be an arbitrary idempotent element of $\mathcal{Z}_G$ (such as p_τ). Then for any $V \in \mathcal{M}(G)$, we have a decomposition $V \cong pV \oplus (1-p)V$ into sub-representations. Moreover this decomposition is functorial in $V \in \mathcal{M}(G)$, i.e., it is preserved by arbitrary morphisms in $\mathcal{M}(G)$. Since p is idempotent, we have $p = \mathrm{id}$ on pV, and $p = 0$ on $(1-p)V$. Conversely, if $p = \mathrm{id}$ (resp. 0) on V, then $V = pV$ (resp. $V = (1-p)V$). Thus if we define $\mathcal{M}(G)^{p=1}$ (resp. $\mathcal{M}(G)^{p=0}$) to be the full subcategory of $\mathcal{M}(G)$ consisting of (V,π) on which $p = \mathrm{id}$ (resp. $p = 0$), then we have a decomposition of category

$$\mathcal{M}(G) \cong \mathcal{M}(G)^{p=1} \oplus \mathcal{M}(G)^{p=0}.$$

Here note that there are no non-zero morphisms between objects from the two subcategories (because they must intertwine p), so a morphism in $\mathcal{M}(G)$ is indeed determined by morphisms in the two subcategories independently. That is to say, we have

$$\mathrm{Hom}_{\mathcal{M}(G)}(V,V') = \mathrm{Hom}_{\mathcal{M}(G)^{p=1}}(pV,pV') \oplus \mathrm{Hom}_{\mathcal{M}(G)^{p=0}}((1-p)V,(1-p)V').$$

Proposition 13.6. *The following statements hold.*

(1) *The subcategory $\mathcal{M}(G)^{p_\tau=0}$ consists of (V,π) all of whose subquotients are not isomorphic to (W,τ).*
(2) *The subcategory $\mathcal{M}(G)^{p_\tau=1}$ consists of those (V,π) which are direct sums of copies of (W,τ).*

Proof. We prove (1) assuming (2). If $(V,\pi) \in \mathcal{M}(G)^{p_\tau=0}$, then any irreducible subquotient of it is also in $\mathcal{M}(G)^{p_\tau=0}$, and hence non-isomorphic to (W,τ). Conversely, suppose $(V,\pi) \in \mathcal{M}(G)$ is such that all its irreducible subquotients are non-isomorphic to (W,τ). We decompose $V = V_1 \oplus V_2$, with $V_1 \in \mathcal{M}(G)^{p_\tau=1}$ and $V_2 \in \mathcal{M}(G)^{p_\tau=0}$. If $V_1 \neq 0$, then by (2), V_1 contains an irreducible sub-representation isomorphic to (W,τ). This contradicts with our assumption on V. Hence $V_1 = 0$ and $V = V_2 \in \mathcal{M}(G)^{p_\tau=0}$.

Now we prove (2). If (V,π) is a direct sum of copies of (W,τ), then clearly it belongs to $\mathcal{M}(G)^{p_\tau=1}$. Conversely, let $(V,\pi) \in \mathcal{M}(G)^{p_\tau=1}$. It suffices to show that V is the sum of sub-representations isomorphic to (W,τ) (since the sum is automatically direct by the irreducibility of (W,τ)). Let $v \in V$, and suppose $v \in V^K$. The map

$$f : W \otimes W^\vee \xrightarrow{\mathrm{MC}_{W,\tau}} \mathcal{H}(G) \xrightarrow{\pi} \mathrm{End}_{\mathbb{C}}(V) \xrightarrow{(\cdot)v} V$$

is G-equivariant, where G acts on $W \otimes W^\vee$ via the factor W. But recall that by construction $e_{K,\tau} \in \mathrm{im}(\mathrm{MC}_{\mathrm{W},\tau})$. Hence $v = p_\tau v = \pi(e_{K,\tau})v \in \mathrm{im}(f)$. Now $W \otimes W^\vee$ as a G-representation on the first factor, is isomorphic to a direct sum of copies (W,τ). Hence $\mathrm{im}(f)$ is generated by sub-representations isomorphic to (W,τ). □

Lect.21, Mar 13

14. A general decomposition result

Let G be an arbitrary td group. The terminology in the following definition is non-standard.

Definition 14.1. A non-zero irreducible smooth representation (W, τ) of G is called *quasi-compact*, if $\mathcal{M}(G)$ decomposes into a direct sum of subcategories $\mathcal{M}(G)_\tau \oplus \mathcal{M}(G)^\tau$, where $\mathcal{M}(G)_\tau$ is the full subcategory consisting of representations that are a direct sum of copies of (W, τ), and $\mathcal{M}(G)^\tau$ is the full subcategory consisting of representations none of whose subquotients are isomorphic to (W, τ).

In the previous section we showed that if G is second countable and unimodular, then irreducible compact representations are quasi-compact. This is the motivation for the above definition.

Recall that an object X of an abelian category $\mathcal{A}$ is called *injective*, if the (contravariant) functor $\mathcal{A} \to \{\text{abelian groups}\}, Y \mapsto \mathrm{Hom}(Y, X)$ is exact. It is called *projective*, if the covariant functor $Y \mapsto \mathrm{Hom}(X, Y)$ is exact.

Lemma 14.2. *Any irreducible quasi-compact representation (W, τ) is simultaneously an injective and a projective object in $\mathcal{M}(G)$.*

Proof. We prove projectivity and leave injectivity as an exercise. By general nonsense, it suffices to check that any surjection $p : V \to W$ in $\mathcal{M}(G)$ admits a section, i.e., a morphism $q : W \to V$ in $\mathcal{M}(G)$ such that $p \circ q = \mathrm{id}_W$. We decompose $V = V_\tau \oplus V^\tau$, where V_τ is a direct sum of copies of (W, τ), and V^τ has no subquotient isomorphic to (W, τ). Clearly $p(V^\tau) = 0$, and so $p(V_\tau) = W$. Write $V_\tau = \bigoplus_{i \in I} W_i$, with each W_i isomorphic to (W, τ). Then there must exist $i \in I$ such that $p(W_i) \neq 0$, i.e., $p(W_i) = W$. Then $p|_{W_i}$ is an isomorphism $W_i \xrightarrow{\sim} W$, and we define q to be the inverse of it. $\square$

Lemma 14.3. *Every non-zero smooth representation has a non-zero irreducible subquotient. Every non-zero finitely generated smooth representation has a non-zero irreducible sub-representation.*

Proof. Since every smooth representation contains a finitely generated sub-representation, the first statement follows from the second. To show the second, we apply Zorn's lemma to the set of proper sub-representations of a finitely generated (V, π). The finitely generated condition guarantees that any totally ordered union of proper sub-representations is still proper. Hence there exists a maximal proper sub-representation $V_1 \subset V$. The quotient V/V_1 is then irreducible. $\square$

From now on, we fix a set F of isomorphism classes of non-zero irreducible quasi-compact representations of G. Here F can be finite or infinite, and it can be the set of all isomorphism classes of such representations.

Definition 14.4. Let $\mathcal{M}(G)_F$ (resp. $\mathcal{M}(G)^F$) be the full subcategory of $\mathcal{M}(G)$ consisting of representations all of whose irreducible subquotients are isomorphic to a member of F (resp. non-isomorphic to any member of F).

Proposition 14.5. *The category $\mathcal{M}(G)_F$ is semi-simple, with simple objects being the irreducible representations whose isomorphism classes belong to F. In other words, every representation in $\mathcal{M}(G)_F$ is a direct sum of irreducible representations whose isomorphism classes belong to F.*

Proof. Let $(V,\pi) \in \mathcal{M}(G)_F$. Let $V_1 \subset V$ be the vector space generated by the irreducible sub-representations of V. It suffices to show that $V = V_1$, since V_1 is necessarily a direct sum of irreducible representations (because irreducible sub-representations of V have no non-zero overlap), and the isomorphism classes of these irreducible representations are clearly in F. Suppose $V \neq V_1$. Then $V/V_1 \neq 0$. Take an irreducible subquotient W of V/V_1. This is also a sub-quotient of V, so its isomorphism class is in F, and in particular W is quasi-compact. But then W is projective, so V/V_1 has a sub-representation isomorphic to W. Again by projectivity, the inclusion $W \to V/V_1$ lifts to a map $W \to V$ whose image is not contained in V_1, a contradiction with the definition of V_1. □

Lect.22, Mar 15

Consider the following finiteness condition:

(FC): For every compact open subgroup K, there are only finitely many isomorphism classes $[(W,\pi)]$ in F such that $W^K \neq 0$.

Theorem 14.6. *Assume (FC). Then $\mathcal{M}(G) = \mathcal{M}(G)_F \oplus \mathcal{M}(G)^F$.*

Proof. It is easy to see that there are no non-zero homomorphisms between objects of the two subcategories. The question is to show that every $(V,\pi) \in \mathcal{M}(G)$ admits a functorial decomposition $V = V_F \oplus V^F$.

Let $(V,\pi) \in \mathcal{M}(G)$. For each isomorphism class $[(W,\tau)]$ in F, we have a decomposition $V = V_\tau \oplus V^\tau$ by the assumption that (W,τ) is quasi-compact. Let $p_\tau : V \to V_\tau$ be the projection. Now define $p : V \to V$ as follows. Let $v \in V$, and assume $v \in V^K$. Define

$$p(v) = \sum_{[(W,\tau)]\in F} p_\tau(v).$$

Note that $p_\tau(v)$ is non-zero only when (W,τ) has non-zero K-fixed vectors. Hence by (FC), we know that the above sum is finite. It is easy to check that p is idempotent, G-equivariant, and functorial in $V \in \mathcal{M}(G)$. In other words, p is an idempotent element of the Bernstein center $\mathcal{Z}_G$.

We check that $p : V \to V$ is idempotent. Let $v \in V$, then

$$p^2 v = \sum_{[\tau],[\tau']\in F} p_\tau p_{\tau'}(v),$$

where the double sum is still finite. If $[\tau] \neq [\tau']$, then $p_\tau p_{\tau'} = 0$ since $p_{\tau'}(V)$ is a direct sum of copies of τ and therefore lies in $\mathcal{M}(G)^{\tau'}$. Hence

$$p^2 v = \sum_{[\tau]\in F} p_\tau^2 v = \sum p_\tau v = p(v).$$

This proves that p is idempotent.

As before we have a decomposition

$$\mathcal{M}(G) = \mathcal{M}(G)^{p=1} \oplus \mathcal{M}(G)^{p=0}.$$

It remains to show $\mathcal{M}(G)^{p=1} = \mathcal{M}(G)_F$ and $\mathcal{M}(G)^{p=0} = \mathcal{M}(G)^F$.

Using Proposition 14.5, it is easy to check $\mathcal{M}(G)_F \subset \mathcal{M}(G)^{p=1}$. Conversely, if $(V,\pi) \in \mathcal{M}(G)^{p=1}$, then any irreducible subquotient of (V,π) is still in $\mathcal{M}(G)^{p=1}$, and therefore its isomorphism class is in F. Hence $(V,\pi) \in \mathcal{M}(G)_F$.

Similarly, it is easy to show $\mathcal{M}(G)^{p=0} \subset \mathcal{M}(G)^F$. For the reverse containment, let $V \in \mathcal{M}(G)^F$. Write $V = V_1 \oplus V_2$ with $V_1 \in \mathcal{M}(G)^{p=1}$ and $V_2 \in \mathcal{M}(G)^{p=0}$. It suffices to show $V_1 = 0$. Suppose not. Then by what we have already shown, we have $V_1 \in \mathcal{M}(G)_F$, and therefore V_1 has an irreducible subquotient in F. But this is also a subquotient of V, a contradiction. ☐

Lect.23, Mar 17

15. Unitary and square integrable representations

Let G be a second countable td group. We call a smooth representation (V, π) *unitary*, if there is a positive definite Hermitian form $\langle\cdot,\cdot\rangle$ on V such that G-acts on V via isometries, i.e., $\langle\pi(g)v, \pi(g)w\rangle = \langle v, w\rangle$ for all $g \in G, v, w \in V$. We also say that G preserves $\langle\cdot,\cdot\rangle$. The motivation for the terminology "unitary" is that each $\pi(g)$ is a unitary operator (with respect to the given Hermitian form).

It turns out that the notion of a unitary representation behaves well when coupled with the admissible condition.

Lemma 15.1. *Let $(V, \pi) \in \mathcal{M}(G)$ be irreducible admissible. Then on V there is at most one, up to scaling, positive definite Hermitian form preserved by G.*

Proof. Any such Hermitian form defines a non-zero G-linear map $V \to V^\vee$, and is determined by the latter. Recall from Exercise 12.6 that $V^\vee$ is irreducible. (This uses that V is admissible.) Hence any non-zero G-linear map $V \to V^\vee$ must be an isomorphism, and is unique up to scaling. ☐

Lemma 15.2. *Let $(V, \pi) \in \mathcal{M}(G)$ be unitary and admissible. Then $V \cong V^\vee$.*

Proof. Let $\langle\cdot,\cdot\rangle$ be a positive definite G-invariant Hermitian form on V. Then we have a G-linear map $\phi : V \to V^\vee, v \mapsto \langle\cdot, v\rangle$. It suffices to check that ϕ is an isomorphism. For this, we only need to show that for each compact open subgroup K, $\phi : V^K \xrightarrow{\sim} (V^\vee)^K$. Recall that $(V^\vee)^K \cong (V^K)^*$, and the induced map $\phi : V^K \to (V^K)^*$ is the usual map arising from the positive-definite form on V^K. Since V^K is finite dimensional, this map is an isomorphism. ☐

Lemma 15.3. *Let $(V, \pi) \in \mathcal{M}(G)$ be unitary and admissible. For any sub-representation $W \subset V$, the orthogonal complement $W^\perp = \{v \in V \mid \langle v, w\rangle = 0, \forall w \in W\}$ is also a sub-representation, and $V = W \oplus W^\perp$.*

Proof. Only the fact that $V = W + W^\perp$ is unclear. Consider a compact open subgroup K. Then V^K is finite dimensional by admissibility. Now V^K inherits the hermitian form from V, and the orthogonal complement of W^K in V^K is $(W^\perp)^K$. Indeed, if $v \in V^K$ is perpendicular to W^K, then for any $w \in W$ we have $\langle v, w\rangle = \langle e_K v, w\rangle = \langle v, e_K w\rangle = 0$ since $e_K w \in W^K$. Hence $V^K = W^K \oplus (W^\perp)^K$, and in particular $V^K \subset W + W^\perp$. Taking the union over K we get the desired result. ☐

Lemma 15.4. *Let $(V, \pi) \in \mathcal{M}(G)$ be unitary, admissible, and non-zero. Then (V, π) has a non-zero irreducible sub-representation.*

Proof. We know that V has an irreducible subquotient. Up to shrinking V, we may assume that it has an irreducible quotient $f : V \to W$. Then $\ker f \neq V$, and by Lemma 15.3, $W' := (\ker f)^\perp \neq 0$. Then W' is a non-zero sub-representation of V, and it is irreducible since it maps injectively into W (hence isomorphic to W). ☐

Proposition 15.5. *Let $(V, \pi) \in \mathcal{M}(G)$ be unitary and admissible. Then (V, π) is semi-simple, i.e., isomorphic to a direct sum of irreducible representations.*

Proof. Let $V_1 \subset V$ be the vector subspace generated by all irreducible sub-representations of V. As we have seen several times, it suffices to show $V = V_1$. Suppose not. Then $V = V_1 \oplus V_1^\perp$ with $V_1^\perp \neq 0$ by Lemma 15.3. Clearly $V_1^\perp$ is also unitary and admissible. By Lemma 15.4, $V_1^\perp$ contains an irreducible sub-representation, a contradiction with the definition of V_1. □

Now we assume that G is unimodular.

Fact 15.6. *If G is unimodular, then $G/Z(G)$ is a unimodular td group. We fix a Haar measure $\mu(\bar{g})$ on it.*

Definition 15.7. By a character on $Z(G)$, we mean a continuous homomorphism $\chi : Z(G) \to \mathbb{C}^\times$, where continuous just means it kills an open subgroup. We say χ is *unitary* if $|\chi(z)| = 1$ for all $z \in Z(G)$. Similarly, we define the notion of a (unitary) character for G.

Definition 15.8. Fix a unitary character χ of $Z(G)$. Let $L^2(G, \chi)$ be the $\mathbb{C}$-vector subspace of $C^\infty(G)$ consisting of f such that

(1) $f(gz) = f(g)\chi(z)$ for all $g \in G, z \in Z(G)$. (We say that f translates under $Z(G)$ via χ.) In particular, $|f|$ descends to a smooth function on $G/Z(G)$.
(2) $|f|$ is square integrable on $G/Z(G)$, i.e., $\int_{G/Z(G)} |f(\bar{g})|^2 \mu(\bar{g}) < \infty$. Here the integral is *a priori* an infinite series of positive terms.

Exercise 15.9. Work out the explicit definition of $\int_{G/Z(G)} |f|$ as an infinite series for general $f \in C^\infty(G/Z(G))$ (using that f takes only countably many different values since $G/Z(G)$ is second countable), and show that this is well-defined (as an element of $\mathbb{R} \cup \{+\infty\}$). Show that $L^2(G, \chi)$ is a sub-representation of the $G \times G$-representation $C^\infty(G)$. Show that on $L^2(G, \chi)$ we have a $G \times G$-invariant positive definite Hermitian form given by

$$\langle f_1, f_2 \rangle = \int_{G/Z(G)} \bar{f}_1(\bar{g}) f_2(\bar{g}) \mu(\bar{g}).$$

In particular, $L^2(G, \chi)$ is a unitary $G \times G$-representation.

Definition 15.10. We call $(V, \pi) \in \mathcal{M}(G)$ *square integrable*, if the following conditions are satisfied:

(1) $Z(G)$ acts on V by a unitary character χ. (We say that (V, π) has unitary central character.)
(2) Every matrix coefficient $\phi_{v,\lambda}$ (for $v \in V, \lambda \in V^\vee$) lies in $L^2(G, \chi)$.

Remark 15.11. Condition (1) above already implies every matrix coefficient $\phi_{v,\lambda} \in C^\infty(G)$ satisfies condition (1) in Definition 15.8.

Remark 15.12. Any compact representation admitting a central character is square integrable. Indeed, the existence of such a representation implies that $Z(G)$ is compact, and hence the central character in question must be unitary since its image in $\mathbb{C}^\times$ is a finite subgroup.

Definition 15.13. We call $(V, \pi) \in \mathcal{M}(G)$ *essentially square integrable*, if there is a character ω of G (trivial on a compact open subgroup) such that $(V, \omega\pi)$ is square integrable. Here $(\omega\pi)(g) := \omega(g) \cdot \pi(g) \in \mathrm{End}(V)$.

Lemma 15.14. *Let $(V, \pi) \in \mathcal{M}(G)$ be irreducible. The following statements hold.*

(1) *If (V, π) is essentially square integrable, then it is admissible.*

(2) *If (V,π) is admissible, has unitary character χ, and there is one pair $(v,\lambda) \in V \times V^\vee$ such that $0 \neq \phi_{v,\lambda} \in L^2(G,\chi)$, then (V,π) is square integrable.*

Proof. (1) We may assume that (V,π) is square integrable, with unitary central character χ. For simplicity assume $Z(G) = 1$; the general case is treated similarly. Let K be a compact open subgroup, and suppose V^K is infinite dimensional for the sake of contradiction. Fix a non-zero $v \in V$. Then V is spanned by $\{\pi(g)v \mid g \in G\}$, and hence $V^K = \pi(e_K)V$ is spanned by $\{\pi(e_K)\pi(g)v \mid g \in G\}$. Hence there exist infinitely many $g_1, g_2, \cdots \in G$ such that $\pi(e_K)\pi(g_n)v \in V^K$ are linearly independent. Recall that $(V^\vee)^K \cong (V^K)^*$. Hence we can find $\lambda \in (V^\vee)^K$ such that $\lambda(\pi(e_K)\pi(g_n)v) = 1$ for all n. One easily shows that

$$\int_G |\phi_{v,\lambda}|^2 = \sum_{g \in G/K} |\lambda(\pi(e_K)\pi(g^{-1})v)|^2 \operatorname{vol}(K).$$

Clearly the images of g_n in $K\backslash G$ are distinct. Hence the above is greater or equal to

$$\operatorname{vol}(K) \sum_n |\lambda(\pi(e_K)\pi(g_n)v)|^2 = \infty,$$

a contradiction.

(2) Since (V,π) is irreducible and admissible, we know $V^\vee$ is irreducible in $\mathcal{M}(G)$, and $V \otimes V^\vee$ is irreducible in $\mathcal{M}(G \times G)$. The matrix coefficient construction is a $G \times G$-linear map $V \otimes V^\vee \to C^\infty(G)$. We conclude by noting that $L^2(G,\chi) \subset C^\infty(G)$ is a sub-representation. □

Proposition 15.15. *Let $(V,\pi) \in \mathcal{M}(G)$ be admissible and square integrable. Then it is unitary (and hence semi-simple by Proposition 15.5).*

Proof. Consider a finitely generated sub-representation $W \subset V$. Let $\operatorname{Ann}(W)$ be the annihilator of W in $V^\vee$. Then $W^\vee \cong V^\vee/\operatorname{Ann}(W)$. Indeed, it suffices to show that the natural map $V^\vee \to W^\vee$ is onto. For this use that $(V^\vee)^K \cong (V^K)^*$ and $(W^\vee)^K \cong (W^K)^*$.

Let $w_1, \cdots, w_n$ be generators of W as a G-representation. For $\lambda_1, \lambda_2 \in W^\vee$, define

$$\langle \lambda_1, \lambda_2 \rangle = \sum_{i=1}^n \int_{G/Z(G)} \bar{\phi}_{w_i,\tilde{\lambda}_1} \phi_{w_i,\tilde{\lambda}_2},$$

where $\tilde{\lambda}_i \in V^\vee$ is a lift of $\lambda_i \in W^\vee \cong V^\vee/\operatorname{Ann}(W)$. Then this is a well-defined, positive-definite Hermitian form on W invariant under G. We only check positive-definite: Suppose $\langle \lambda, \lambda \rangle = 0$. Then $\phi_{w_i,\tilde{\lambda}} = 0$ for each i. Hence $\tilde{\lambda}$ kills everything in the G-orbit of w_i, and therefore $\tilde{\lambda} \in \operatorname{Ann}(W)$. This means $\lambda = 0$.

Thus we have shown that $W^\vee$ is unitary. Since V is admissible, so are W and $W^\vee$. We conclude that W is unitary by Lemma 15.2. Now V is the union of all finitely generated subrepresentations, and the latter are unitary admissible and therefore semi-simple by Proposition 15.5. It easily follows that V is semi-simple. Writing $V = \bigoplus_{i \in I} V_i$ with V_i irreducible, we have each V_i is unitary since it is finitely generated. Let $\langle \cdot, \cdot \rangle_i$ be a G-invariant positive-definite Hermitian form on V_i. Then we can define $\langle \sum_i v_i, \sum_i v_i' \rangle := \sum_i \langle v_i, v_i' \rangle_i$ (all sums being finite), which is a G-invariant positive-definite Hermitian form on V. □

Next we discuss Schur orthogonality for square integrable representations. As usual, for any function f on G, we write $\check{f}$ for $g \mapsto f(g^{-1})$.

Theorem 15.16. *Let $(V,\pi) \in \mathcal{M}(G)$ be irreducible, admissible, square integrable. The following statements hold.*

(1) *There exists* $d(\pi) \in \mathbb{R}_{>0}$ *(depending on the choice of Haar measure on* $G/Z(G)$*), called the* formal degree *of* (V,π)*, with the property that for all* $v_1, v_2 \in V, \lambda_1, \lambda_2 \in V^\vee$*, we have*

$$\int_{G/Z(G)} \phi_{v_1,\lambda_1}\check{\phi}_{v_2,\lambda_2} = d(\pi)^{-1}\lambda_1(v_2)\lambda_2(v_1).$$

(2) *Let* (V_2,π_2) *satisfy the same assumptions as* (V,π)*, but non-isomorphic to* (V,π)*. Assume that* V_2 *has the same central character as* V*. Write* (V_1,π_1) *for* (V,π)*. For any* $v_i \in V_i, \lambda_i \in V_i^\vee$ $(i = 1,2)$*, we have*

$$\int_{G/Z(G)} \phi_{v_1,\lambda_1}\check{\phi}_{v_2,\lambda_2} = 0.$$

Sketch of proof. The integrals all make sense by the square integrable assumptions and the assumption on the central characters.

By Proposition 15.15, (V,π) is unitary. We may then identify $V \cong V^\vee$ as in Lemma 15.2. Sending $(v_1 \otimes \lambda_1, \lambda_2 \otimes v_2) \in (V \otimes V) \times (V \otimes V)$ to the two sides of the equation in (1) respectively, defines two positive-definite Hermitian forms on $V \otimes V$. They are both $G \times G$-invariant. Since $V \otimes V$ is an irreducible $G \times G$-representation, these Hermitian forms must differ by a positive scalar, by Lemma 15.1. This proves (1).

For (2), fix v_2 and λ_1. Define $V_1 \to V_2^{\vee\vee}$ by sending $v_1 \in V_1$ to the map

$$V_2 \ni v_2 \longmapsto \int_{G/Z(G)} \phi_{v_1,\lambda_1}\check{\phi}_{v_2,\lambda_2}.$$

This map is G-linear and hence must be 0 since $V_2^\vee \cong V_2$ is non-isomorphic to V_1. □

Exercise 15.17. Let $(V_1,\pi_1), \cdots, (V_n,\pi_n) \in \mathcal{M}(G)$ be pairwise non-isomorphic, irreducible admissible square integrable. Let ϕ_i be a matrix coefficient for (V_i,π_i). Then $\phi_1, \cdots, \phi_n$ are linearly independent in $C^\infty(G)$.

Lect.24, Mar 27

16. Reductive groups

We recall some aspects of the general theory of reductive groups over a field. Standard references in this subject are [Spr09], [Spr79], [Bor91]. We also recommend [Mil17].

Let F be a field of characteristic zero, and fix an algebraic closure $\overline{F}$. By an *algebraic variety over* F, we mean a reduced scheme of finite type over F such that each connected component is irreducible. In particular, an affine F-variety is of the form $\operatorname{Spec} A = \coprod_{i=1}^n \operatorname{Spec} A_i$, where $A = \prod_{i=1}^n A_i$ and each A_i is a finitely generated F-algebra and is an integral domain. We often identify a variety V with its set of $\overline{F}$-points $V(\overline{F})$ (which is also in bijection with the set of *closed* points of the scheme V), when no confusion arises. Note that we have a natural action of $\Gamma_F := \operatorname{Gal}(\overline{F}/F)$ on $V = V(\overline{F})$. The fixed points are exactly the F-points of V, denoted by $V(F)$. We denote the base change of V to $\overline{F}$ by $V_{\overline{F}}$.

Recall that a *linear algebraic group* over F is an affine F-variety G together with F-variety maps $m : G \times_F G \to G, i : G \to G, e : \operatorname{Spec} F \to G$ satisfying the usual axioms of multiplication, inversion, identity, for a group. For any F-algebra R, the R-points of G is then an actual group $G(R)$. As usual, by Yoneda, G as a linear algebraic group is determined by the functor $R \mapsto G(R)$ from F-algebras to groups. Similarly, a homomorphism $G \to G'$ is determined by a functorial family of homomorphisms $G(R) \to G'(R)$ for all F-algebras R.

Example 16.1. We have a linear algebraic group $G = \mathrm{GL}_n$ over F. The functor of points is given by $R \mapsto \mathrm{GL}_n(R) = \{n \times n \text{ invertible matrices over } R\} = \{A \in M_{n\times n}(R) \mid \det A \in R^\times\}$. We can also describe the variety G and the structure maps m, e, i explicitly. Namely, $G = \operatorname{Spec} F[a_{11}, a_{12}, \cdots, a_{nn}, \det(a_{ij})^{-1}]$. The map $m : G \times G \to G$ is given by, at the level of the rings,

$$F[a_{ij}, \det(a_{ij})^{-1}] \to F[b_{ij}, \det(b_{ij})^{-1}] \otimes_F F[c_{ij}, \det(c_{ij})^{-1}], a_{ij} \mapsto \sum_k b_{ik} \otimes c_{kj}.$$

That is, it sends the ij-th coordinate function to its pull-back under the matrix multiplication $(B, C) \mapsto B \cdot C$. We omit the description of i and e.

Fact 16.2. *Every linear algebraic group over F is a closed subgroup of GL_n (over F) for some n. The converse is also true. So we can think of a linear algebraic group concretely as a subvariety of GL_n closed under the group operations.*

Let G be a linear algebraic group over F. An element $g \in G$ (meaning $g \in G(\overline{F})$) is called *unipotent*, if for every algebraic homomorphism $G_{\overline{F}} \to \mathrm{GL}_{n,\overline{F}}$ (for all n), the image of g is a unipotent matrix in $\mathrm{GL}_n(\overline{F})$, that is, a matrix A such that $A - I_n$ is nilpotent. Similarly, an element $g \in G$ is called *semi-simple*, if for every algebraic homomorphism $G_{\overline{F}} \to \mathrm{GL}_{n,\overline{F}}$ (for all n), the image of g is a diagonalizable matrix (one that is conjugate to a diagonal matrix). We call G *unipotent*, if every $g \in G$ is unipotent. Note that all these definitions depend only on $G_{\overline{F}}$, not on G.

Fact 16.3 (Jordan decomposition)**.** *Let G be a linear algebraic group over F, and $g \in G$. Then there is a unique decomposition $g = g_s g_u$ with $g_s, g_u \in G$ such that $g_s g_u = g_u g_s$, g_s is semi-simple, and g_u is unipotent.*

Fact 16.4. *A linear algebraic group over $\overline{F}$ is unipotent if and only if it isomorphic to a closed subgroup of U_n for some n, where U_n is the closed subgroup of GL_n (over $\overline{F}$) consisting of the upper-triangular matrices with 1's on the diagonal.*

Fact 16.5. *Let G be a linear algebraic group over F. Then $G_{\overline{F}}$ has a maximal closed subgroup which is connected, normal, and unipotent. Moreover, this subgroup is defined over F.*[11] *It is called the* unipotent radical of G*, and denoted by $R_u(G)$.*

Definition 16.6. A linear algebraic group G is called *reductive*, if $R_u(G) = 1$.

Remark 16.7. We allow a reductive group to be disconnected.

Remark 16.8. By definition G is reductive if and only if $G_{\overline{F}}$ is reductive.

Lect.25, Mar 29

Remark 16.9. For a linear algebraic group G over F, the connectedness of the F-scheme G is equivalent to the connectedness of the $\overline{F}$-scheme $G_{\overline{F}}$, since G has a section over F (namely the identity $e : \operatorname{Spec} F \to G$).

Example 16.10. The group GL_n is reductive. Note that GL_n has the closed subgroup U_n which is connected and reductive, but this is not a contradiction, because U_n is not normal in GL_n.

[11] We say that a closed subgroup H of $G_{\overline{F}}$ is defined over F, if the following two equivalent conditions are satisfied: (1) Viewed as a subgroup of $G(\overline{F})$, H is $\mathrm{Gal}(\overline{F}/F)$-stable. (2) Viewed as a closed $\overline{F}$-subvariety of $G_{\overline{F}}$, H arises as the base change of a (unique) closed F-subvariety of G.

Example 16.11. Consider the closed subgroup B_n of GL_n consisting of the upper triangular invertible matrices. Then B_n contains U_n as a closed normal subgroup. In fact $R_u(B_n) = U_n$. Hence B_n is not reductive for $n \geq 2$.

Example 16.12. Let G_1 and G_2 be reductive (resp. connected), then $G_1 \times G_2$ is reductive (resp. connected).

In the beginning of the course we already gave some examples of reductive groups. (More precisely, we gave examples of the $\mathbb{Q}_p$-points of reductive groups over $\mathbb{Q}_p$.) We somewhat repeat them here.

Example 16.13. Let V be a finite dimensional F-vector space, and let $\langle \cdot, \cdot \rangle$ be a symmetric or anti-symmetric non-degenerate bilinear form on V. We have a closed subgroup G of $\mathrm{GL}(V)$ (isomorphic to GL_n) such that for each F-algebra R, $G(R)$ is the subgroup of $\mathrm{GL}(V)(R) = \mathrm{Aut}_{R\text{-lin}}(V \otimes_F R)$ consisting of $g \in \mathrm{GL}(V)(R)$ preserving the R-bilinear form on $V \otimes_F R$ induced by $\langle \cdot, \cdot \rangle$ and such that $\det(g) = 1$. (The condition $\det(g) = 1$ is automatic in the anti-symmetric case.) We usually write $G = \mathrm{SO}(V, \langle \cdot, \cdot \rangle)$ in the symmetric case, and write $G = \mathrm{Sp}(V, \langle \cdot, \cdot \rangle)$ in the anti-symmetric case. They are called the special orthogonal group and the symplectic group respectively. In both cases G is connected and reductive.

Example 16.14. Let E/F be a finite extension, and G be a linear algebraic group over E. Then there is a linear algebraic group $\mathrm{Res}_{E/F}\, G$ over F, called the *Weil restriction of scalars of G from E to F*, characterized by that $\mathrm{Res}_{E/F}(G)(R) = G(R \otimes_F E)$ for all F-algebras R. Here $G(R \otimes_F E)$ makes sense as $R \otimes_F E$ is an E-algebra and G is a linear algebraic group over E.

Exercise 16.15. Prove that this functor is indeed given by a linear algebraic group $\mathrm{Res}_{E/F}(G)$ over F. (For $G = \mathrm{GL}_n$, this was hinted at in Example 1.1.)

Exercise 16.16. Show that $(\mathrm{Res}_{E/F}\, G)_{\overline{F}} \cong \prod_{\sigma \in \mathrm{Hom}_F(E, \overline{F})} G_{\sigma, \overline{F}}$, where $G_{\sigma, \overline{F}}$ denotes the base change of G from E to $\overline{F}$ along $\sigma : E \hookrightarrow \overline{F}$. In particular, G is a reductive (resp. connected reductive) group over E if and only if $\mathrm{Res}_{E/F}\, G$ is a reductive (resp. connected reductive) group over F.

Example 16.17. Let E/F be a finite extension, and let V be a finite dimensional E-vector space. For every F-algebra R, we have $(\mathrm{Res}_{E/F}\, \mathrm{GL}(V))(R) = \mathrm{GL}(V)(E \otimes_F R) = \mathrm{Aut}_{E\text{-lin}}(V \otimes_E E \otimes_F R) = \mathrm{Aut}_{E\text{-lin}}(V \otimes_F R)$. Thus if we write V_0 for the underlying F-vector space of V (whose F-dimension is $[E : F] \dim_E V$), then $\mathrm{Res}_{E/F}\, \mathrm{GL}(V)$ is the F-subgroup of $\mathrm{GL}(V_0)$ defined by the E-linear condition.

In the case where V is equipped with a non-degenerate E/F-hermitian form $\langle \cdot, \cdot \rangle$, we have the unitary group $G = \mathrm{U}(V, \langle \cdot, \cdot \rangle)$. This is the closed subgroup of $\mathrm{Res}_{E/F}\, \mathrm{GL}(V)$ such that $G(R)$ consists of E-linear automorphisms of $V \otimes_F R$ preserving $\langle \cdot, \cdot \rangle$. We know that G is connected reductive.

Example 16.18. Let C be a central simple algebra over F. We have a linear algebraic group G over F such that $G(R) = (C \otimes_F R)^\times$. Since $C \otimes_F \overline{F} \cong M_{n \times n}(\overline{F})$ as $\overline{F}$-algebras, it is easy to see that $G_{\overline{F}} \cong \mathrm{GL}_n$. We say that G is a *form of* GL_n, meaning that they become isomorphic over $\overline{F}$ but may not be isomorphic over F. In fact, in the previous example, the unitary group is also a form of GL_n (for $n = \dim_E V$). In general the unitary group, the current G associated to C, and GL_n, are pairwise non-isomorphic over F.

17. Diagonalizable groups and tori

Definition 17.1. A linear algebraic group G over F is called diagonalizable (resp. a torus) if and only if $G_{\overline{F}}$ is isomorphic to a closed subgroup of $\mathbb{G}_m^n$ (resp. isomorphic to $\mathbb{G}_m^n$) for some n. Here $\mathbb{G}_m = \mathrm{GL}_1$. A torus is called *split*, if it is isomorphic to $\mathbb{G}_m^n$ over F.

Remark 17.2. We can identify $\mathbb{G}_m^n$ with the closed subgroup of GL_n consisting of diagonal matrices. This explains the terminology "diagonalizable".

Fact 17.3. *Let G be a linear algebraic group over F. Then G is a torus if and only if it is connected reductive commutative, if and only if G is connected and diagonalizable. If G' is diagonalizable, then the identity connected component of G' is a torus.*

Write Γ_F for $\mathrm{Gal}(\overline{F}/F)$.

Definition 17.4. For any linear algebraic group G over F, define $X^*(G) = \mathrm{Hom}_{\overline{F}\text{-gps}}(G_{\overline{F}}, \mathbb{G}_m)$, namely the set of morphisms $G_{\overline{F}} \to \mathbb{G}_m$ in the category of linear algebraic groups over $\overline{F}$. This has a natural abelian group structure, given by the abelian group variety structure on $\mathbb{G}_m$.

There is a continuous Γ_F-action on $X^*(G)$. Here continuity just means that the action factors through $\mathrm{Gal}(E/F)$ for some finite Galois extensions E/F. We explain this action. There is a natural injection from $X^*(G)$ to the abelian group of actual group homomorphisms $\chi : G(\overline{F}) \to \mathbb{G}_m(\overline{F}) = \overline{F}^\times$. (The image consists of those homomorphisms that are "algebraic".) We only describe the Γ_F-action on the latter set. Given $\sigma \in \Gamma_F$ and $\chi : G(\overline{F}) \to \overline{F}^\times$, we define $\sigma(\chi)$ to be $G(\overline{F}) \to \overline{F}^\times, g \mapsto \sigma(\chi(\sigma^{-1}(g)))$. Here $g \mapsto \sigma^{-1}(g)$ is the action of σ^{-1} on $G(\overline{F})$.

Remark 17.5. We often denote the abelian group structure on $X^*(G)$ additively, although it is defined by the multiplication on $\mathbb{G}_m$.

Fact 17.6. *The functor $G \mapsto X^*(G)$ defines an anti-equivalence from the abelian category of diagonalizable groups over F to the abelian category of finitely generated abelian groups together with continuous Γ_F-actions. We have G is a torus of dimension n if and only if $X^*(G)$ is free of rank n (i.e., isomorphic to $\mathbb{Z}^n$), and G is a split torus if and only if the above condition holds and Γ_F acts trivially on $X^*(G)$.*

Remark 17.7. The category of diagonalizable groups is abelian, where kernels and cokernels are special cases of kernels and quotients of linear algebraic groups, which will be discussed slightly later. We point out here that the subcategory of tori is not abelian, as it is not closed under taking kernels. (e.g. the kernel of $\mathbb{G}_m \to \mathbb{G}_m, z \mapsto z^n$ is μ_n, whose R-points is the group of n-th roots of unity in $R^\times$. This group is not connected.) Similarly, the category of finite rank free $\mathbb{Z}$-modules together with a continuous Γ_F-action is not abelian, as it is not closed under taking cokernels.

Lect.26, Mar 31

Example 17.8. Let E/F be a finite extension. Consider $T = \mathrm{Res}_{E/F}\,\mathbb{G}_m$. For every F-algebra R, $T(R) = (R \otimes_F E)^\times$. We know that T is a torus, and $X^*(T)$ is the finite free $\mathbb{Z}$-module with basis given by the F-embeddings $E \to \overline{F}$, and the Γ_F-action is given by the natural Γ_F-action on the set of F-embeddings $E \to \overline{F}$.

For instance, for $E/F = \mathbb{C}/\mathbb{R}$, we can identify $X^*(\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m)$ with $\mathbb{Z}^2$, with complex conjugation in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acting on $\mathbb{Z}^2$ by $(a, b) \mapsto (b, a)$.

In general, we have a homomorphism of F-tori

$$N : \operatorname{Res}_{E/F} \mathbb{G}_m \longrightarrow \mathbb{G}_m,$$

which on R-points is the map

$$(R \otimes_F E)^\times \to R^\times, \sum_i r_i \otimes e_i \mapsto \sum_i r_i \otimes N_{E/F} e_i.$$

Here $N_{E/F}$ is the norm map $E \to F$.

Now under the functor X^*, the map N induces the map

$$X^*(N) : X^*(\mathbb{G}_m) \longrightarrow X^*(\operatorname{Res}_{E/F} \mathbb{G}_m)$$

given by $\mathbb{Z} \to \prod_{\sigma \in \operatorname{Hom}_F(E, \overline{F})} \mathbb{Z}, a \mapsto (a, \cdots, a)$. Since X^* is an equivalence of abelian categories, if we let T^1 be the kernel of $N : T \to \mathbb{G}_m$, then $X^*(T^1)$ is the cokernel of $X^*(N)$, which one can check is free as a $\mathbb{Z}$-module. It follows that T^1 is actually a torus (as opposed to merely a diagonalizable group).

18. Center, derived subgroup, and abelianization

Fact 18.1. *Let G be a connected reductive group over F. Then there is a closed subgroup Z_G of G, called the* center of G*, such that for every F-algebra R, $Z_G(R) =$ center of $G(R)$. Moreover, Z_G is a diagonalizable group over F. We call G* semi-simple*, if G is connected reductive and Z_G has dimension 0.*

Fact 18.2. *Let G be a linear algebraic group, and $H \subset G$ a closed normal subgroup (both defined over F). Here normal means that $H(R)$ is normal in $G(R)$ for every F-algebra R. Then we can form the quotient G/H, which is still a linear algebraic group over F. The map $G \to G/H$ is characterized as universal among all homomorphisms $G \to G'$ between linear algebraic groups over F killing H.*

Remark 18.3. We have $(G/H)(\overline{F}) \cong G(\overline{F})/H(\overline{F})$, but for a general F-algebra R, for instance $R = F$, we only have an injection $G(F)/H(F) \to (G/H)(F)$.

Fact 18.4. *Given any homomorphism $\phi : G \to G'$ between linear algebraic groups over F, the kernel of it is a closed normal subgroup $K \subset G$ defined over F. It is characterized by $K(R) = \ker(G(R) \to G'(R))$ for all F-algebras R. Then ϕ induces a homomorphism $\bar{\phi} : G/K \to G'$ which is injective at the level of R-points for all F-algebras R. We say that ϕ is* injective*, if the kernel is trivial, or equivalently ϕ is injective at the level of R-points for all R.*

We say that ϕ is surjective*, if it is surjective at the level of $\overline{F}$-points. This is if and only if $\bar{\phi}$ is an isomorphism. Thus our previous remark on the quotient applies here, namely for a surjective $\phi : G \to G'$, the induced map $G(R) \to G'(R)$ may not be surjective.*

Fact 18.5. *Let G be a connected reductive group over F. Then G/Z_G is also a connected reductive group over F, and its center is trivial. We denote G/Z_G by G^{ad}, and call it the* adjoint group of G*. There is a minimal closed normal subgroup G_{der} of G, called the* derived subgroup*, such that G/G_{der} is abelian. We know that G_{der} is connected reductive and semi-simple, and G/G_{der} is a torus. We denote the latter by G^{ab}, called the abelianization of G. We know that G is generated by G_{der} and Z_G, in the sense that there is no proper closed subgroup of G containing both G_{der} and Z_G. We have $Z_{G_{\mathrm{der}}} \subset Z_G$.*

Remark 18.6. The F-group homomorphism $G \to G^{\mathrm{ab}}$ is surjective, but the induced map $G(F) \to G^{\mathrm{ab}}(F)$ may not be surjective.

Example 18.7. For $G = \mathrm{GL}_n$, we have $Z_G = \mathbb{G}_m =$ the invertible scalar matrices, and $G_{\mathrm{der}} = \mathrm{SL}_n$. Note that $Z_{\mathrm{SL}_n} = \mu_n$, the kernel of the n-th power map $\mathbb{G}_m \to \mathbb{G}_m$. This is disconnected, even though Z_{GL_n} is connected. We have $G^{\mathrm{ab}} \cong \mathbb{G}_m$, and the natural map $G \to G^{\mathrm{ab}}$ is identified with the determinant map $\mathrm{GL}_n \to \mathbb{G}_m$. Note that in this case, $\mathrm{GL}_n(F) \to \mathbb{G}_m(F)$ is in fact surjective (which is elementary to see).

Lect.27, Apr 3

Definition 18.8. Let G be a connected reductive group over F. Let $X^*(G) = \mathrm{Hom}_{\overline{F}\text{-gps}}(G, \mathbb{G}_m)$. This is a finite rank free abelian group together with a continuous Γ_F-action. In fact it is identified with $X^*(G^{\mathrm{ab}})$.

Definition 18.9. We write $X^*(G(F))$ for $X^*(G)^{\Gamma_F} = \{\chi \in X^*(G) \mid \sigma(\chi) = \chi, \forall \sigma \in \Gamma_F\}$.

Note that $X^*(G(F))$ consists exactly of F-group homomorphisms $G \to \mathbb{G}_m$. Hence for $\chi \in X^*(G(F))$, we indeed obtain a character $G(F) \to F^\times$, which partially justifies the notation. Alternatively, we can identify an element $\chi \in X^*(G)$ with an actual homomorphism $G(\overline{F}) \to \overline{F}^\times$ (which should be "defined by polynomials over $\overline{F}$"). Then χ is fixed by Γ_F if and only if the homomorphism $G(\overline{F}) \to \overline{F}^\times$ is Γ_F-equivariant, and in this case we obtain a homomorphism between the Γ_F-fixed points on the two sides, namely a homomorphism $G(F) \to F^\times$.

Moreover, it is a fact that $G(F)$ is Zariski dense in G (which holds for arbitrary connected linear algebraic groups over F), and hence an element $\chi \in X^*(G(F))$ is uniquely determined by the homomorphism $G(F) \to F^\times$ induced by itself. We sometimes call elements of $X^*(G(F))$ "algebraic characters on $G(F)$".

19. Local fields

We now recall the notion of a non-archimedean local field. Let F be a field. By a non-archimedean absolute value on F, we mean a function $|\cdot| : F \to \mathbb{R}_{\geq 0}$ satisfying the following axioms:

(1) $|xy| = |x||y|$.
(2) $|x| = 0$ if and only if $x = 0$.
(3) $|x + y| \leq \max(|x|, |y|)$.

Given $|\cdot|$, we can define a subring $\mathcal{O}_F$ of F by

$$\mathcal{O}_F = \{x \in F \mid |x| \leq 1.\}$$

This is a local ring, and its unique maximal ideal is

$$\mathfrak{m}_F = \{x \in F \mid |x| < 1\}.$$

It follows that

$$\mathcal{O}_F^\times = \{x \in F \mid |x| = 1\}.$$

We write k_F for the residue field $\mathcal{O}_F/\mathfrak{m}_F$.

We say that F together with a non-archimedean absolute value $|\cdot|$ is a *non-archimedean local field* (or simply a *local field*), if the following conditions are satisfied:

(1) (Completeness.) The space F is complete with respect to the topology defined by $|\cdot|$, i.e., every Cauchy sequence with respect to $|\cdot|$ converges in F.[12]

[12] Note that the notion of a Cauchy sequence depends only on a neighborhood basis of 0. Hence it indeed depends only on the topology, not on the metric.

(2) (Discreteness.) The image of the group homomorphism $|\cdot| : F^\times \to (\mathbb{R}_{>0}, \times)$ is a discrete subgroup. Note that every discrete subgroup of $\mathbb{R}_{>0}$ is cyclic. We take the unique $0 < \alpha < 1$ such that $|F^\times| = \alpha^{\mathbb{Z}}$.
(3) (Local compactness.) The topology on F defined by $|\cdot|$ is locally compact. Equivalently, the residue field k_F is finite.

Let $(F, |\cdot|)$ be a non-archimedean local field. Note that if we replace $|\cdot|$ by $|\cdot|^u$ for some $u \in \mathbb{R} - \{0\}$, then all the axioms are still satisfied, the topology on F remains the same, and the subring $\mathcal{O}_F \subset F$ stays the same. The generator α for $|F^\times|$ will be replaced by α^u. Hence we can always arrange that α is our favorite number between 0 and 1. The *canonical normalization* refers to the choice that $\alpha = q^{-1}$, where q is the cardinality of the finite field k_F. In the sequel we shall always take the canonical normalization, and denote the absolute value thus normalized by $|\cdot|_F$.

Definition 19.1. A *uniformizer* in F means any element $\pi \in F$ such that $|\pi|_F = q^{-1}$.

Example 19.2. $F = \mathbb{Q}_p, \mathcal{O}_F = \mathbb{Z}_p, k_F = \mathbb{F}_p$, and p is a uniformizer.

Lect.28, Apr 5

Then $\mathfrak{m}_F = \pi\mathcal{O}_F$ for any uniformizer π. For any $x \in F^\times$, we have

$$|x|_F = q^{-v_F(x)},$$

where $v_F(x)$ is the unique integer such that $x\pi^{-v_F(x)} \in \mathcal{O}_F^\times = \{x \in F \mid |x|_F = 1\}$. For instance $v_F(\pi) = 1$.

For every F-variety V, $V(F)$ has a canonical topology, which we call the non-archimedean topology: It is the coarsest topology such that for every Zariski open F-subvariety $U \subset V$, the subset $U(F)$ is open in $V(F)$, and for every F-variety morphism $f : U \to \mathbb{A}^n_F$, the induced map $U(F) \to \mathbb{A}^n_F(F) = F^n$ is continuous. Here F^n is equipped with the product topology coming from the natural topology on F (defined by $|\cdot|_F$).

On $V(F)$ we can also consider the *Zariski topology*, where open sets are precisely of the form $U(F)$ for Zariski open F-subvarieties U of V. Equivalently, this is the topology on $V(F)$ inherited from the Zariski topology on the F-scheme V, and also the same as the topology inherited from the Zariski topology on the $\overline{F}$-scheme $V(\overline{F})$.

Exercise 19.3. Show that the last two topologies on $V(F)$ are indeed the same.

Remark 19.4. By definition, the non-archimedean topology on $V(F)$ is finer than the Zariski topology.

Exercise 19.5. Let V be an affine F-variety. Then V is a closed F-subscheme of $\mathbb{A}^n_F = \operatorname{Spec} F[X_1, \cdots, X_n]$, and in particular $V(F) \subset \mathbb{A}^n_F(F) = F^n$. Show that the non-archimedean topology on $V(F)$ is the subspace topology inherited from F^n.

For a linear algebraic group G over F, if we realize G as a Zariski closed subgroup of GL_n, then $G(F)$ is closed in $\mathrm{GL}_n(F)$ when the two are equipped with the non-archimedean topology, and $\mathrm{GL}_n(F)$ is open in $M_{n\times n}(F) \cong F^{n^2}$. Under the non-archimedean topology, $G(F)$ is a td group and second countable. More specifically, a neighborhood basis of 1 in G consisting of compact open subgroups is given by the intersections $G(F) \cap 1 + \pi^k M_{n\times n}(\mathcal{O}_F), n \geq 1$. Note that $G(F)$ is totally disconnected, despite that G may be connected in the Zariski topology.

Fact 19.6. *If G is a connected reductive group over F, then $G(F)$ is unimodular.*

20. Unramified characters

Let G be a connected reductive group over F. For each $g \in G$ and $\chi \in X^*(G(F))$, by considering χ as a homomorphism $G(F) \to F^\times$ we obtain $v_F(\chi(g)) \in \mathbb{Z}$. Sending χ to $v_F(\chi(g))$ is a homomorphism $X^*(G(F)) \to \mathbb{Z}$, i.e., an element $H_G(g) \in X_*(G(F)) = \mathrm{Hom}(X^*(G(F)), \mathbb{Z})$. Moreover, the map $g \mapsto H_G(g)$ is a homomorphism

$$H_G : G(F) \longrightarrow X_*(G(F)).$$

We denote by ${}^0G(F)$ the kernel of H_G. In other words,

$${}^0G(F) = \{g \in G(F) \mid \forall \chi \in X^*(G(F)), \chi(g) \in \mathcal{O}_F^\times\}.$$

We denote by $\Lambda(G)$ the image of H_G.

Example 20.1. Let $G = \mathrm{GL}_n$. Then $X^*(G(F)) = X^*(G^{\mathrm{ab}})^{\Gamma_F} = X^*(G^{\mathrm{ab}}) = \mathbb{Z}$, where a generator corresponds to the homomorphism $\det : G(F) \to F^\times$. Hence ${}^0G(F) = \{g \in G(F) \mid \det g \in \mathcal{O}_F^\times\}$.

Example 20.2. If G is semi-simple, then $G(F) = {}^0G(F)$.

Lemma 20.3. *Let $\chi \in X^*(G(F))$. Consider the homomorphism $|\chi|_F : G(F) \to \mathbb{R}_{>0}$ that is the composition of $\chi : G(F) \to F^\times$ and $|\cdot|_F : F^\times \to \mathbb{R}_{>0}$. Then $|\chi|_F$ is smooth (i.e., locally constant), and its kernel contains all compact subgroups of $G(F)$.*

Proof. Obviously the second statement implies the first, since there exist compact open subgroups. By the definition of the non-archimedean topology, the map $\chi : G(F) \to F^\times$ continuous. Hence the image of any compact subgroup under $|\chi|_F$ is a compact subgroup of the discrete group $q^{\mathbb{Z}} \cong \mathbb{Z}$, which must be finite and hence trivial. □

Proposition 20.4. *Let G be a connected reductive group over F. Then ${}^0G(F)$ is open, contains all compact subgroups of $G(F)$, and contains $G_{\mathrm{der}}(F)$.*

Proof. Immediate from the previous lemma. □

Definition 20.5. We call a homomorphism $G(F) \to \mathbb{C}^\times$ *unramified* if it kills ${}^0G(F)$. Let $\chi(G)$ be the abelian group of all unramified homomorphisms $G(F) \to \mathbb{C}^\times$.

Example 20.6. For $G = \mathbb{G}_m$, a homomorphism $\mathbb{G}_m(F) = F^\times \to \mathbb{C}^\times$ is unramified if and only if it kills $\mathcal{O}_F^\times$. This agrees with the usual terminology in local class field theory.

By definition, $\chi(G)$ is identified with the group of homomorphisms from $\Lambda(G)$ to $\mathbb{C}^\times$, where $\Lambda(G)$ is the image of $H_G : G(F) \to X_*(G(F))$. Recall that $X^*(G) = X^*(G^{\mathrm{ab}})$ is a finite rank free $\mathbb{Z}$-module since G^{ab} is a torus. It follows that $X^*(G(F)), X_*(G(F)), \Lambda(G)$ are all finite rank free $\mathbb{Z}$-modules. There is a unique (up to canonical isomorphism) torus $\mathbb{U}$ over $\mathbb{C}$ such that $X^*(\mathbb{U}) = \Lambda(G)$. Concretely, $\mathbb{U} = \mathrm{Spec}\, \mathbb{C}[\Lambda(G)]$ where $\mathbb{C}[\Lambda(G)]$ is the group algebra of $\Lambda(G)$ (isomorphic to $\mathbb{C}[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]$ if we fix isomorphism $\Lambda(G) \xrightarrow{\sim} \mathbb{Z}^n$), and the group structure on $\mathbb{U}$ is given by the identifications $\mathbb{U} \xrightarrow{\sim} \mathrm{Spec}\, \mathbb{C}[X_1^{\pm 1}, \cdots, X_n^{\pm 1}] \xrightarrow{\sim} (\mathrm{Spec}\, \mathbb{C}[X^{\pm 1}])^n \cong \mathbb{G}_m^n$. (See the next exercise for more details.) Now note that a group homomorphism $\Lambda(G) \to \mathbb{C}^\times$ is the same thing as a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[\Lambda(G)] \to \mathbb{C}$, and the same thing as a $\mathbb{C}$-point of the torus $\mathbb{U}$. We conclude that

$$\chi(G) \cong \mathbb{U}(\mathbb{C}).$$

Thus we say that $\chi(G)$ has the structure of a complex torus. Lect.29, Apr 10

Exercise 20.7. Let F be a field of characteristic zero. Recall that we have an anti-equivalence $X^*(\cdot)$ from the category of split tori over F to the category of finite free $\mathbb{Z}$-modules (thought of as having the trivial Γ_F-action). Show that a quasi-inverse of this functor is given by $\Lambda \mapsto \operatorname{Spec} F[\Lambda]$, where $F[\Lambda]$ is the group algebra of Λ over F, and the group structure on $\operatorname{Spec} F[\Lambda]$ is given by $F[\Lambda] \to F[\Lambda] \otimes F[\Lambda], [\lambda] \mapsto [\lambda] \otimes [\lambda]$ for all $\lambda \in \Lambda$ representing $[\lambda] \in F[\Lambda]$. Show that if we fix an isomorphism $\Lambda \cong \mathbb{Z}^n$ and hence an isomorphism $F[\Lambda] \cong F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]$, the resulting isomorphism of F-varieties $\operatorname{Spec} F[\Lambda] \cong \mathbb{G}_m^n$ is a group variety isomorphism.

Lemma 20.8. *Let A be a split torus over a non-archimedean local field F. Then $X^*(A) = X^*(A(F)), X_*(A) = X_*(A(F))$. Moreover, $\Lambda(A) = X_*(A)$, and if we identify A with $\mathbb{G}_m^n$ so that $A(F) \cong (F^\times)^n$, the subgroup ${}^0A(F) \subset A(F)$ is given by $(\mathcal{O}_F^\times)^n$. In particular, ${}^0A(F)$ is compact.*

Proof. The first claim follows from the fact that Γ_F acts trivially on $X^*(A)$. Identify $A \cong \mathbb{G}_m^n$ and $X^*(A) \cong \mathbb{Z}^n$. Then $X_*(A)$ is also identified with $\mathbb{Z}^n$. Under these identifications, the map $H_A : A(F) \to X_*(A)$ is the map

$$(F^\times)^n \to \mathbb{Z}^n, \quad (x_1, \cdots, x_n) \mapsto (v(x_1), \cdots, v(x_n)).$$

The statements about $\Lambda(A)$ and ${}^0A(F)$ follow immediately. $\square$

21. The split component

Let F be a field of characteristic zero.

Fact 21.1. *Let Z be a diagonalizable group over F, and let Z^0 be the identity connected component of Z. Then Z has a maximal (with respect to containment) split sub-torus A. The inclusions $A \subset Z^0 \subset Z$ corresponds to surjections $X^*(Z) \to X^*(Z^0) \to X^*(A)$ described as follows: The group $X^*(Z^0)$ is the maximal free quotient of $X^*(Z)$ (i.e., the quotient by the torsion subgroup), equipped with the induced Γ_F-action. (Note that the Γ_F-action on $X^*(Z)$ automatically stabilizes the torsion subgroup.) The group $X^*(A)$ is the maximal free quotient of $X^*(Z^0)_{\Gamma_F}$, equipped with the trivial Γ_F-action. Here for any Γ_F-module M, we write M_{Γ_F} for the group of Γ_F-coinvariants, namely the quotient of M by the submodule generated by $\{gm - m \mid g \in \Gamma_F, m \in M\}$.*

Exercise 21.2. Show that $X^*(A)$ is also the maximal free quotient of $X^*(Z)_{\Gamma_F}$.

Definition 21.3. Let G be a connected reductive group over F. By the *split component of* G, we mean the maximal split sub-torus of Z_G. We denote it by A_G.

Example 21.4. Let G be the unitary group over F associated to a quadratic extension E/F and a hermitian space over E. Then $Z_G = Z_G^0 = (\operatorname{Res}_{E/F} \mathbb{G}_m)^1$, i.e., the kernel of the norm map $\mathrm{N}_{E/F} : \operatorname{Res}_{E/F} \mathbb{G}_m \to \mathbb{G}_m$. (In particular, $Z_G(F) = \{x \in E^\times \mid \mathrm{N}_{E/F}(x) = 1\}$.) Recall that under $X^*(\cdot)$, the norm map $\mathrm{N}_{E/F}$ corresponds to

$$X^*(\mathrm{N}_{E/F}) : X^*(\mathbb{G}_m) = \mathbb{Z} \longrightarrow X^*(\operatorname{Res}_{E/F} \mathbb{G}_m) = \mathbb{Z} \oplus \mathbb{Z}, \quad a \longmapsto (a, a).$$

Here Γ_F acts on $\mathbb{Z} \oplus \mathbb{Z}$ by its quotient $\operatorname{Gal}(E/F)$ and the non-trivial element $\sigma \in \operatorname{Gal}(E/F)$ acts by $(a, b) \mapsto (b, a)$. Hence

$$X^*(Z_G) = \operatorname{coker}(X^*(\mathrm{N}_{E/F})) \cong \mathbb{Z},$$

where Γ_F acts by the quotient $\operatorname{Gal}(E/F)$ and σ acts by $a \mapsto -a$. Then $X^*(Z_G)_{\Gamma_F} = \mathbb{Z}/2\mathbb{Z}$, and its maximal free quotient $X^*(A_G)$ is trivial. We conclude that A_G is trivial in this case.

Lemma 21.5. *Let M be a finitely generated abelian group together with a continuous Γ_F-action. Then the natural maps $M^{\Gamma_F} \hookrightarrow M \to M_{\Gamma_F}$ induce an isomorphism of $\mathbb{Q}$-vector spaces*

$$M^{\Gamma_F} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} M_{\Gamma_F} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Proof. Let Γ be a finite quotient of Γ_F such that the Γ_F-action on M factors through Γ. Then the inverse map is given by $[m] \mapsto (\sum_{g\in\Gamma} gm) \otimes |\Gamma|^{-1}$, for any $m \in M$ representing $[m] \in M_{\Gamma_F}$. □

Lect.30, Apr 12

Proposition 21.6. *The following statements hold.*

(1) *The natural map $X^*(G(F)) \to X^*(A_G)$ (given by restriction along $A_G(F) \to G(F)$ if we think of the elements as homomorphisms $G(F) \to F^\times$ and $A_G(F) \to F^\times$) is injective and the image has finite index.*
(2) *If F is a non-archimedean local field, then ${}^0A_G(F) = A_G(F) \cap {}^0G(F)$.*

Proof. From $G = Z_G G_{\mathrm{der}}$, we know that the map $Z_G \to G^{\mathrm{ab}}$ is surjective and with finite kernel (in the category of diagonalizable groups). Thus the map $X^*(G^{\mathrm{ab}}) = X^*(G) \to X^*(Z_G)$ is injective and the image has finite index. In particular, the map $X^*(G)^{\Gamma_F} \otimes \mathbb{Q} \to X^*(Z_G)^{\Gamma_F} \otimes \mathbb{Q}$ is an isomorphism. Now $X^*(A_G)$ is identified with the free quotient of $X^*(Z_G)_{\Gamma_F}$, and $X^*(G(F))$ is identified with $X^*(G)^{\Gamma_F}$. Hence by Lemma 21.5 the natural map $X^*(G(F)) \otimes \mathbb{Q} \to X^*(A_G) \otimes \mathbb{Q}$ is an isomorphism. Since $X^*(G(F))$ is torsion free, part (1) follows.

For (2), the direction "$\subset$" follows immediately from the fact that any $\chi \in X^*(G(F))$ restricts to an element of $X^*(A_G)$. Conversely, let $a \in A_G(F) \cap {}^0G(F)$ and let $\chi \in X^*(A_G)$. We need to show that $|\chi(a)| = 1$. Now by part (1) there exists n such that χ^n lies in the image of $X^*(G(F)) \to X^*(A_G)$. Hence $|\chi^n(a)| = 1$, and it follows that $|\chi(a)| = 1$. □

Corollary 21.7. *The natural map $X_*(A_G) \to X_*(G(F))$ is injective and the image has finite index.*

Proof. By Proposition 21.6 (1) or its proof, the natural map $X^*(G(F)) \otimes \mathbb{Q} \to X^*(A_G) \otimes \mathbb{Q}$ is an isomorphism. By dualizing, we know that hte natural map $X_*(A_G) \otimes \mathbb{Q} \to X_*(G(F)) \otimes \mathbb{Q}$ is an isomorphism. Since $X_*(A_G)$ is torsion free (being dual to the finite free $\mathbb{Z}$-module $X^*(A_G)$), the corollary follows. □

Lemma 21.8. *Let $S \to T$ be a surjection of tori over a non-archimedean local field F. Then the cokernel of $S(F) \to T(F)$ is finite.*

Sketch of proof. We have $X^*(T) \hookrightarrow X^*(S)$. First assume that the cokernel $X^*(S)/X^*(T)$ is torsion free. Then $\ker(S \to T)$ is a torus U. Associated to the short exact sequence $1 \to U \to S \to T \to 1$ we have a long exact sequence

$$1 \to U(F) \to S(F) \to T(F) \to \mathbf{H}^1(F, U) \to \cdots$$

Since F is non-archimedean local, by Tate–Nakayama duality, the Galois cohomology group $\mathbf{H}^1(F, U)$ is isomorphic to the torsion part of $X_*(U)_{\Gamma_F}$, and is hence finite.

In the general case, let T_1 be the quotient torus of S such that $X^*(T_1) \subset X^*(S)$ is the saturation of $X^*(T)$, i.e., $X^*(T_1) = \{x \in X^*(S) \mid \exists n \in \mathbb{Z}, nx \in X^*(T)\}$. (This makes sense since the saturation of $X^*(T)$ in $X^*(S)$ is indeed a Γ_F-stable $\mathbb{Z}$-submodule.) We have surjections $S \to T_1 \to T$. Then by the previous paragraph, the cokernel of $S(F) \to T_1(F)$ is finite. It remains to show that the cokernel of $T_1(F) \to T(F)$ is finite. Write ϕ for the surjection $T_1 \to T$. We have $n \cdot X^*(T_1) \subset X^*(T) \subset X^*(T_1)$ for some $n \in \mathbb{Z}$. Hence the n-th power map $[n] : T_1 \to T_1$ factors (uniquely) through $\phi : T_1 \to T$. Write $[n] = \psi \circ \phi$, where $\psi : T \to T_1$. Since $\ker \psi$ is finite, by the snake lemma applied to

$$\begin{array}{ccccccccc} 1 & \longrightarrow & 1 & \longrightarrow & T_1(F) & \xrightarrow{\mathrm{id}} & T_1(F) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle [n]} & & \\ 1 & \longrightarrow & \ker \psi(F) & \longrightarrow & T(F) & \xrightarrow{\psi} & \psi(T(F)) & \longrightarrow & 1 \end{array}$$

we have reduced the problem to showing that $[n] : T_1(F) \to T_1(F)$ has finite cokernel.

Let E/F be a finite Galois extension such that the Γ_F-action on $X^*(T_1)$ factors through $\mathrm{Gal}(E/F)$. Consider

$$M := \mathbb{Z}[\mathrm{Gal}(E/F)] \otimes_{\mathbb{Z}} X^*(T_1) = X^*(\mathrm{Res}_{E/F}\, \mathbb{G}_m) \otimes_{\mathbb{Z}} X^*(T_1),$$

equipped with Γ_F-action *only on the first factor via left translation*. Then we have a Γ_F-equivariant map $M \to X^*(T_1), [g] \otimes x \mapsto gx$. Note that M is just the direct sum of $r = \mathrm{rk}(X^*(T_1))$ many copies of $\mathbb{Z}[\mathrm{Gal}(E/F)] = X^*(\mathrm{Res}_{E/F}\, \mathbb{G}_m)$. Hence we have a closed F-embedding $T_1 \to L := \mathrm{Res}_{E/F}\, \mathbb{G}_m^r$. By the snake lemma, in order to show that $[n] : T_1(F) \to T_1(F)$ has finite cokernel, it suffices to show that $[n] : L(F)/T_1(F) \to L(F)/T_1(F)$ has finite kernel, and that $[n] : L(F) \to L(F)$ has finite cokernel. For the first statement, we have $L(F)/T_1(F) \subset (L/T_1)(\overline{F})$, and the latter is just a direct sum of copies of $\overline{F}^\times$, so clearly the n-th power map on that has finite kernel. For the second statement, it suffices to show that the n-th power map on $E^\times$ has finite cokernel. This is a basic property of non-archimedean local fields. □

Remark 21.9. Over $\mathbb{Q}$, we have a surjection of tori $[2] : \mathbb{G}_m \to \mathbb{G}_m$, but the map $[2] : \mathbb{Q}^\times \to \mathbb{Q}^\times$ has infinite cokernel. This does not happen for a non-archimedean local field.

Fact 21.10. *Let $\phi : G \to H$ be a map of linear algebraic groups over a non-archimedean local field F. Then $\phi(G(F)) \subset H(F)$ is closed with respect to the non-archimedean topology. If ϕ is surjective, then $\phi(F) : G(F) \to H(F)$ is an open map with respect to the non-archimedean topology, and in particular $\phi(G(F)) \subset H(F)$ is open (and hence closed).*

Sketch of proof. To show the second statement, we know that $\phi_{\overline{F}} : G_{\overline{F}} \to H_{\overline{F}}$ induces a surjection of $\overline{F}$-vector spaces $\mathrm{Lie}\, G_{\overline{F}} \to \mathrm{Lie}\, H_{\overline{F}}$ by the surjectivity of ϕ. It follows that ϕ induces a surjection of F-vector spaces $\mathrm{Lie}\, G \to \mathrm{Lie}\, H$. By a suitable version of the inverse function theorem, this implies that $\phi(F)$ is an open map (cf. [PR94, p. 133, Prop. 3.2]). The first statement follows from the second statement applied to the image of ϕ, which is a Zariski closed subgroup of H such that ϕ is a surjection from G to it. □

Lect.31, Apr 14

Definition 21.11. A torus T over a field F of characteristic zero is called *anisotropic*, if $X^*(T)^{\Gamma_F} = 0$.

Lemma 21.12. *Let T be a torus over F. Then there is a maximal anisotropic sub-torus $B \subset T$. Moreover, if A is the maximal split sub-torus of T, then the multiplication map $A \times B \to T$ is surjective with finite kernel.*

Proof. If $B' \subset T$ is anisotropic, then the map $X^*(T) \to X^*(B')$ is a surjection killing $X^*(T)^{\Gamma_F}$. On the other hand, $M = X^*(T)/X^*(T)^{\Gamma_F}$ is free and $M^{\Gamma_F} = 0$ (exercise, using that the Γ_F-action factors through a finite quotient). Therefore B exists and corresponds to the quotient $X^*(T) \to M$.

To show the second statement, it suffices to show that the map $X^*(T) \to (X^*(A) \otimes \mathbb{Q}) \oplus (X^*(B) \otimes \mathbb{Q})$ is an isomorphism. This follows immediately from the previous construction of B, and Lemma 21.5. $\square$

Example 21.13. Let $T = \mathrm{Res}_{E/F}\,\mathbb{G}_m$. Then the maximal anisotropic sub-torus of T is the norm-1 torus $(\mathrm{Res}_{E/F}\,\mathbb{G}_m)^1$.

Lemma 21.14. *Let T be an anisotropic torus over a non-archimedean local field F. Then $T(F)$ is compact.*

Proof. As in the proof of Lemma 21.8, we have a closed F-embedding $T \hookrightarrow \mathrm{Res}_{E/F}\,\mathbb{G}_m^n$. Since T is anisotropic, it must be contained in the maximal anisotropic subgroup of the right hand side, namely the n-th power of the norm-1 torus. Hence it suffices to show that $\{x \in E^\times \mid N_{E/F}x = 1\}$ is compact, which is clear. $\square$

Corollary 21.15. *Let T be a torus over a non-archimedean local field F. There is a finite-index, closed (and hence open) subgroup of $T(F)$ of the form $A(F)B(F)$, where $A \subset T$ is the maximal split sub-torus, and $B \subset T$ is the maximal anisotropic sub-torus. Moreover $B(F)$ is compact.*

Proof. Finite index follows from Lemma 21.8 and Lemma 21.12. Closedness follows from Fact 21.10. Compactness of $B(F)$ is Lemma 21.14. $\square$

Proposition 21.16. *Let G be a connected reductive group over a non-archimedean local field F. The subgroup ${}^0G(F)A_G(F) \subset G(F)$ is normal and of finite index. The group ${}^0G(F) \cap Z_G(F)$ is compact. If $Z_G(F)$ is compact, then ${}^0G(F) = G(F)$.*

Proof. Normality is clear. To show finite index, we have

$$G(F)/\,{}^0G(F)A_G \cong \mathrm{coker}(A_G(F) \to \Lambda(G)) \leq \mathrm{coker}(A_G(F) \to X_*(G(F))).$$

Now the map $A_G(F) \to X_*(G(F))$ factors as $A_G(F) \to X_*(A_G) \to X_*(G(F))$, where the first map is H_{A_G} and surjective by Lemma 20.8, and the second map is the natural map which is injective and the image has finite index by Corollary 21.7. Hence $G(F)/{}^0G(F)A_G$ is finite.

To show the second statement, first note that $Z_G^0(F)$ is of finite index in $Z_G(F)$ since the quotient injects into $(Z_G/Z_G^0)(F)$ which is finite. Also $Z_G^0(F)$ is closed in $Z_G(F)$ since it is Zariski closed, and therefore also open. By Lemmas 21.8 and 21.14 applied to the torus Z_G^0, we know that $Z_G^0(F)$, and hence $Z_G(F)$, has a finite-index open subgroup of the form $A_G(F)B(F)$, with $B(F)$ compact. It suffices to show that ${}^0G(F) \cap A_G(F)B(F)$ is compact. Since ${}^0G(F)$ contains all compact subgroups of $G(F)$, we have $B(F) \subset {}^0G(F)$. Hence it suffices to show that ${}^0G(F) \cap A_G(F)$ is compact. This follows from Proposition 21.6 (2) and Lemma 20.8.

We now show the third statement. If $Z_G(F)$ is compact, then $A_G(F)$ is compact, and hence $A_G(F) \subset {}^0G(F)$. Hence by the first statement, ${}^0G(F)$ is of finite index in $G(F)$. This implies that they are equal since the quotient $G(F)/{}^0G(F) = \Lambda(G)$ is torsion-free. $\square$

22. Inertia classes of representations

Let G be a connected reductive group over a non-archimedean local field F. Denote by $\Pi(G)$ the set of isomorphism classes of smooth representations of $G(F)$, and denote by $\mathrm{Irr}(G) \subset \Pi(G)$ the set of isomorphism classes of irreducible smooth representations. In general, given $(\pi, V) \in \mathcal{M}(G(F))$ and a one-dimensional $\omega \in \mathcal{M}(G(F))$, we have $\pi \otimes \omega \in \mathcal{M}(G(F))$. Concretely, if we think of ω as a smooth character $G(F) \to \mathbb{C}^\times$, then the representation $\pi \otimes \omega$ is on the same vector space V, and the map $G(F) \to \mathrm{Aut}(V)$ is given by $g \mapsto \omega(g) \cdot \pi(g)$. Clearly a subspace of V is a sub-representation with respect to π if and only if it is a sub-representation with respect to $\pi \otimes \omega$. Hence π is irreducible if and only if $\pi \otimes \omega$ is irreducible. Also, when ω is fixed, the isomorphism class of π and the isomorphism class of $\pi \otimes \omega$ determine each other. We conclude that the abelian group of all smooth characters $G(F) \to \mathbb{C}^\times$ acts on $\Pi(G)$, stabilizing $\mathrm{Irr}(G)$.

Recall that $\chi(G)$ is the abelian group of characters $\omega : G \to \mathbb{C}^\times$ killing ${}^0G(F)$, and each such ω is smooth. Thus we have a twisting action of $\chi(G)$ on $\Pi(G)$ and $\mathrm{Irr}(G)$.

Proposition 22.1. *For each $\pi \in \mathrm{Irr}(G)$, the stabilizer of π in $\chi(G)$ is finite.*

Proof. Recall that $\chi(G) \cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda(G), \mathbb{C}^\times)$, where $\Lambda(G)$ is the image of $H_G : G(F) \to X_*(G(F))$. Moreover, we have a commutative diagram

$$\begin{array}{ccccc} A_G(F) & \hookrightarrow & G(F) & & \\ \downarrow{\scriptstyle H_{A_G}} & & \downarrow{\scriptstyle H_G} & & \\ \Lambda(A_G) = X_*(A_G) & \longrightarrow & \Lambda(G) & \hookrightarrow & X_*(G(F)) \end{array}$$

Identify A_G with $\mathbb{G}_m^n$. Then the homomorphism $H_{A_G} : A_G(F) \to X_*(A_G)$ is identified with the map $(F^\times)^n \to \mathbb{Z}^n, (x_1, \cdots, x_n) \mapsto (v_F(x_1), \cdots, v_F(x_n))$. Thus if we fix a uniformizer $\varpi \in F$, then H_{A_G} has a section (which is a homomorphism) $(k_1, \cdots, k_n) \mapsto (\varpi^{k_1}, \cdots, \varpi^{k_n})$. Denote the image of this section by $C \subset A_G(F)$. By Schur's Lemma, C acts on π via a character $\lambda : C \to \mathbb{C}^\times$. Now if $\omega \in \chi(G)$ stabilizes π, then it must stabilize λ. On the other hand, ω sends λ to $\lambda + \delta$, where δ is the image of ω under the restriction map

$$\chi(G) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda(G), \mathbb{C}^\times) \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda(A_G), \mathbb{C}^\times) \cong \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{C}^\times).$$

We conclude that ω must lie in the kernel of the above map, i.e., ω is identified with a character $\Lambda(G)/\Lambda(A_G) \to \mathbb{C}^\times$. By Corollary 21.7, $\Lambda(G)/\Lambda(A_G)$ is finite, so there are only finitely many choices of such ω. □

Lect.32, Apr 17

We call each $\chi(G)$-orbit in $\Pi(G)$ an *inertia class*. The above shows that each inertia class in $\mathrm{Irr}(G)$ is a principal homogeneous space under $\chi(G)/\Delta$, where Δ is a finite subgroup of $\chi(G)$ (depending on the inertia class). Recall that $\chi(G)$ has the canonical structure of a complex torus, isomorphic to $\mathrm{Spec}\, \mathbb{C}[\Lambda(G)]$. We leave it as an exercise to show that $\chi(G)/\Delta$ also has the canonical structure of a complex torus. Thus each inertia class in $\mathrm{Irr}(G)$ has the canonical structure of a principal homogeneous space under a complex torus.

23. Parabolic and Levi subgroups

There is an inductive scheme of studying the representation theory of $G(F)$, for a connected reductive group G over a non-archimedean local field F. Namely, inside the F-algebraic group G there are distinguished subgroups called parabolic subgroups and Levi subgroups, the two classes being closely related to each other. The Levi subgroups are connected reductive groups of smaller dimensions than G. We shall define a smooth representation of $G(F)$ to be *supercuspidal* if, roughly speaking, it does not "interact" with representations of $M(F)$ for proper Levi subgroups M of G. The representation theory for $G(F)$ is then divided into understanding the supercuspidal representations and understanding those representations that are constructed inductively from $M(F)$ for proper Levi subgroups M. In general, the construction and classification of supercuspidal representations is indeed difficult. Nevertheless, they enjoy nice structural properties. Most notably, by a theorem of Harish-Chandra, they are exactly those representations that are "almost compact", in the sense that the restrictions to ${}^0G(F)$ are compact.

We now discuss parabolic and Levi subgroups in general. Let G be a connected reductive group over a field F. In the following, by a *subgroup* of linear algebraic group over F we always mean a (Zariski) closed F-subgroup variety.

Definition 23.1. By a *parabolic subgroup of* G, we mean a connected subgroup $P \subset G$ such that the homogeneous space G/P (with the natural algebraic variety structure) is projective.

Fact 23.2. *Let $P \subset G$ be a parabolic subgroup. Then there exists a subgroup $M \subset P$ such that $P = M \ltimes R_u P$. The choice of M is unique up to conjugation by elements of $(R_u P)(F)$. Such M is called a* Levi factor *of P.*

Remark 23.3. We have $M \cong P/R_u P$, from which we know M is connected reductive.

It is customary to write N_P or simply N for $R_u P$, when P is fixed. We shall also write simply $P = MN$ to indicate that we have made the choice of a Levi factor M of P. The decomposition $P = MN$ is usually referred to as a Levi decomposition.

Example 23.4. In $G = \mathrm{GL}_n$, the parabolic subgroups are those subgroups that are $G(F)$-conjugate to $P_{n_1,\cdots,n_k}$, where $(n_1, \cdots, n_k)$ is an ordered partition of n, and $P_{n_1,\cdots,n_k}$ is the group of invertible block upper triangular matrices with block sizes $n_1, \cdots, n_k$ on the diagonal (counting from the left upper corner to the right lower corner). In $P_{n_1,\ldots,n_k}$, the unipotent radical $N_{n_1,\cdots,n_k}$ is the subgroup with identities matrices on the block diagonal. A choice of Levi factor is given by $M_{n_1,\cdots,n_k}$ consisting of invertible block diagonal matrices.

Definition 23.5. By a *Levi subgroup of* G, we mean a Levi factor of a parabolic subgroup of G.

Fact 23.6. *There is a inclusion-reversing bijection from the set of all Levi subgroups of G to the set of all split tori inside G containing A_G, sending M to A_M. The inverse map sends A to the centralizer of A in G.*

Fact 23.7. *All maximal split tori in G (which automatically contain A_G) are conjugate to each other under $G(F)$. Equivalently, all minimal Levi subgroups of G are conjugate under $G(F)$.*

Fact 23.8. *All minimal parabolic subgroups of G are conjugate to each under $G(F)$. If we fix a minimal parabolic subgroup P_0, then every parabolic subgroup P of G is $G(F)$-conjugate to a unique parabolic subgroup P' containing P_0. (The normalizer of P in G is P, so the element of $G(F)$ conjugating P to P' is also unique up to multiplication by $P(F)$.) If we further fix a Levi decomposition $P_0 = M_0 N_0$, then for every parabolic P containing P_0, P has a unique Levi factor containing M_0.*

Definition 23.9. Fix a minimal parabolic subgroup and a Levi decomposition $P_0 = M_0 N_0$. Then parabolic subgroups of G containing P_0 are called standard. For a standard parabolic P, the unique Levi factor of it containing M_0 is called the standard Levi factor of P. A Levi subgroup of G is called standard, if it is the standard Levi factor of a standard parabolic subgroup.

It is clear from the above discussion that every parabolic subgroup (resp. Levi subgroup) of G is $G(F)$-conjugate to a standard one. Moreover, we have a surjective map from the set of standard parabolic subgroups to the set of standard Levi subgroups, sending P to the standard Levi factor of P, i.e., the unique Levi factor of P containing M_0. This map is also injective. There is a combinatorial classification of all standard parabolic subgroups (hence standard Levi subgroups) in terms of root systems. We will come back to this later when needed.

However, we have the following subtlety: Every parabolic subgroup is $G(F)$-conjugate to a unique standard one, but a Levi subgroup of G may be $G(F)$-conjugate to multiple standard ones. In other words, there may exist two distinct standard parabolic subgroups such that their standard Levi factors are $G(F)$-conjugate.

Definition 23.10. Two parabolic subgroups of G are called *associated*, if their Levi factors are $G(F)$-conjugate.

By the above discussion, association is an equivalence relation between parabolic subgroups that is weaker than $G(F)$-conjugation.

Lect.33, Apr 19

Fact 23.11. *Fix a parabolic subgroup $P \subset G$ and a Levi factor M. Then there is a unique parabolic subgroup $\bar{P} \subset G$, called* the opposite of P (with respect to M)*, with the property that $P \cap \bar{P} = M$. In this case,* $\mathrm{Lie}\, G = \mathrm{Lie}\, N_P \oplus \mathrm{Lie}\, M \oplus \mathrm{Lie}\, N_{\bar{P}}$. *The adjoint representation of A_M on* $\mathrm{Lie}\, G$ *(as an algebraic representation, by which we mean a homomorphism from a linear algebraic group to the general linear group of a vector space; in this case $A_M \to \mathrm{GL}(\mathrm{Lie}\, G)$) is trivial on* $\mathrm{Lie}\, M$ *and stabilizes* $\mathrm{Lie}\, N_P$ *and* $\mathrm{Lie}\, N_{\bar{P}}$. *Moreover, the representations of on* $\mathrm{Lie}\, N_P$ *and* $\mathrm{Lie}\, N_{\bar{P}}$ *decompose into one-dimensional representations (just as* any *algebraic representation of a split torus)*

$$\mathrm{Lie}\, N_P \cong \bigoplus_{\alpha \in R(A_M, N_P)} X_\alpha^{\oplus m(\alpha)}, \qquad \mathrm{Lie}\, N_{\bar{P}} \cong \bigoplus_{\alpha \in R(A_M, N_{\bar{P}})} X_\alpha^{\oplus m(\alpha)}$$

where $R(A_M, N_P), R(A_M, N_{\bar{P}})$ are subsets of $X^*(A_M) - \{0\}$, and X_α is the one-dimensional representation $A_M \to \mathrm{GL}_1$ corresponding to α. Moreover, we have

$$R(A_M, N_{\bar{P}}) = -R(A_M, N_P).$$

Definition 23.12. Elements of $R(A_M, N_P) \cup R(A_M, N_{\bar{P}})$ are called *roots*, whereas elements of $R(A_M, N_P)$ are called *positive roots*.

Example 23.13. Consider $G = \mathrm{GL}(V), \dim V = n$. The parabolic subgroups of G are in bijection with filtrations of V, where a filtration means a string of subspaces $0 \subset V_1 \subset \cdots \subset V_k = V$. The parabolic subgroup corresponding to a filtration is the stabilizer of that filtration in G. Let P be a parabolic subgroup, corresponding to a filtration $0 \subset V_1 \subset \cdots \subset V_k = V$. Then the set of Levi factors of P correspond to splittings of this filtration, i.e., choices of a complement of V_i in V_{i+1} for all i. Given such a splitting, we obtain a gradation of V: $V = W_1 \oplus W_2 \cdots \oplus W_k$, where $W_i \cong V_i / V_{i-1}$. The corresponding Levi factor of P is $\mathrm{GL}(W_1) \times \cdots \mathrm{GL}(W_k)$ embedded in $P \subset G$ in the obvious way. The Levi subgroups of G are in bijection with gradations of V up to permuting the indexing of the summands.

The choice of a minimal Levi subgroup is equivalent to the choice of a basis of V up to reordering and scaling its members. The choice of a minimal parabolic subgroup is equivalent to the choice of a complete flag, i.e., a filtration $0 \subset V_1 \subset \cdots \subset V_n = V$ with $\dim V_i = i$. The choice of a minimal parabolic P_0 *together with* the choice of a Levi factor M_0, is equivalent to the choice of an ordered basis of V up to scaling its members. Fix an ordered basis compatible with P_0 and M_0 in this sense. Then G is identified with GL_n, and P_0 is the group of invertible upper triangular matrices, M_0 is the group of invertible diagonal matrices, and $N_0 = N_{P_0}$ is the group of upper triangular matrices with 1's on the diagonal.

Standard parabolic subgroups (and hence Levi subgroups) are classified by ordered partitions of n. Given such a partition $n = n_1 + \cdots + n_k$, the corresponding standard parabolic $P = P_{n_1,\cdots,n_k}$ is the group of invertible block upper triangular matrices of block sizes $n_1, \cdots, n_k$. The standard Levi M is the group of invertible block diagonal matrices, and the unipotent radical of P is the group of block upper triangular matrices with identity matrices on the block diagonal. The opposite parabolic $\bar{P}$ is the group of invertible block lower triangular matrices, and its unipotent radical is the subgroup with identity matrices on the block diagonal. We have $M \cong \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_K}$, and $A_M \cong \mathbb{G}_m^k, \mathrm{diag}(z_1 I_{n_1}, \cdots, z_k I_{n_k}) \mapsto (z_1, \cdots, z_k)$.

We can canonically identify $\mathrm{Lie}\, N_P$ with the affine space of all block upper triangular matrices with 0's on the block diagonal. It has basis $\{E_{i,j}\}$ where (i,j) runs through the positions in the $n \times n$ matrix strictly above the block diagonal, and $E_{i,j}$ is the elementary matrix with 1 on the (i,j)-entry and 0 elsewhere. (E.g., $P = P_{1,2} \subset \mathrm{GL}_3$, then $\mathrm{Lie}\, N_P$ has basis $\{E_{12}, E_{13}\}$.) Each $\mathrm{Span}(E_{i,j})$ is an eigenspace for the action of $A_M \cong \mathbb{G}_m^k$, and the corresponding root is $(z_1, \cdots, z_k) \mapsto z_{u(i)}/z_{u(j)}$, where $u(i)$ is such that $n_1 + \cdots + n_{u(i)-1} < i \leq n_1 + \cdots + n_{u(u)}$, and similarly for $u(j)$. From this we see that $R(A_M, N_P) = \{(z_1, \cdots, z_k) \mapsto z_i/z_j \mid 1 \leq i < j \leq k\}$. Similarly, we check that

$$R(A_M, N_{\bar{P}}) = -R(A_M, N_P) = \{(z_1, \cdots, z_k) \mapsto z_j/z_i \mid 1 \leq i < j \leq k\}.$$

Two standard parabolics are associated if and only if the corresponding partitions are obtained from each other by re-ordering.

24. Decompositions involving compact open subgroups

Let G be a connected reductive group over a non-archimedean local field F. Fix a minimal parabolic and a Levi decomposition $P_0 = M_0 N_0$, and we shall use this to talk about standard parabolic subgroups and standard Levi subgroups.

Consider a standard parabolic $P = MN$ with standard Levi factor M. Recall that $X_*(A_M) = \Lambda(A_M) \hookrightarrow \Lambda(M) \hookrightarrow X_*(M(F))$, and both inclusions have finite index. (E.g., $G = \mathrm{GL}_n$, $M = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k}$. Then $\Lambda(M) = X_*(M(F))$, and the index of $\Lambda(A_M)$ in it is $n_1 \cdots n_k$.) Thus we have a canonical identification $X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R} \cong X_*(M(F)) \otimes_{\mathbb{Z}} \mathbb{R}$, and we denote them commonly by $\mathfrak{a}_M$. Inside the $\mathbb{R}$-vector space $\mathfrak{a}_M$, we have a cone $\mathfrak{a}_M^+$ defined by the inequalities $\langle \cdot, \alpha \rangle \geq 0$, for all $\alpha \in R(A_M, N)$. Let A_M^+ be the inverse image of this cone under $H_{A_M} : A_M(F) \to \mathfrak{a}_M$, and let M^+ be the inverse image of this cone under $H_M : M(F) \to \mathfrak{a}_M$. Similarly, we define A_M^{++} and M^{++} using the interior $\mathfrak{a}_M^{++}$ of the cone $\mathfrak{a}_M^+$, i.e., replacing the non-strict inequalities by the strict inequalities.

Explicitly, A_M^+ (resp. A_M^{++}) is the set of $a \in A_M(F)$ such that all eigenvalues of a acting on $\mathrm{Lie}\, N$ (which are in F) satisfy $v_F(\cdot) \geq 0$ (resp. > 0). However, the sets M^+ and M^{++} are harder to describe in this manner.

Example 24.1. Consider $P = P_{1,2} \subset G = \mathrm{GL}_3$. Then $A_M \cong \mathbb{G}_m^2 = \{\mathrm{diag}(z_1, z_2, z_2) \in \mathrm{GL}_3\}$ and $M \cong \mathrm{GL}_1 \times \mathrm{GL}_2 = \{\mathrm{diag}(g_1, g_2) \in \mathrm{GL}_3 \mid g_i \in \mathrm{GL}_i\}$. We have

$$R(A_M, N_P) = \{(z_1, z_2) \mapsto z_1/z_2\}.$$

Hence $A_M^+ = \{(z_1, z_2) \mid v_F(z_1) \geq v_F(z_2)\}$, and $M^+ = \{(g_1, g_2) \mid v_F(\det g_1) \geq \frac{1}{2} v_F(\det g_2)\}$. Indeed, we can identify $X_*(A_M)$ with $\mathbb{Z}^2$ such that the map $H_{A_M} : A_M(F) \to \mathbb{Z}^2$ is $(F^\times)^2 \ni (x_1, x_2) \mapsto (v_F(x_1), v_F(x_2))$, and we can identify $X_*(M(F))$ with $\mathbb{Z}^2$ such that the map $H_M : M(F) \to \mathbb{Z}^2$ is $\mathrm{GL}_1(F) \times \mathrm{GL}_2(F) \ni (g_1, g_2) \mapsto (v_F(\det g_1), v_F(\det g_2))$. But then the map $X_*(A_M) \hookrightarrow X_*(M(F))$ is not the identity map on $\mathbb{Z}^2$, but rather $(a, b) \mapsto (a, 2b)$.

Lect.34, Apr 21

The discussion below on compact open subgroups of $G(F)$ belongs to the subject of Bruhat–Tits theory, which has a vast literature. An entry point is [Tit79], with a more recent update in [Yu09]. See also the very recently published book [KP23]. Our discussion follows [Ren10, V.5].

Fact 24.2 (Iwasawa and Cartan decomposition)**.** *There exists a (maximal) compact open subgroup $K_0 \subset G(F)$ satisfying the following conditions:*

(1) *(Iwasawa decomposition.) We have $G(F) = P_0(F)K_0 = K_0P_0(F)$.*
(2) *For each standard parabolic $P = MN$ with standard Levi M, we have $P(F) \cap K_0 = (M(F) \cap K_0)(N(F) \cap K_0)$.*
(3) *(Cartan decomposition.) We have*

$$G(F) = \bigcup_{a \in M_0^+} K_0 a K_0.$$

Moreover, for $a, b \in M_0^+$, we have $K_0aK_0 = K_0bK_0$ if and only if a, b have the same image under $H_{M_0} : M_0(F) \to \Lambda(M_0) \subset X_*(M_0(F))$, i.e., $a \equiv b \mod {}^0M_0(F)$. In particular, if for every $\lambda \in \Lambda(M_0) \cap \mathfrak{a}^+_{M_0}$ we fix a lift $\tilde{\lambda} \in M_0(F)$, then we have

$$G(F) = \coprod_{\lambda \in \Lambda(M_0) \cap \mathfrak{a}^+_{M_0}} K_0 \tilde{\lambda} K_0.$$

Remark 24.3. We say that K_0 is special and adapted to M_0, or to A_{M_0}. (This notion depends only on the choice of a minimal Levi subgroup $M_0 \subset G$, not on P_0.)

Exercise 24.4. Let K_0 be as above. Using the Iwasawa decomposition and the fact that all minimal parabolic subgroups are $G(F)$-conjugate, show that $G(F) = P_0'(F)K_0 = K_0P_0'(F)$ for arbitrary minimal parabolic $P_0' \subset G$.

Remark 24.5. In the language of Bruhat–Tits theory, here K_0 is the maximal compact open subgroup associated with a special vertex in the apartment of A_{M_0}. From the modern point of view it is often more useful to replace K_0 by the parahoric subgroup associated to that vertex, which is a compact open subgroup of finite index in K_0. Then the Iwasawa and Cartan decompositions still hold, but in the Cartan decomposition the condition for two double cosets to be the same is more refined: Instead of a, b having the same image under H_{M_0}, we require that they have the same image under the Kottwitz map $M_0(F) \to \pi_1(M_0)$, which is a stronger condition.

Corollary 24.6. *For any parabolic subgroup $P \subset G$, $G(F)/P(F)$ with the quotient topology (induced by the non-archimedean topology on $G(F)$) is compact.*

Fact 24.7 (Iwahori decomposition)**.** *There exists a neighborhood basis of 1 in $G(F)$ consisting of compact open subgroups $K \subset G(F)$ satisfying the following properties:*

(1) *K is a normal subgroup of K_0 in the previous fact.*
(2) *(Iwahori decomposition.) For each standard $P = MN$, let $\bar{N}$ be the unipotent radical of the opposite parabolic. Then we have*

$$K = K_{\bar{N}} K_M K_N,$$

where $K_{\bar{N}} = K \cap \bar{N}(F), K_M = K \cap M(F), K_N = K \cap N(F)$. Moreover, for all $g \in M^+$, we have $gK_Ng^{-1} \subset K_N$ and $g^{-1}K_{\bar{N}}g \subset K_{\bar{N}}$. Also K_M is normal in $M(F)$.

(3) *Let $m \in M^{++}$. Then $m^l K_N m^{-l} \to 1$ for $l \to +\infty$ and $m^l K_N m^{-l} \to N(F)$ for $l \to -\infty$. Similarly for K_N replaced by $K_{\bar{N}}$, with "$l \to +\infty$" and "$l \to -\infty$" switched.*

(4) *More generally, for every $\epsilon > 0$, let $M^{++}(\epsilon)$ be the set of $m \in M^{++}$ satisfying the "ϵ-strengthened" inequalities used to define M^{++}. That is, in the definition of the cone $\mathfrak{a}_M^{++}$, we replace the inequalities $\langle \cdot, \alpha \rangle > 0$ by $\langle \cdot, \alpha \rangle > \epsilon, \forall \alpha \in R(A_M, N_P)$. Then for any open neighborhood U of 1 in $N(F)$ and every compact subset V in $N(F)$, there exists (a very large) ϵ such that for all $m \in M^{++}(\epsilon)$, we have $mK_N m^{-1} \subset U$ and $m^{-1} K_N m \supset V$.*

Remark 24.8. Here is the heuristics of why $m^l (K \cap N(F))^{-l} \to 1$ as $l \to +\infty$, for $m \in M^+ +$. By definition, all eigenvalues λ of the adjoint action of m on $\operatorname{Lie} N_P$ satisfy $v_F(\lambda) > 0$, i.e., $|\lambda|_F < 1$. Hence at the group level, the adjoint action of m on $N(F)$ is "shrinking" everything to 1.

Example 24.9. For $G = \mathrm{GL}_n$ and the standard choice of $P_0 = M_0 N_0$, we can take $K_0 = \mathrm{GL}_n(\mathcal{O}_F)$, and take the K's in the Iwahori decomposition to be the principal congruence subgroups $1 + \varpi^k M_n(\mathcal{O}_F), k \geq 1$, where ϖ is a uniformizer. The Cartan decomposition has the explicit form

$$\mathrm{GL}_n(F) = \coprod_{k_1 \geq \cdots \geq k_n} \mathrm{GL}_n(\mathcal{O}_F) \operatorname{diag}(\varpi^{k_1}, \cdots, \varpi^{k_n}) \mathrm{GL}_n(\mathcal{O}_F),$$

where ϖ is a uniformizer, and the disjoint union is over all n-tuples of non-increasing integers.

Exercise 24.10. Prove the three decompositions for GL_n, with the above choices of K_0, K. (For the Cartan decomposition, use the Smith normal form of matrices in $M_n(\mathcal{O}_F)$.)

Corollary 24.11. *For any parabolic subgroup $P \subset G$, $N_P(F)$ is the union of an increasing sequence of compact open subgroups.*

Lect.35, Apr 24

25. Parabolic induction and the Jacquet module

Let P be a subgroup of an abstract group G. Let (π, V) be a representation of P, i.e., V is a $\mathbb{C}$-vector space (of arbitrary dimension) and π is a homomorphism $P \to \operatorname{Aut}_{\mathbb{C}} V$. We define a representation $\operatorname{algInd}_P^G \pi$ of G, called the *algebraic induction of π from P to G*, as follows. The underlying vector space is the space of all functions $f : G \to V$ satisfying $f(pg) = \pi(p) f(g)$ for all $p \in p, g \in G$. The action is given by right translation, i.e., for $g \in G$ and $f \in \operatorname{algInd}_P^G \pi$,

$$g \cdot f : G \longrightarrow V, \quad h \longmapsto f(hg).$$

This construction is functorial in π in the following way: If we have a P-linear map $\phi : (\pi, V) \to (\pi', V')$, then we obtain a G-linear map $\operatorname{Ind}_P^G(\phi) : \operatorname{Ind}_P^G \pi \to \operatorname{Ind}_P^G \pi', f \mapsto \phi \circ f$.

Now let G be a connected reductive group over a non-archimedean local field F. Let $P \subset G$ be a parabolic subgroup. In the following we write $\mathcal{M}(G)$ for $\mathcal{M}(G(F))$ = the category of smooth $G(F)$-representations, and similarly for subgroups of G. We define a functor

$$\operatorname{Ind}_P^G : \mathcal{M}(P) \longrightarrow \mathcal{M}(G)$$

by sending $\pi \in \mathcal{M}(P)$ to the smooth part of the $G(F)$-representation $\operatorname{algInd}_{P(F)}^{G(F)} \pi$. Thus the underlying vector space of $\operatorname{Ind}_P^G \pi$ consists of functions $f : G(F) \to V$ such that $f(pg) = \pi(p) f(g)$ for all $p \in P(F), g \in G(F)$ and such that f is right invariant by some compact open subgroup of $G(F)$ (which depends on f).

Proposition 25.1. *The functor* Ind_P^G *preserves admissibility, and is exact.*

Proof. Let $(\pi, V) \in \mathcal{M}(P)$, and let $K \subset G(F)$ be a compact open subgroup. If $f \in (\mathrm{Ind}_P^G \pi)^K$, then f is a right K-invariant function $G(F) \to V$ satisfying $f(pg) = \pi(p)f(g)$ for all $p \in P(F), g \in G(F)$. Thus f is determined by $f(g_i)$ where $\{g_i\} \subset G(F)$ is a family of representatives of $P(F)\backslash G(F)/K$. By Corollary 24.6, $P(F)\backslash G(F)/K$ is finite, so we can write $\{g_i\}_{i=1}^n$. Now for each $i = 1, \cdots, n$, define $K_i := P(F) \cap g_i K g_i^{-1}$, which is a compact open subgroup of $P(F)$. For pK_i, we have

$$\pi(p)f(g_i) = f(pg_i) = f(g_i g_i^{-1} p g_i) = f(g_i),$$

where the last equality is because $g_i^{-1} p g_i \in K$. Hence we have obtained an injective linear map

$$(\mathrm{Ind}_P^G \pi)^K \longrightarrow \bigoplus_{i=1}^n V^{K_i}, \quad f \longmapsto (f(g_i))_{i=1}^n. \tag{25.1}$$

It is easy to see that this map is also bijective.

Now if (π, V) is admissible, then the RHS of (25.1) is finite dimensional, and so $\mathrm{Ind}_P^G \pi$ is admissible since K in the above discussion is arbitrary.

We now show that Ind_P^G preserves is exact. Let $0 \to V_1 \to V_2 \to V_3 \to 0$ be a short exact sequence in $\mathcal{M}(P)$. Fix a compact open subgroup K, and let $\{g_i\}$ be as above. It suffices to show that $0 \to (\mathrm{Ind}_P^G V_1)^K \to (\mathrm{Ind}_P^G V_2)^K \to (\mathrm{Ind}_P^G V_3)^K \to 0$ is exact. Using the isomorphism (25.1) for V_1, V_2, V_3 respectively, we reduce to showing that for each $i = 1, \cdots, n$, the sequence $0 \to V_1^{K_i} \to V_2^{K_i} \to V_3^{K_i} \to 0$ is exact. Only the surjectivity of $V_2^{K_i} \to V_3^{K_i}$ is unclear. For this, use the averaging operator $\pi_2(e_{K_i})$. □

Proposition 25.2 (Frobenius reciprocity)**.** *The restriction functor* $\mathrm{Res}_P^G : \mathcal{M}(G) \to \mathcal{M}(P)$ *is left adjoint to* Ind_P^G*. That is, given* $(\sigma, W) \in \mathcal{M}(G)$ *and* $(\pi, V) \in \mathcal{M}(P)$*, there is a canonical bijection*

$$\Phi : \mathrm{Hom}_{P(F)}(\sigma, \pi) \xrightarrow{\sim} \mathrm{Hom}_{G(F)}(\sigma, \mathrm{Ind}_P^G \pi)$$

which is functorial in σ *and* π*.*

Proof. Given $\alpha \in$ LHS, we define $\Phi(\alpha) \in$ RHS as follows. For $w \in (\sigma, W)$, define $\Phi(\alpha)(w) \in \mathrm{Ind}_P^G \pi$ to be the function $f : G(F) \to V, g \mapsto \alpha(gw)$. For $p \in P(F)$, we have $f(pg) = \alpha(pgw) = p \cdot \alpha(gw) = p \cdot f(g)$, where the second equality is because α is $P(F)$ linear. Hence f indeed lies in $\mathrm{algInd}_P^G \pi$. Moreover, since (σ, W) is smooth, f is clearly right K-invariant for some compact open subgroup K (as long as K fixes w). Hence f lies in $\mathrm{Ind}_P^G \pi$.

We now need to check that $\Phi(\alpha)$ is $G(F)$-linear. Thus let $g_0 \in G$. Then $\Phi(\alpha)(g_0 w) : g \mapsto \alpha(g g_0 w) = f(g g_0) = (g_0 \cdot f)(g)$.

To define Φ^{-1}, let $\beta \in$ RHS. Define $\Phi^{-1}(\beta) : (\sigma, W) \to (\pi, V), w \mapsto \beta(w)(1)$. To check it is $P(F)$-linear, let $p \in P(F)$. Then $\Phi^{-1}(\beta)(pw) = \beta(pw)(1) = (p \cdot \beta(w))(1) = \beta(w)(1 \cdot p) = \beta(w)(p \cdot 1) = p \cdot [\beta(w)(1)]$.

We leave it as an exercise to show that Φ and Φ^{-1} are indeed inverse maps, and that Φ is functorial in σ, π. □

Let P be a parabolic subgroup of G, and fix a Levi decomposition $P = MN$. We have a functor $\mathcal{M}(M) \to \mathcal{M}(P)$ as follows: For each $(\pi, V) \in \mathcal{M}(M)$, we extend the $M(F)$-action on V trivially along $N(F)$ to get a $P(F)$-action on V; equivalently, we identify

$M(F)$ as the quotient $P(F)/N(F)$, and then pull-back the $M(F)$-representation to a $P(F)$-representation. We then consider the composite functor

$$\mathcal{M}(M) \longrightarrow \mathcal{M}(P) \xrightarrow{\mathrm{Ind}_P^G} \mathcal{M}(G).$$

By abuse of notation, we still denote this functor by Ind_P^G. This functor is called the *parabolic induction*, and it obviously still preserves admissibility and exact sequences as in Proposition 25.1 (since the functor $\mathcal{M}(M) \to \mathcal{M}(P)$ trivially has these properties).

Remark 25.3. One can also define $\mathrm{Ind}_M^G : \mathcal{M}(M) \to \mathcal{M}(G)$ by taking the smooth part of $\mathrm{algInd}_{M(F)}^{G(F)}$. This functor is *different* from $\mathrm{Ind}_P^G : \mathcal{M}(M) \to \mathcal{M}(G)$, and will never be considered in the sequel. For instance, for a compact open subgroup K of $G(F)$, $(\mathrm{Ind}_M^G(\mathrm{triv}))^K$ is the space of all functions $M(F)\backslash G(F)/K \to \mathbb{C}$, and this is in general infinite dimensional. Thus Ind_M^G does not preserve admissibility.

In the reverse direction, we consider the following composite functor

$$J_N : \mathcal{M}(G) \xrightarrow{\mathrm{Res}_P^G} \mathcal{M}(P) \xrightarrow{(\cdot)_N} \mathcal{M}(M),$$

where the first functor is restriction from $G(F)$ to $P(F)$, and the second is taking $N(F)$-coinvaiants. Let us explain the second functor: Let $(V,\pi) \in \mathcal{M}(P)$. Then

$$V(N) := \mathrm{span}\{\pi(n)v - v \mid n \in N(F), v \in V\}$$

is a sub-representation, and so the coinvariant space

$$V_N := V/V(N)$$

is naturally a smooth $P(F)$-representation. Moreover, this $P(F)$-representation factors through the quotient $P(F)/N(F) \cong M(F)$. We thus have the functor $(\cdot)_N : \mathcal{M}(P) \to \mathcal{M}(M)$. We call $J_N = (\cdot)_N \circ \mathrm{Res}_P^G$ the *Jacquet module functor*. For $\pi \in \mathcal{M}(G)$, the representation $J_N(\pi) \in \mathcal{M}(M)$ is called the *Jacquet module* of π with respect to $P = MN$.

Proposition 25.4. *The functor $J_N : \mathcal{M}(G) \to \mathcal{M}(M)$ is left adjoint to $\mathrm{Ind}_P^G : \mathcal{M}(M) \to \mathcal{M}(G)$.*

Proof. Let $\sigma \in \mathcal{M}(G), \pi \in \mathcal{M}(M)$. By Proposition 25.2, we have

$$\mathrm{Hom}_{P(F)}(\sigma, \pi) \cong \mathrm{Hom}_{G(F)}(\sigma, \mathrm{Ind}_P^G \pi).$$

Here on the two sides π is viewed as in $\mathcal{M}(P)$. But since by definition $N(F)$ acts trivially on π, the LHS is canonically identified with $\mathrm{Hom}_{M(F)}(\sigma_N, \pi) = \mathrm{Hom}_{M(F)}(J_N(\sigma), \pi)$. □

Lect.36, Apr 26

Lemma 25.5. *Let (V, π) be a smooth representation of $N(F)$, and as before let $V(N) = \mathrm{span}\{\pi(n)v - v \mid n \in N(F), v \in V\}$. Then $V(N) = \bigcup_K \ker \pi(e_K)$, where K runs over compact open subgroups of $N(F)$.*

Proof. We use the fact that $N(F)$ is the union of an increasing sequence of compact open subgroups, as in Corollary 24.11. Let $v \in V(N)$. Then $v = \sum_{i=1}^r n_i v_i - v_i$ for $n_i \in N(F), v_i \in W$. Find a compact open subgroup $K \subset N(F)$ such that all n_i are in K. Then $\pi(e_K) n_i v_i = \pi(e_K) v_i$, and hence $\pi(e_K)v = 0$. Conversely, let $v \in V$ such that $\pi(e_K)v = 0$ for some compact open subgroup $K \subset N(F)$. Recall that $\pi(e_K)v$ is the average over the finite K-orbit of v. Say the K-orbit of v is $\{k_1 v, \cdots, k_r v\}$. Then $\sum_i k_i v = 0$, and hence

$$v = \frac{-1}{r} \sum_{i=1}^r (k_i v - v) \in V(N).$$

□

Proposition 25.6. *The functor $J_N : \mathcal{M}(G) \to \mathcal{M}(M)$ is exact.*

Proof. In general, taking co-invariants is right exact, so we only need to check that J_N preserves injections. Thus for $V \in \mathcal{M}(G)$ and a sub-representation $W \subset V$, we need $W \cap V(N) = W(N)$. This follows immediately from Lemma 25.5. □

26. Normalization and transitivity

Let $P = MN$ be a parabolic subgroup of G. Let $\delta_P : P(F) \to \mathbb{R}_{>0}$ be the modulus character on $P(F)$.

Fact 26.1. *The character δ_P factors through $P(F)/N(F) \cong M(F)$. Moreover, for $m \in M(F)$, $\delta_P(m) = |\det(\mathrm{Ad}(m) \mid \mathrm{Lie}\, N)|_F^{-1}$. Here $|\cdot|_F$ is the canonical normalization of the absolute value on F.*

If for $\pi \in \mathcal{M}(M)$ we denote by π' its image in $\mathcal{M}(P)$, then $(\pi \otimes \delta_P^{-1/2})' \cong \pi' \otimes \delta_P^{-1/2}$. Here on the LHS δ_P is viewed as a character on $M(F)$. In the following we write both of them simply as $\pi \otimes \delta_P^{-1/2}$. We define the *normalized parabolic induction functor* to be

$$i_P^G : \mathcal{M}(M) \longrightarrow \mathcal{M}(G), \quad \pi \longmapsto \mathrm{Ind}_P^G(\pi \otimes \delta_P^{-1/2}).$$

Similarly, we define the *normalized Jacquet module functor* to be

$$r_P^G : \mathcal{M}(G) \longrightarrow \mathcal{M}(M), \quad \pi \longmapsto J_N(\pi) \otimes \delta_P^{1/2}.$$

Corollary 26.2. *The functor r_P^G is left adjoint to i_P^G. Both are exact, and i_P^G preserves admissible representations.*

The following is the main motivation for introducing i_P^G in place of Ind_P^G.

Fact 26.3. *The functor i_P^G is compatible with taking contragredient (i.e., $i_P^G(\pi^\vee) \cong (i_P^G\pi)^\vee$), and it preserves unitary representations.*

Remark 26.4. We explain the rough idea for compatibility with contragredient; the perseverance of unitarity is similar. Let us try and define a pairing between $\mathrm{Ind}_P^G \pi$ and $\mathrm{Ind}_P^G(\pi^\vee)$ in order to identify the latter with the contragredient of the former. If $f \in \mathrm{Ind}_P^G \pi$ and $f' \in \mathrm{Ind}_P^G(\pi^\vee)$, then we obtain a function $F : G \to \mathbb{C}, g \mapsto \langle f(g), f'(g)\rangle$. This function is left $P(F)$-invariant. We would like to define $\langle f, f'\rangle$ to be the "integration of F over the compact set $P(F)\backslash G(F)$". However, there is not a well-behaved right $G(F)$-invariant functional "$\int_{P(F)\backslash G(F)}$" that takes a smooth function $P(F)\backslash G(F) \to \mathbb{C}$ to a complex number. The problem is precisely that $G(F)$ is unimodular while $P(F)$ is not. Instead, we have such a right $G(F)$-invariant integral if the integrand is a function $F : G(F) \to \mathbb{C}$ that is not left $P(F)$-invariant but rather satisfies $F(pg) = \delta_P^{-1}(p)F(g)$ for $p \in P(F), g \in G(F)$. Now if we take $f \in i_P^G\pi$ and $f' \in i_P^G(\pi^\vee)$, then the function $F(g) = \langle f(g), f'(g)\rangle$ does satisfy $F(pg) = \delta_P^{-1}(p)F(g)$.

Fact 26.5 (Transitivity)**.** *Let $P = MN, Q = LU$ be standard parabolic subgroups of G with standard Levi decompositions. Suppose that $P \subset Q$. Then $M \subset L$ and $N \supset U$. Moreover, $P \cap L$ is a parabolic subgroup of L and M is a Levi factor of it. We have $i_P^G = i_Q^G \circ i_{P\cap L}^L$, and $r_P^G = r_{L\cap P}^L \circ r_Q^G$.*

27. Supercuspidal representations

Definition 27.1. We call $(\pi, V) \in \mathcal{M}(G)$ *supercuspidal*, if for all proper parabolic subgroups $P \subset G$, we have $V_{N_P} = 0$.

Remark 27.2. Clearly this is equivalent to the condition that for any *standard* proper parabolic $P = MN$, we have $r_P^G \pi = 0$, or equivalently, $J_N \pi = 0$. By adjunction, this is equivalent to the condition that for all $\sigma \in \mathcal{M}(M)$, we have $\mathrm{Hom}_{G(F)}(\pi, i_P^G \sigma) = 0$, and equivalently for Ind_P^G in place of i_P^G. Thus π is supercuspidal if and only if there is no non-zero $G(F)$-map from π into any properly parabolically induced representations.

Theorem 27.3 (Harish-Chandra)**.** *Let $(\pi, V) \in \mathcal{M}(G)$. The following statements are equivalent.*

1. *(π, V) is supercuspidal.*
2. *The restriction of (π, V) to ${}^0G(F) \subset G(F)$ is a compact representation.*
3. *Every matrix coefficient of (π, V) is compactly supported modulo $Z_G(F)$. That is, the support has compact image in $G(F)/Z_G(F)$.*

Remark 27.4. If $Z_G(F)$ is compact, then ${}^0G(F) = G(F)$ by Proposition 21.16. Thus in this case supercuspidal representations are the same as compact representations. The reason that we do not want to make this assumption in our theory is that even if we do so, the proper Levi subgroups M of G (if they exist) will no longer satisfy this assumption, because A_M is non-trivial.

Proof. Since ${}^0G(F)$ is open in $G(F)$, a vector in V^* is smooth with respect to the $G(F)$-action if and only if it is smooth with respect to the ${}^0G(F)$-action. Hence the set of matrix coefficients for $\pi|_{{}^0G(F)}$ is obtained by restricting the matrix coefficients for π to ${}^0G(F)$. Hence the equivalence of (2) and (3) would follow if we show that a smooth function $G(F) \to \mathbb{C}$ is compactly supported modulo $Z_G(F)$ if and only if its restriction to ${}^0G(F)$ is compactly supported. The "only if" direction is true because $\ker({}^0G(F) \to G(F)/Z_G(F)) = Z_G(F) \cap {}^0G(F)$ is compact (Proposition 21.16). The "if" direction is true because $Z_G(F)\,{}^0G(F)$ has finite index in $G(F)$ (Proposition 21.16). Lect.37, Apr 28

"(2) $\Rightarrow$ (1)". Let $P = MN$ be a standard proper parabolic, and we show that $V_N = 0$. Let $v \in V$. We need to show that $v \in V(N)$.

Let K be a compact open subgroup of $G(F)$ satisfying Fact 24.7 and fixing v. For any compact subgroup L of $G(F)$, we write $\gamma(L) : V \to V$ for the map sending a vector to the average of the (finite) L-orbit of that vector. (If L is open, then $\gamma(L)$ is just $\pi(e_L)$.) By Lemma 25.5, in order to show that $v \in V(N)$ it suffices to find a compact open subgroup $U \subset N(F)$ such that $\gamma(U)v = 0$.

We claim that ${}^0G(F) \cap A_M^{++} \neq \emptyset$. Indeed, let $s \in A_M^{++}$. Then any positive power of s still lies in A_M^{++}, while one of them satisfies $H_G(s^n) \in \Lambda(A_G) \subset \Lambda(G)$, since $\Lambda(A_G)$ is of finite index in $\Lambda(G)$. Pick $a \in A_G(F)$ such that $H_G(a) = H_{A_G}(a) = H_G(s^n)$. Then $t := a^{-1}s^n \in {}^0G(F) \cap A_M^{++}$.

Let $t \in {}^0G(F) \cap A_M^{++}$. The set $\{H_{A_M}(t^n) \mid n \geq 1\}$ is unbounded in $\mathfrak{a}_M$. The map H_{A_M} is continuous (with respect to the real vector space topology on $\mathfrak{a}_M$), so $\{t^n \mid n \geq 1\}$ is not contained in any compact subset of $A_M(F)$. But $\pi|_{{}^0G(F)}$ is a compact representation, so the map ${}^0G(F) \to V, g \mapsto \pi(e_K)\pi(g)v$ is compactly supported. (Recall that $K \subset {}^0G(F)$.) Therefore there exists $n \geq 1$ such that $\pi(e_K)\pi(t^n)v = 0$. In particular $\pi(e_{t^{-n}Kt^n})v = 0$. Now

$$t^{-n}Kt^n = (t^{-n}K_N t^n)(t^{-n}K_M t^n)(t^{-n}K_{\bar{N}} t^n),$$

where $t^{-n}K_N t^n, t^{-n}K_M t^n, t^{-n}K_{\bar{N}}t^n$ are compact open subgroups of $N(F), M(F), \bar{N}(F)$ respectively. By Exercise 27.6 below, we have

$$0 = \pi(e_{t^{-n}Kt^n})v = \gamma(t^{-n}Kt^n)v = \gamma(t^{-n}K_N t^n)\gamma(t^{-n}K_M t^n)\gamma(t^{-n}K_{\bar{N}}t^n)v.$$

Now $t^{-n}K_{\bar{N}}t^n \subset K_{\bar{N}} \subset K$ (because $\mathrm{Ad}(t^{-1})$ is "shrinking" on $N(F)$) and $t^{-n}K_M t^n = K_M \subset K$ because t is central in $M(F)$. Hence both $\gamma(t^{-n}K_M t^n)$ and $\gamma(t^{-n}K_{\bar{N}}t^n)$ fix v. Therefore

$$\gamma(t^{-n}K_N t^n)v = 0.$$

But $t^{-n}K_N t^n$ is a compact open subgroup of $N(F)$, so we are done.

"(1) $\Rightarrow$ (2)". Suppose (2) is false. Then there exist $v \in V$ and $\lambda \in V^\vee$ such that the function $\phi : {}^0G(F) \to \mathbb{C}, g \mapsto \langle \lambda, \pi(g)v\rangle$ is not compactly supported on ${}^0G(F)$.[13] Let $\{m_i\}_{i\geq 1}$ and $P = MN$ be as in Lemma 27.5, applied to the support of ϕ. Since ϕ is bi-invariant under a compact open subgroup K' of $G(F)$ (as long as K' fixes v and λ), and since we may assume that K' is normal and of finite index in K_0, we can find $r, s \in K_0$ such that up to extracting a sub-sequence we have $\phi(rm_i s) \neq 0$ for all i. Replacing v, λ by $sv, r^{-1}\lambda$, we then have $\phi(m_i) \neq 0$ for all i. We now find a sufficiently small compact open subgroup $K \subset G(F)$ which fixes v and λ and satisfies Fact 24.7.

Since $V_N = 0$, by Lemma 25.5 there is a compact open subgroup $U \subset N(F)$ satisfying $\gamma(U)v = 0$. For sufficiently large i, we have $m_i^{-1}K_N m_i \supset U$ by Fact 24.7 (4), so we have $\gamma(m_i^{-1}K_N m_i)v = 0$. In particular, $\gamma(K_N)\pi(m_i)v = 0$. But λ is fixed by K and in particular fixed by K_N, so we have

$$0 \neq \phi(m_i) = \langle \lambda, \pi(m_i)v\rangle = \langle \lambda, \gamma(K_N)\pi(m_i)v\rangle = 0,$$

a contradiction. □

Lemma 27.5. *Let C be a non-compact closed subset of ${}^0G(F)$. Let K_0 be as in Fact 24.2. Then there is a standard parabolic $P = MN$ and a sequence $\{m_i\}_{i\geq 1}$ in $M^{++} \cap {}^0G(F)$ such that $m_i \in M^{++}(\epsilon_i)$ for a sequence $\epsilon_i \in \mathbb{R}_{>0}$ tending to $+\infty$ (see Fact 24.7 (4) for the notation) and such that C meets $K_0 m_i K_0$ for all i.*

Proof. Let $P_0 = M_0 N_0$ be the fixed minimal parabolic with M_0 the fixed Levi factor. We use the Cartan decomposition

$$G(F) = \bigcup_{m\in M_0^+} K_0 m K_0.$$

Note that since $K_0 \subset {}^0G(F)$, a double coset $K_0 m K_0$ as above is a subset of ${}^0G(F)$ if and only if $m \in M_0^+ \cap {}^0G(F)$, if and only if $K_0 m K_0 \cap {}^0G(F) \neq \emptyset$.

Since C is non-compact and closed, it is not contained in any compact set, and so it meets $K_0 m_i K_0$ for a sequence $\{m_i\}_{i\geq 1} \subset M_0^+ \cap {}^0G(F)$ such that $\{m_i\}$ is not contained in any compact subset of $G(F)$. Since $\Lambda(A_{M_0})$ is of finite index in $\Lambda(M_0)$, up to extracting a sub-sequence we may assume that all m_i are of the form $m_0 a_i$ for some fixed $m_0 \in M_0^+ \cap {}^0G(F)$ and for $a_i \in A_{M_0}(F) \cap {}^0G(F)$. We now use without proof the fact that

$$[\bigcap_{\alpha\in R(A_{M_0},N_0)} \ker(\alpha : A_{M_0} \to \mathbb{G}_m)]^0 = A_G. \tag{27.1}$$

[13]This function is $g \mapsto \phi_{v,\lambda}(g^{-1})$ where $\phi_{v,\lambda}$ is the matrix coefficient associated with v, λ.

Here $[\cdot]^0$ denotes taking the identity connected component under the Zariski topology. In particular, the kernel of the map

$$\bigoplus_{\alpha \in R(A_{M_0}, N_0)} \alpha : A_{M_0}(F) \cap {}^0G(F) \to (F^\times)^{\oplus R(A_{M_0}, N_0)}$$

contains $A_G(F) \cap {}^0G(F)$ with finite index, and is therefore compact by Proposition 21.16. Since $\{a_i\}_{i\geq 1}$ is not contained in any compact set, there exists $\alpha \in R(A_{M_0}, N_0)$ such that $\{\alpha(a_i)\}_{i\geq 1}$ does not lie in a compact subset of $F^\times$. We have a (split) short exact sequence

$$1 \to \mathcal{O}_F^\times \to F^\times \xrightarrow{v_F} \mathbb{Z} \to 0$$

with $\mathcal{O}_F^\times$ compact. Hence $v_F(\alpha(a_i))$ takes infinitely many different values. But they are bounded from below (as $m_0 a_i \in M_0^+$). Hence up to extracting a sub-sequence, we may assume that $v_F(\alpha(a_i)) \to +\infty$. Now $R(A_{M_0}, N_0)$ has a minimal subset Δ, called the set of simple roots, with the property that all elements of $R(A_{M_0}, N_0)$ are sums of elements of Δ. We may assume that $\alpha \in \Delta$. Then by the classification of standard parabolic subgroups, there exists a (maximal proper) standard parabolic subgroup $P = MN$ such that $R(A_M, N)$ consists of only positive multiples of $\alpha|_{A_M}$. Then up to extracting a sub-sequence, we have $m_i \in M^{++} \cap {}^0G(F)$ and $m_i \in M^{++}(\epsilon_i)$ for all $i \geq 1$. □

Exercise 27.6. In general, for arbitrary two compact subgroups L_1, L_2 of $G(F)$ such that L_2 normalizes L_1, we have a compact subgroup $L_1 L_2 = L_2 L_1$ of $G(F)$ and we have

$$\gamma_{L_1 L_2} = \gamma_{L_2} \circ \gamma_{L_1} = \gamma_{L_1} \circ \gamma_{L_2}.$$

Exercise 27.7. Prove (27.1) for $G = \mathrm{GL}_n$.

Exercise 27.8. For $G = \mathrm{GL}_n$ and the standard choices of P_0, M_0, we have $A_{M_0} = M_0 =$ the diagonal torus, $R(A_{M_0}, N_0) = \{\alpha_{i,j} \mid 1 \leq i < j \leq n\}$ where $\alpha_{i,j}(\mathrm{diag}(z_1, \cdots, z_n)) = z_i/z_j$. Moreover $\Delta = \{\alpha_{1,2}, \alpha_{2,3}, \cdots, \alpha_{n-1,n}\}$, and the standard parabolic corresponding to $\alpha_{i,i+1}$ is $P_{i,n-i}$.

Lect.38, May 1

Lemma 27.9. *The functor $r_P^G : \mathcal{M}(G) \to \mathcal{M}(M)$ sends a finitely generated representation to a finitely generated representation.*

Proof. Let $(\pi, V) \in \mathcal{M}(G)$ be generated by $v_1, \cdots, v_n$. Let $K \subset G(F)$ be a compact open subgroup fixing all v_i. Let $g_1, \cdots, g_k \in G(F)$ be representatives of $P(F)\backslash G(F)/K$ (which is finite). Then as a $P(F)$-representation, (π, V) is generated by $\{g_j v_i\}$. □

Proposition 27.10. *Let $(\pi, V) \in \mathcal{M}(G)$ be irreducible. Then there is a standard parabolic $P = MN$ such that the following are satisfied:*

1. *$r_P^G \pi \neq 0$ and is supercuspidal.*
2. *There exists an irreducible supercuspidal $\sigma \in \mathcal{M}(M)$ such that π is a sub-representation of $i_P^G \sigma$.*

Proof. By transitivity of the Jacquet module functor, we can find $P = MN$ satisfying (1) by taking P to be minimal such that $r_P^G \pi \neq 0$. (If π is supercuspidal, then we take $P = G$.) By Lemma 27.9, $r_P^G \pi$ is finitely generated, so it admits an irreducible quotient σ. By the (right) exactness of the Jacquet module functor, σ is also supercuspidal. By adjunction,

$$\mathrm{Hom}_{G(F)}(\pi, i_P^G \sigma) \cong \mathrm{Hom}_{M(F)}(r_P^G \pi, \sigma) \neq 0.$$

Since π is irreducible, this means that π is a sub-representation of $i_P^G \sigma$. □

Remark 27.11. The second statement says that all irreducible representations "can be built" from supercuspidal representations of Levi subgroups.

28. Results on admissibility

Theorem 28.1. *Let $(\pi, V) \in \mathcal{M}(G)$ be irreducible. Then it is admissible.*

Proof. By Proposition 27.10, we may assume that $\pi = i_P^G \sigma$ for an irreducible supercuspidal $(\sigma, W) \in \mathcal{M}(M)$. Since i_P^G preserves admissibility, it suffices to show that σ is admissible. Since every compact open subgroup of $M(F)$ is contained in ${}^0M(F)$. It suffices to show that σ is admissible as a ${}^0M(F)$-representation. We write σ' for $\sigma|_{{}^0M(F)}$. We already know that σ' is compact, and recall that any finitely generated compact representation is admissible. It remains to show that σ' is finitely generated. Let $v_0 \in W - \{0\}$. Then σ is generated by v_0 as a $M(F)$-representation. Moreover, since σ is irreducible, by Schur's lemma, $Z_M(F)$ acts on σ by a character $\chi : Z_M(F) \to \mathbb{C}^\times$. By Proposition 21.16, $Z_M(F)\ {}^0M(F)$ is normal and of finite index in $M(F)$, so we can fix representatives $g_1, \cdots, g_r \in M(F)$ of $(Z_M(F)\ {}^0M(F))\backslash M(F)$. Every $v \in W$ is a $\mathbb{C}$-linear combination of $M(F)$-translates of v_0, so

$$v = \sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{t} c_{i,j,k} z_j h_k g_i v_0 = \sum c_{i,j,k}\chi(z_j) h_k g_i v_0$$

for $c_{i,j,k} \in \mathbb{C}, z_j \in Z_M(F), h_k \in {}^0M(F)$. Thus v is a $\mathbb{C}$-linear combination of ${}^0M(F)$-translates of $g_1 v_0, \cdots, g_r v_0$. Hence the ${}^0M(F)$-representation σ' is finitely generated. □

Our next goal is to prove the following stronger result:

Theorem 28.2 (Uniform admissibility, Bernstein)**.** *Fix a compact open subgroup $K \subset G(F)$. Then there exists a constant N such that for every irreducible $(\pi, V) \in \mathcal{M}(G)$ we have $\dim V^K \le N$.*

We need some preparation. Let K_0 be as in Fact 24.2, and K be as in Fact 24.7.

Proposition 28.3. *We have a decomposition*

$$\mathcal{H}(G(F))_K = \mathcal{H}(K_0)_K \cdot D \cdot C \cdot \mathcal{H}(K_0)_K,$$

where D is a finite dimensional $\mathbb{C}$-vector subspace, and C is a finitely generated commutative $\mathbb{C}$-algebra. Here the decomposition means that every element of $\mathcal{H}(G(F))_K$ can be written as $xyzw$ with $x, w \in \mathcal{H}(K_0)_K, y \in D, z \in C$. Moreover, the number of generators of C as a $\mathbb{C}$-algebra is bounded from the above by the number of generators of $X_*(A_{M_0})^+ := X_*(A_{M_0}) \cap \mathfrak{a}_{M_0}^+$ as a monoid.[14]

If $A_{M_0} = M_0$, then $D = \mathbb{C}$. In general, D accounts for the possible failure of the existence of a group-theoretic section of $M_0(F) \to \Lambda(M_0)$. We will sketch a proof of Proposition 28.3 later. We also need the following fact, whose proof is found in [BZ76, Lem. 4.10] or [Ren10, C.I].

Fact 28.4 (Bernstein–Zelevinsky)**.** *Any commutative subalgebra of $M_{k\times k}(\mathbb{C})$ generated by l elements has dimension $\le k^{2-2^{1-l}}$.*

[14] The fact that $X_*(A_{M_0})^+$ is indeed finitely generated as a monoid follows from Gordan's lemma: Let $\lambda_1, \cdots, \lambda_k \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Z})$. Then the monoid $\{x \in \mathbb{Z}^n \mid \lambda_i x \ge 0, \forall i\}$ is finitely generated. (The statement is equivalent if we allow $\lambda_i \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Q})$, but it becomes false if we allow $\lambda_i : \mathbb{Z}^n \to \mathbb{R}$ to take irrational values.)

Proof of Theorem 28.2. Recall that V^K is a simple unital $\mathcal{H}(G)_K$-module. Let $k = \dim V^K$, which is finite by Theorem 28.1. By Burnside's theorem, for every algebraically closed field C, every unital C-algebra R, and every simple R-module M that is finite dimensional over C, the map $R \to \mathrm{End}_C(M), r \mapsto r \cdot (\cdot)$ is surjective. Hence $\pi : \mathcal{H}(G)_K \to \mathrm{End}_{\mathbb{C}}(V^K)$ is surjective. Without loss of generality, we may assume that K is as in Fact 24.7. Then we can apply Proposition 28.3. Clearly $\mathcal{H}(K_0)_K$ is finite dimensional (of dimension $[K_0 : K]$). Hence

$$k^2 = \dim \pi(\mathcal{H}(G)_K) \le d \dim \pi(C)$$

for a constant $d = [K_0 : K]^2 \dim D > 0$. By Fact 28.4, if l is the number of generators of C, then

$$k^2 \le d k^{2-2^{1-l}},$$

from which

$$k \le d^{2^{l-1}}.$$

□

Exercise 28.5. For GL_n, we have $X_*(A_{M_0})^+ \cong \{(a_1, \cdots, a_n) \in \mathbb{Z}^n \mid a_1 \ge \cdots \ge a_n\}$. This monoid has generators $(1, 0, \cdots, 0), (1, 1, 0, \cdots, 0), \cdots, (1, \cdots, 1), (-1, \cdots, -1)$. Also $A_{M_0} = M_0$. Hence in the above proof we can take $d = [K_0 : K]^2$, and $l = n + 1$. Thus for K as in Fact 24.7 and for irreducible $(\pi, V) \in \mathcal{M}(G)$ we have

$$\dim V^K \le [K_0 : K]^{2^{n+1}}.$$

Lect.39, May 3

We now discuss the proof of Proposition 28.3. For each $g \in G(F)$, let

$$a_g = a_{g,K} := e_K * \delta_g * e_K = e_K * (l(g) e_K) \in \mathcal{H}(G(F))_K.$$

If we fix a left Haar distribution on G and identify $\mathcal{H}(G(F))$ with the K-bi-invariant functions in $C_c^\infty(G)$, then

$$a_g = \mathrm{vol}(KgK)^{-1} 1_{KgK}.$$

Hence $\mathcal{H}(G(F))_K$ has a $\mathbb{C}$-basis $\{a_g\}$ where g runs over a set of representatives of $K \backslash G(F)/K$. The key to the proof of Proposition 28.3 is the following lemma, which will also be needed later independently.

Lemma 28.6. *For $g_1, g_2 \in M_0^+$, we have $a_{g_1} a_{g_2} = a_{g_1 g_2}$.*

Proof. We claim that for any $(\pi, V) \in \mathcal{M}(G)$, we have

$$\pi(e_K)\pi(g_1)\pi(e_K)\pi(e_K)\pi(g_2)\pi(e_K) = \pi(e_K)\pi(g_1)\pi(g_2)\pi(e_K).$$

The lemma then follows since we can take (π, V) to be $\mathcal{H}(G(F))$ itself. Similar to the proof of Theorem 27.3, we compute (omitting the subscript 0 in $M_0, N_0, \bar{N}_0$):

$$\begin{aligned} \text{LHS } &= \pi(e_K)\pi(g_1)\gamma(K_N)\gamma(K_M)\gamma(K_{\bar{N}})\pi(g_2)\pi(e_K) \\ &= \pi(e_K)\gamma(g_1 K_N g_1^{-1})\gamma(g_1 K_M g_1^{-1})\pi(g_1 g_2)\gamma(g_2^{-1} K_{\bar{N}} g_2)\pi(e_K). \end{aligned}$$

Now $g_1 K_N g_1^{-1} \subset K_N \subset K$ and $g_1 K_M g_1^{-1} = K_M \subset K$, so $\pi(e_K)\gamma(g_1 K_N g_1^{-1})\gamma(g_1 K_M g_1^{-1}) = \pi(e_K)$. Also $g_2^{-1} K_{\bar{N}} g_2 \subset K_{\bar{N}} \subset K$, so $\gamma(g_2^{-1} K_{\bar{N}} g_2)\pi(e_K) = \pi(e_K)$. □

Sketch of proof of Proposition 28.3. For simplicity, assume that $A_{M_0} = M_0$. We fix a group theoretic section of $H_{A_{M_0}} : A_{M_0}(F) \to X_*(A_{M_0})$ (which exists since this map is isomorphic to $(F^\times)^n \to \mathbb{Z}^n, (x_i) \mapsto (v_F(x_i))$), and in the following we use this section to identify

$X_*(A_{M_0})$ with a subgroup of $A_{M_0}(F)$. Write X^+ for $X_*(A_{M_0})^+$, identified with a subgroup of $A^+_{M_0}$. Under our simplifying assumption, the Cartan decomposition is of the form

$$G(F) = \coprod_{\lambda \in X^+} K_0 \lambda K_0.$$

Fix $\{\lambda_1, \cdots, \lambda_r\}$ to be a set of generators of the monoid X^+. By the Cartan decomposition, any $g \in G(F)$ is of the form $k_1 \lambda_1^{n_1} \cdots \lambda_r^{n_r} k_2$ for $k_1, k_2 \in K_0$ and $n_1, \cdots, n_r \in \mathbb{N}$. Write h for $\lambda_1^{n_1} \cdots \lambda_r^{n_r}$. Then in $C_c^{-\infty}(G(F))$ we have

$$a_g = e_K * \delta_{k_1} * \delta_h * \delta_{k_2} * e_K = (e_K * \delta_{k_1} * e_K) * (e_K * \delta_h * e_K) * (e_K * \delta_{k_2} * e_K) = a_{k_1} * a_h * a_{k_2},$$

where the second equality follows from the fact that k_1, k_2 normalize K (from which $e_K * \delta_{k_i} = \delta_{k_i} * e_K$). We have $a_{k_i} \in \mathcal{H}(K_0)_K$. Define $D = \mathbb{C}$, and define C to be the linear span of a_λ for $\lambda \in X^+$. Then $a_h \in C$, so by the above we get the decomposition

$$\mathcal{H}(G(F))_K = \mathcal{H}(K_0)_K \cdot C \cdot \mathcal{H}(K_0)_K.$$

By Lemma 28.6, we have $a_{\lambda_1^{n_1} \cdots \lambda_r^{n_r}} = a_{\lambda_i}^{n_1} \cdots a_{\lambda_r}^{n_r} \in C$, from which C is a commutative algebra generated by a_{λ_i}. □

Remark 28.7. In general, if there exists a group theoretic section of $H_{M_0} : M_0(F) \to \Lambda(M_0)$, then our proof of Proposition 28.3 immediately generalizes and gives rise to the following variant of Proposition 28.3: We have

$$\mathcal{H}(G(F))_K = \mathcal{H}(K_0)_K \cdot C \cdot \mathcal{H}(K_0)_K,$$

where C is a finitely generated commutative sub-algebra, and the number of generators is bounded by the number of generators of the monoid $\Lambda(M_0)^+ := \Lambda(M_0) \cap \mathfrak{a}^+_{M_0}$. It is expected by experts that such a group theoretic section always exists, but it seems that a complete proof has not appeared in the literature.

Corollary 28.8. *Fix a compact open subgroup $K \subset G(F)$. There is a compact subset $\Omega \subset {}^0G(F)$ such that for every irreducible supercuspidal representation $(\pi, V) \in \mathcal{M}(G)$ and every K-bi-invariant matrix coefficient ϕ for π, we have* $\mathrm{supp}(\phi) \cap {}^0G(F) \subset \Omega$.

Sketch of proof. For simplicity assume that $Z_G = 1$ (in particular $G(F) = {}^0G(F)$) and assume that $A_{M_0} = M_0$. For the proof in the general case see [Ren10, V.5.3]. Without loss of generality, we may assume that K is as in Fact 24.7. We use the notation in the proof of Proposition 28.3, and as in that proof we identify X^+ with a subgroup of $A^+_{M_0}$. In particular, the Cartan decomposition is of the form

$$G(F) = \coprod_{\lambda \in X^+} K_0 \lambda K_0.$$

If ϕ is a K-bi-invariant matrix coefficient for (π, V), then the function $\phi(g^{-1})$ is of the form $g \mapsto \langle \lambda, \pi(g) v \rangle = \langle \lambda, \pi(a_g) v \rangle$ for $v \in V^K, \lambda \in (V^\vee)^K$. Hence we only need to show that for every irreducible supercuspidal (π, V), the map $G(F) \to \mathrm{End}_{\mathbb{C}}(V), g \mapsto \pi(a_g)$ is supported on some compact Ω which is independent of (π, V).

We claim that there is a constant integer $N \geq 1$ independent of (π, V) such that for every non-trivial $\lambda \in X^+$ we have $\pi(a_\lambda)^N = 0$.

Assuming the claim, we finish the proof as follows. Pick a set of generators $\{\lambda_1, \cdots, \lambda_r\}$ of the monoid X^+ with each λ_j non-trivial, and take Ω to be the union of $K_0 \lambda_1^{n_1} \cdots \lambda_r^{n_r} K_0$ over $n_1, \cdots, n_r \in \{0, 1, \cdots, N\}$. To see that this works, let $g \notin \Omega$. Then $g = k_1 \lambda_1^{n_1} \cdots \lambda_r^{n_r} k_2$

with $n_1, \cdots, n_r \geq 0$ and some $n_j > N$, and with $k_1, k_2 \in K_0$. Since K_0 normalizes K, and by Lemma 28.6, we have

$$\pi(a_g) = \pi(e_K)\pi(k_1)\pi(\lambda_1^{n_1} \cdots \lambda_r^{n_r})\pi(k_2)\pi(e_K) = \pi(k_1)\pi(e_K)\pi(\lambda_1^{n_1} \cdots \lambda_r^{n_r})\pi(e_K)\pi(k_2)$$
$$= \pi(k_1)\pi(a_{\lambda_1^{n_1} \cdots \lambda_r^{n_r}})\pi(k_2) = \pi(k_1)\pi(a_{\lambda_1})^{n_1} \cdots \pi(a_{\lambda_r})^{n_r}\pi(k_2).$$

This is zero because $\pi(a_{\lambda_j})^{n_j} = 0$.

To prove the claim, note that $\pi(a_\lambda)^k = \pi(a_{\lambda^k})$ by Lemma 28.6. When $k \to +\infty$, $\lambda^k \in A^+_{M_0}$ leaves any given compact set because its image in $X^+ \subset \mathfrak{a}_{M_0}$, namely $k \cdot \lambda$, gets arbitrarily far away from the origin. Therefore, by the fact that π is supercuspidal (= compact), and by the definition of compact representations, for every $v \in V$ we have $\pi(a_\lambda)^k v = \pi(a_{\lambda^k})v = 0$ for sufficiently large k (in a way that *a priori* depends on (π, V) and v). But $\pi(a_\lambda)$ is determined by its restriction to V^K, and the latter has dimension bounded by some N which is independent of (π, V) by Theorem 28.2. Hence $\pi(a_\lambda)^N = 0$. □

Lect.40, May 5

29. The supercuspidal part of the category of smooth representations

Proposition 29.1. *Let $\pi \in \mathcal{M}(G)$ be irreducible. Write π_0 for $\pi|_{^0G(F)} \in \mathcal{M}(^0G(F))$. The following statements hold.*

(1) *π_0 is a direct sum of finitely many irreducible representations of $^0G(F)$, and their isomorphism classes form precisely one $G(F)$-orbit. (There may be multiplicities in the irreducible decomposition of π_0.) Here $G(F)$ acts on the set of isomorphism classes of representations of $^0G(F)$ by its conjugation action on $^0G(F)$.*

(2) *Let $\pi' \in \mathcal{M}(G)$ be another irreducible representation. The following statements are equivalent:*
   (a) $\pi_0 \cong \pi'_0$.
   (b) *$\pi \cong \pi' \otimes \omega$ for some $\omega \in \chi(G) = \mathrm{Hom}(G/^0G(F), \mathbb{C}^\times)$, i.e., π and π' lie in the same inertia class.*
   (c) $\mathrm{Hom}_{^0G(F)}(\pi_0, \pi'_0) \neq 0$.

Proof. Part (1) is an easy consequence of the fact that $^0G(F)Z_G(F)$ is of finite index in $G(F)$, and the fact that $Z_G(F)$ acts on π by a character by Schur's lemma. We leave the proof as an exercise. For (2), the implications (b) ⇒ (a) ⇒ (c) are trivial. We show (c) ⇒ (b). By (1) and Schur's lemma, the space $H = \mathrm{Hom}_{^0G(F)}(\pi_0, \pi'_0)$ is finite dimensional. The group $G(F)$ acts on it by $g \cdot f = \pi'(g) \circ f \circ \pi(g^{-1})$. This action factors through the abelian group $\Lambda(G) = G(F)/^0G(F)$, and therefore there is a common eigenvector $f \in H$. That is, $\pi(g)f\pi'(g^{-1}) = \omega(g)f$ for some $\omega \in \chi(G)$. But then f is an isomorphism $\pi' \otimes \omega \xrightarrow{\sim} \pi$. □

Exercise 29.2. Prove part (1).

Definition 29.3. Let Irr_{sc} be the set of isomorphism classes of irreducible supercuspidal representations $\pi \in \mathcal{M}(G)$. Let Irr^0_{sc} be the set of isomorphism classes of irreducible representations of $^0G(F)$ which appear in $\pi|_{^0G(F)}$ for some $\pi \in \mathrm{Irr}_{sc}$.

We know that every element of Irr^0_{sc} is (the isomorphism class of) an irreducible compact representation. The converse is also true, but we will not need it.

Corollary 29.4. *Fix a compact open subgroup $K \subset G(F)$. There are only finitely many $\chi(G)$-orbits of $\pi \in \mathrm{Irr}_{sc}$ such that $\pi^K \neq 0$. There are only finitely many elements $\sigma \in \mathrm{Irr}^0_{sc}$ such that $\sigma^K \neq 0$.*

Proof. The two statements are equivalent to each other in view of Proposition 29.1. For each $\sigma \in \mathrm{Irr}^0_{sc}$ satisfying $\sigma^K \neq 0$, we can take a K-bi-invariant matrix coefficient for σ (because $(\sigma^\vee)^K \cong (\sigma^K)^* \neq 0$), and for varying σ these functions on ${}^0G(F)$ are linearly independent of each other by Exercise 15.17. However, all these functions are K-bi-invariant and supported on a common compact subset $\Omega \subset^0 G(F)$ which depends only on K, by Corollary 28.8. The space of all K-bi-invariant functions on Ω is finite dimensional. Hence there are only finitely many elements σ of Irr^0_{sc} such that $\sigma^K \neq 0$. □

Theorem 29.5. *We have*

$$\mathcal{M}({}^0G(F)) \cong \left(\bigoplus_{\sigma \in \mathrm{Irr}^0_{sc}} \mathcal{M}({}^0G(F))_\sigma \right) \oplus \mathcal{M}({}^0G(F))_{nsc},$$

where $\mathcal{M}({}^0G(F))_\sigma$ is the full subcategory consisting of representations which are direct sums of copies of σ, and $\mathcal{M}({}^0G(F))_{nsc}$ is the full subcategory of representations none of whose subquotients are in Irr^0_{sc}.

Proof. Since Irr^0_{sc} is a subset of the set of isomorphism classes of irreducible compact representations of ${}^0G(F)$, we only need to check that Irr^0_{sc} satisfies condition (FC) and then apply Theorem 14.6. By Corollary 29.4, Irr^0_{sc} satisfies condition (FC). □

By Theorem 29.5 and some more work, we obtain the following:

Theorem 29.6. *We have*

$$\mathcal{M}(G) \cong \mathcal{M}(G)_{sc} \oplus \mathcal{M}(G)_{ind} \cong \left(\bigoplus_{[\pi] \in \mathrm{Irr}_{sc}/\chi(G)} \mathcal{M}(G)_{[\pi]} \right) \oplus \mathcal{M}(G)_{ind}.$$

Here $\mathcal{M}(G)_{sc}$ (resp. $\mathcal{M}(G)_{ind}$, resp. $\mathcal{M}(G)_{[\pi]}$) is the full subcategory consisting of objects all of whose irreducible subquotients are supercuspidal (resp. not supercuspidal, resp. members of $[\pi]$).

Let $\pi \in \mathrm{Irr}_{sc}$. Next we state without proof a structure theorem for the category $\mathcal{M}(G)_{[\pi]}$. Fix an irreducible ${}^0G(F)$ sub-representation (σ, W) of $\pi|_{{}^0G(F)}$, and let Π be the compact induction of σ from ${}^0G(F)$ to $G(F)$. That is, consider the space of functions $f : G(F) \to W$ satisfying $f(hg) = \sigma(h)f(g)$ for all $h \in {}^0G(F), g \in G(F)$, and such that f is compactly supported modulo ${}^0G(F)$ (meaning that the support of f has finite image in $\Lambda(G)$). Equip this space with a $G(F)$-action by right translation. Then define Π to be the smooth part of this $G(F)$-representation.

Theorem 29.7. *The functor from $\mathcal{M}(G)_{[\pi]}$ to the category of right unital modules of the ring $\mathrm{End}_{G(F)}(\Pi)$, sending V to $\mathrm{Hom}_{G(F)}(\Pi, V)$, is an equivalence. In particular, the center of the category $\mathcal{M}(G)_{[\pi]}$ is identified with the center of the ring $\mathrm{End}_{G(F)}(\Pi)$.*

After more work, one obtains a geometric description of the center of $\mathcal{M}(G)_{[\pi]}$ as follows. Recall that the inertia class $[\pi] = \chi(G) \cdot \pi$ is a principal homogeneous space under $\chi(G)/\operatorname{Stab}_\pi(\chi(G))$, which is a complex torus. In particular, the set $[\pi]$ has a canonical structure of (the complex points of) a complex affine algebraic variety. Let $\mathcal{O}([\pi])$ denote the ring of regular functions on this variety.

Theorem 29.8. *The center of the category $\mathcal{M}(G)_{[\pi]}$ is canonically identified with $\mathcal{O}([\pi])$. More precisely, let $f \in \mathcal{O}([\pi])$ and $\pi' \in [\pi]$. Clearly π' is an (irreducible) object of $\mathcal{M}(G)_{[\pi]}$. The endomorphism of $\pi' \in$ induced by f is the scalar $f(\pi') \in \mathbb{C}$.*

Lect.41, May 8

30. The Bernstein decomposition theorem

Definition 30.1. By a cuspidal datum for G, we mean a pair (M, σ) where M is a Levi subgroup of G (not necessarily standard) and σ is an irreducible supercuspidal representation of $M(F)$. Two such pairs $(M, \sigma), (M', \sigma')$ are called associated, if there exists $g \in G(F)$ such that $gMg^{-1} = M'$, and the isomorphism $\mathrm{Int}(g) : M(F) \xrightarrow{\sim} M'(F)$ takes σ to σ'. Write $\Omega(G)$ for the set of cuspidal data modulo association.

Theorem 30.2. *Let $(M, \sigma), (M', \sigma')$ be cuspidal data. Let P, P' be parabolic subgroups of G having M and M' as Levi factors, respectively. Let $\pi = i_P^G \sigma$ and $\pi' = i_{P'}^{G'} \sigma'$. Then π and π' are of finite length, and the Jordan–Hölder factors of π (counting multiplicities) depend only on (M, σ), not on P. Moreover, the following are equivalent:*

1. *(M, σ) and (M', σ') are associated.*
2. *$\mathrm{Hom}_{G(F)}(\pi, \pi') \neq 0$.*
3. *π and π' have the same Jordan–Hölder factors (counting multiplicities).*
4. *π and π' have at least one Jordan–Hölder factor in common.*

We thus have a well defined map $\mathrm{CS} : \mathrm{Irr}(G) \to \Omega(G)$ sending π to (M, σ) such that π is a subquotient of $i_P^G \sigma$ for some P having M as a Levi factor. (Here CS stands for "cuspidal support".)

We now introduce an equivalence relation on the set of cuspidal data, which is coarser than association. It is the equivalence relation generated by association, and the requirement that $(M, \sigma) \sim (M, \sigma \otimes \omega)$ for all cuspidal data (M, σ) and all $\omega \in \chi(M)$. Write $\mathscr{B}(G)$ for the set of cuspidal data modulo this equivalence relation. We thus have a natural map $\Omega(G) \to \mathscr{B}(G)$. Write IS for the composition $\mathrm{Irr}(G) \xrightarrow{\mathrm{CS}} \Omega(G) \to \mathscr{B}(G)$, standing for "inertia support".

Theorem 30.3. *We have*

$$\mathcal{M}(G) = \bigoplus_{\mathfrak{s} \in \mathscr{B}(G)} \mathcal{M}(G)_{\mathfrak{s}},$$

where $\mathcal{M}(G)_{\mathfrak{s}}$ is the full subcategory consisting of representations such that each irreducible subquotient π satisfies $\mathrm{IS}(\pi) = \mathfrak{s}$.

Remark 30.4. If $\mathfrak{s}$ is represented by a cuspidal datum of the form (G, π), then

$$\mathfrak{s} = \{(G, \pi') \mid \pi' \in [\pi] = \chi(G) \cdot \pi\},$$

and $\mathcal{M}(G)_{\mathfrak{s}}$ is the same as $\mathcal{M}(G)_{[\pi]}$ as in Theorem 29.6.

Let $\mathfrak{s} \in \mathscr{B}(G)$, and let (M, σ) be a representative of $\mathfrak{s}$. With significantly more work, we have the following structure theorem for $\mathcal{M}(G)_{\mathfrak{s}}$, generalizing Theorem 29.7. Fix an irreducible sub-representation τ of $\sigma|_{{}^0M(F)}$, and let Σ be the compact induction of τ from ${}^0M(F)$ to $M(F)$. Let $P \subset G$ be a parabolic subgroup having M as a Levi factor.

Theorem 30.5. *The functor from $\mathcal{M}(G)_{\mathfrak{s}}$ to the category of right unital modules of the ring $\mathrm{End}_{G(F)}(i_P^G \Sigma)$, sending V to $\mathrm{Hom}_{G(F)}(i_P^G \Sigma, V)$, is an equivalence. In particular, the center of $\mathcal{M}(G)_{\mathfrak{s}}$ is identified with the center of the ring $\mathrm{End}_{G(F)}(i_P^G \Sigma)$.*

We also have a geometric description of the center generalizing Theorem 29.8. We state this result. The inverse image $\Omega(G)_{\mathfrak{s}}$ of $\mathfrak{s}$ in $\Omega(G)$ is the quotient of $[\sigma] = \chi(M) \cdot \sigma$ by the action of the stabilizer of $[\sigma]$ in the finite group $W_M^G := N_{G(F)}(M(F))/M(F)$. Here W_M^G acts on $\mathrm{Irr}(M)$ via the conjugation action of $N_{G(F)}(M(F))$ on $M(F)$. In general, if $\gamma \in \mathrm{Aut}_F(M)$, then γ permutes $X^*(M(F))$ and hence γ stabilizes ${}^0M(F)$. Therefore γ

permutes $\chi(M)$ by an automorphism of the torus $\chi(M)$ (induced by the automorphism of the free abelian group $\Lambda(M)$ induced by γ). Thus if we pick a base point σ in an inertia class $[\sigma]$, then γ stabilizes $[\sigma]$ if and only if $\gamma(\sigma) = \sigma \otimes \omega_0$ for some $\omega_0 \in \chi(M)$, and in that case $\gamma(\sigma \otimes \omega) = \sigma \otimes (\omega_0 \cdot \gamma(\omega))$ for all $\omega \in \chi(M)$. From this we see that γ acts on $[\sigma]$ via an algebraic variety automorphism of $[\sigma]$. Thus $\Omega(G)_{\mathfrak{s}}$ is the quotient of the affine algebraic variety $[\sigma]$ by a finite subgroup of the automorphism group of the algebraic variety $[\sigma]$. As such $\Omega(G)_{\mathfrak{s}}$ obtains the structure of an affine algebraic variety. One checks that this structure is independent of all choices.

Theorem 30.6. *The center of $\mathcal{M}(G)_{\mathfrak{s}}$ is identified with the ring of regular functions on $\Omega(G)_{\mathfrak{s}}$. Let f be such a function and let π be an irreducible object of $\mathcal{M}(G)_{\mathfrak{s}}$. Then the endomorphism of π induced by f is the scalar $f(\mathrm{CS}(\pi))$.*

Corollary 30.7. *The category $\mathcal{M}(G)_{\mathfrak{s}}$ is indecomposable, i.e., not a direct sum of two sub-categories.*

Proof. The center of this category has no idempotents, since the variety $\Omega(G)_{\mathfrak{s}}$ is connected. If $\mathcal{M}(G)_{[\pi]}$ were decomposable, then the projection to a direct summand defines an idempotent in its center. □

31. Illustration of the geometric lemma

Finally, we say a few words on the proof of Theorem 30.2. The key is to understand, when given two parabolic subgroups with Levi decompositions $P = MN$ and $Q = LU$ (none assumed to be standard), and given σ an irreducible supercuspidal representation of $M(F)$, the structure of $r_Q^G i_P^G \sigma$ as an $L(F)$-representation. To this end we have the following result. By general theory, there is a set $W^{Q,P} \subset G(F)$ of representatives for $P(F)\backslash G(F)/Q(F)$ such that for each $w \in W^{Q,P}$, $M \cap w \cdot Q$ is a parabolic subgroup of M with Levi factor $M \cap w \cdot L$ and unipotent radical $M \cap w \cdot U$, and $L \cap w^{-1} \cdot P$ is a parabolic subgroup of L with Levi factor $L \cap w^{-1} \cdot M$ and unipotent radical $L \cap w^{-1} \cdot N$. (Here $w \cdot (\cdots)$ and $w^{-1} \cdot (\cdots)$ denote the conjugation action.)

Theorem 31.1 (Geometric lemma)**.** *The $L(F)$-representation $r_Q^G i_P^G \sigma$ has a filtration, whose associated graded pieces are*

$$i^L_{L\cap w^{-1}\cdot P} \circ w^* \circ r^M_{M\cap w\cdot Q}(\sigma),$$

where w^* runs through $W^{Q,P}$, and w^* denotes the functor $\mathcal{M}(M \cap w \cdot L) \to \mathcal{M}(L \cap w^{-1} \cdot M)$ induced by the isomorphism of groups $L \cap w^{-1} \cdot M \xrightarrow{\sim} M \cap w \cdot L$ given by conjugation by w.

We illustrate the theorem in the case where $G = \mathrm{GL}_2, P = Q =$ the group of upper triangular invertible matrices, and $M = L =$ the diagonal torus. Then $W^{P,P} = \{1, w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$, and the automorphism $w : M \to M$ is $\mathrm{diag}(a, b) \mapsto \mathrm{diag}(b, a)$. An irreducible supercuspidal representation of $M(F)$ is just a one dimensional representation $\mathrm{diag}(a, b) \mapsto \chi_1(a)\chi_2(b)$, where χ_1, χ_2 are smooth characters $F^\times \to \mathbb{C}^\times$. We denote this representation by σ_{χ_1,χ_2}. The functor w^* sends σ_{χ_1,χ_2} to σ_{χ_2,χ_1}. Thus the theorem says that $r_P^G i_P^G \sigma_{\chi_1,\chi_2}$ has a filtration with graded pieces

$$\sigma_{\chi_1,\chi_2}, \qquad \sigma_{\chi_2,\chi_1}.$$

We now prove this in the special case where $\chi_1 = |\cdot|^{-1/2}, \chi_2 = |\cdot|^{1/2}$, so that $\sigma_{\chi_1,\chi_2} = \delta_P^{1/2}$. Then $i_P^G \sigma_{\chi_1,\chi_2} = \mathrm{Ind}_P^G \mathrm{triv}$, and this is the space of smooth functions on $P(F)\backslash G(F)$

equipped with the $G(F)$-action by right translation. The right $N(F)$-action on $P(F)\backslash G(F)$ has two orbits, represented by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In fact, $P(F)\backslash G(F) \cong \mathbb{P}^1(F)$, and the two orbits can be identified with $\mathbb{A}^1(F)$ and $\{\infty\}$. Up to suitably choosing coordinates, the action of $N(F)$ on $\mathbb{A}^1(F)$ is given by

$$N(F) \cong F \ni \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} : \mathbb{A}^1(F) = F \longrightarrow \mathbb{A}^1(F) = F, \qquad y \longmapsto x + y.$$

We have an exact sequence

$$0 \to C_c^\infty(\mathbb{A}^1(F)) \to C_c^\infty(\mathbb{P}^1(F)) \to C_c^\infty(\{\infty\}) \to 0.$$

Taking $N(F)$-coinvariants, we get

$$0 \to C_c^\infty(\mathbb{A}^1(F))_{N(F)} \to J_N i_P^G \sigma_{\chi_1,\chi_2} \to \mathrm{triv} \to 0.$$

The space $C_c^\infty(\mathbb{A}^1(F))_{N(F)}$ is identified with $V := C_c^\infty(N(F))_{N(F)}$. Clearly V is 1-dimensional and generated by 1_K for any compact open subgroup $K \subset N(F)$. If we fix a Haar distribution μ on $N(F)$, then the dual space of V has a basis $l : V \to \mathbb{C}, f \mapsto \langle \mu, f \rangle$. The action of $M(F)$ on V is given as follows: For $t = \mathrm{diag}(a, b) \in M(F)$, it acts on V by $f \mapsto f \circ \mathrm{Ad}(t^{-1})$, where $\mathrm{Ad}(t^{-1})$ is the automorphism of N given by conjugation by t^{-1}. Thus $t : V \to V$ maps 1_K to $1_{tKt^{-1}}$, which is equal to $[tKt^{-1} : K]1_K$ in V. (Here, for two compact open subgroups K_1, K_2 of $N(F)$, we write $[K_1 : K_2]$ for $[K_1 : K_1 \cap K_2][K_2 : K_1 \cap K_2]^{-1}$.) We have

$$[tKt^{-1} : K] = |a/b| = \delta_P^{-1}(t).$$

We conclude that the $M(F)$-representation V is isomorphic to δ_P^{-1}.

In conclusion, $J_N i_P^G \sigma_{\chi_1,\chi_2}$ has a filtration with graded pieces δ_P^{-1} and triv. Twisting by $\delta_P^{1/2}$, we see that $r_P^G i_P^G \sigma_{\chi_1,\chi_2}$ has a filtration with graded pieces $\delta_P^{-1/2}$ and $\delta_P^{1/2}$, that is, σ_{χ_1,χ_2} and σ_{χ_2,χ_1}.

References

[Bor91] Armand Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991. 38

[BZ76] I. N. Bernšteĭn and A. V. Zelevinskiĭ. Representations of the group $GL(n, F)$, where F is a local non-Archimedean field. *Uspehi Mat. Nauk*, 31(3(189)):5–70, 1976. 62

[CN09] Clifton Cunningham and Monica Nevins. *Ottawa lectures on Admissible Representations of Reductive p-adic Groups*, volume 26. American Mathematical Soc., 2009. 3

[KP23] Tasho Kaletha and Gopal Prasad. *Bruhat-Tits theory—a new approach*, volume 44 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2023. 54

[Mil17] J. S. Milne. *Algebraic groups*, volume 170 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017. The theory of group schemes of finite type over a field. 38

[Ngô16] BC Ngô. Harmonic analysis and representation theory of p-adic reductive groups, 2016. available at https://math.uchicago.edu/~ngo/Rep-p-adic.pdf. 3

[PR94] Vladimir Platonov and Andrei Rapinchuk. *Algebraic groups and number theory*, volume 139 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen. 48

[Ren10] David Renard. *Représentations des groupes réductifs p-adiques*. Société mathématique de France Paris, 2010. 3, 8, 30, 54, 62, 64

[Spr79] T. A. Springer. Reductive groups. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 3–27. Amer. Math. Soc., Providence, R.I., 1979. 38

[Spr09] T. A. Springer. *Linear algebraic groups*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, second edition, 2009. 38

[Tit79] J. Tits. Reductive groups over local fields. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 29–69. Amer. Math. Soc., Providence, R.I., 1979. 54

[Yu09] Jiu-Kang Yu. Bruhat-Tits theory and buildings. In *Ottawa lectures on admissible representations of reductive p-adic groups*, volume 26 of *Fields Inst. Monogr.*, pages 53–77. Amer. Math. Soc., Providence, RI, 2009. 54