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Stabilizers of irreducible components of affine Deligne–Lusztig varieties

Abstract. We study the $J_b(F)$-action on the set of top-dimensional irreducible components of affine Deligne–Lusztig varieties in the affine Grassmannian. We show that the stabilizer of any such component is a parahoric subgroup of $J_b(F)$ of maximal volume, verifying a conjecture of X. Zhu. As an application, we give a description of the set of top-dimensional irreducible components in the basic locus of Shimura varieties.

Keywords. affine Deligne–Lusztig varieties, irreducible components, very special parahoric subgroups

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1. Introduction

1.1. Affine Deligne–Lusztig varieties and their irreducible components

Affine Deligne–Lusztig varieties were introduced by Rapoport in [34]. In the equal characteristic setting, affine Deligne–Lusztig varieties are related to the moduli space of local shtukas. In the mixed characteristic setting, they are related to the geometry of Rapoport–Zink spaces and hence to the geometry of certain distinguished loci in the special fiber of Shimura varieties via the p-adic uniformization. Therefore studying the geometry of affine Deligne–Lusztig varieties can give useful information on the geometry of special cycles on Shimura varieties.

This paper is concerned with studying the set of top-dimensional irreducible components of affine Deligne–Lusztig varieties. To state our main results we fix some notation. Let F be a local field with ring of integers O_F, and let $\breve{F}$ be the completion of the maximal unramified extension of F. Let G be a reductive group over F, which we assume is unramified in the introduction for simplicity. For $b \in G(\breve{F})$ and μ a cocharacter of G, we have the affine Deligne–Lusztig variety $X_\mu(b)$ which is a locally closed subscheme of the affine Grassmannian. We refer to §2.4.1 for the precise definition.

If F is of equal characteristic, $X_\mu(b)$ is locally of finite type. If F is of mixed characteristic, $X_\mu(b)$ is a perfect scheme and is locally of perfectly finite type. In either case, it is known that $X_\mu(b)$ is finite dimensional. We write $\Sigma^{\mathrm{top}}(X_\mu(b))$ for the set of top-dimensional irreducible components of $X_\mu(b)$.

The scheme $X_\mu(b)$ is equipped with an action of $J_b(F)$, the F-rational points of a certain reductive group J_b over F (the Frobenius-centralizer of b). This induces an action of $J_b(F)$ on $\Sigma^{\mathrm{top}}(X_\mu(b))$. The goal of this paper is to understand the $J_b(F)$-set $\Sigma^{\mathrm{top}}(X_\mu(b))$. This amounts to considering the following two problems.

(i) Classify the $J_b(F)$-orbits in $\Sigma^{\mathrm{top}}(X_\mu(b))$.

(ii) For each $Z \in \Sigma^{\mathrm{top}}(X_\mu(b))$, determine the stabilizer of Z in $J_b(F)$.

For (i), M. Chen and X. Zhu conjectured (see [12, Conjecture 1.3]) that the set of the $J_b(F)$-orbits in $\Sigma^{\mathrm{top}}(X_\mu(b))$ should be in natural bijection with the Mirkovic–Vilonen basis $\mathbb{MV}_\mu(\lambda_b)$ for a certain weight space of a representation of the dual group $\widehat{G}$. (See

§4.1 for the definition of $\mathbb{MV}_\mu(\lambda_b)$.) Special cases of this conjecture was proved by Xiao–Zhu [43], Hamacher–Viehmann [12] and Nie [31]. The conjecture was finally proved by Nie [32], and by the second and third authors [45] using different methods.

It is known that the stabilizer of every irreducible component of $X_\mu(b)$ is a parahoric subgroup (see [45, Theorem 3.1.1]). For (ii), Xiao–Zhu [43, Theorem 4.4.14] showed that if the element $b \in G(\breve{F})$ is unramified, then the stabilizer of every $Z \in \Sigma^{\mathrm{top}}(X_\mu(b))$ is a hyperspecial subgroup of $J_b(F)$ (see also [45, Theorem 6.2.2]). For general b, it was conjectured by X. Zhu[1] that every stabilizer should be a parahoric subgroup of $J_b(F)$ of maximal volume. [2] Our first main result confirms this conjecture.

Theorem A (See Theorem 4.1.2 and Corollary 4.1.4). *For each $Z \in \Sigma^{\mathrm{top}}(X_\mu(b))$, the stabilizer of Z in $J_b(F)$ is a* very special *parahoric subgroup of $J_b(F)$. In particular, there is an isomorphism of $J_b(F)$-sets*

$$\Sigma^{\mathrm{top}}(X_\mu(b)) \cong \coprod_{\mathbf{a}\in\mathbb{MV}_\mu(\lambda_b)} J_b(F)/\mathcal{J}^{\mathbf{a}},$$

where $\mathcal{J}^{\mathbf{a}} \subset J_b(F)$ is a very special parahoric subgroup.

We refer to §2.2.1 for the definition of very special parahoric subgroups, and Proposition 2.2.5 for the equivalence of this condition with that of having maximal volume. After this result was announced, S. Nie informed us that he could also prove this result using a different method.

For a reductive group over F with no factors of type C-BC_n, the condition that a parahoric is very special determines the parahoric up to conjugation in the adjoint group. Thus when J_b has no factors of type C-BC_n, Theorem A determines the stabilizers up to conjugation by $J_b^{\mathrm{ad}}(F)$. It is an interesting problem to determine the stabilizers up to $J_b(F)$-conjugacy. However Theorem A is already enough for some important applications explained below.

1.2. Application to Shimura varieties

Let $(\mathbf{G}, X)$ be a Shimura datum, and let $\mathrm{K} \subset \mathbf{G}(\mathbb{A}_f)$ be a sufficiently small compact open subgroup. Then we have the associated Shimura variety $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)$ which is an algebraic variety defined over a number field $\mathbf{E}$. Let $p > 2$ be a prime. We assume that $(\mathbf{G}, X)$ is of Hodge type, and that $\mathrm{K} = \mathrm{K}_p\mathrm{K}^p$ where K^p is a compact open subgroup of $\mathbf{G}(\mathbb{A}_f^p)$ and K_p is a hyperspecial subgroup of $\mathbf{G}(\mathbb{Q}_p)$. Then by work of Kisin [21], for any prime $v|p$ of $\mathbf{E}$, there is a smooth canonical integral model $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ of $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)$ over $\mathcal{O}_{\mathbf{E}_{(v)}}$. We write $\underline{\mathrm{Sh}}_{\mathrm{K}}$ for its special fiber.

Write G for $G = \mathbf{G}_{\mathbb{Q}_p}$. There is a stratification of $\underline{\mathrm{Sh}}_{\mathrm{K}}$ indexed by the Kottwitz set $B(G, \mu)$ (cf. §2.4.4). We let $[b]_{\mathrm{bas}}$ denote the unique basic element of $B(G, \mu)$, and we

[1]Private communication.

[2]This conjecture implies that all the stabilizers have the same volume. The latter statement was also conjectured by M. Rapoport.

write $\underline{\mathrm{Sh}}_{\mathrm{K,bas}}$ for the stratum corresponding to $[b]_{\mathrm{bas}}$. This is known as the *basic locus*, and is a generalization of the supersingular locus in the special fiber of a modular curve. The Rapoport–Zink uniformization (see e.g. [43, Corollary 7.2.16]) implies that there is an isomorphism of perfect schemes

$$\underline{\mathrm{Sh}}^{\mathrm{pfn}}_{\mathrm{K,bas}} \cong I(\mathbb{Q})\backslash X_\mu(b) \times \mathbf{G}(\mathbb{A}_f^p)/\mathrm{K}^p. \tag{†}$$

Here I is a certain reductive group over $\mathbb{Q}$ with $I \otimes_{\mathbb{Q}} \mathbb{A}_f^p \cong \mathbf{G} \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ and $I \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong J_b$, and the left hand side denotes the perfection of $\underline{\mathrm{Sh}}_{\mathrm{K,bas}}$. The following theorem then follows immediately from Theorem A and the above isomorphism.

Theorem B (See Corollary 5.2.3). *There exists a bijection between the set of irreducible components of $\underline{\mathrm{Sh}}_{\mathrm{K,bas}}$ of top dimension and the set*

$$\coprod_{\mathbf{a}\in\mathbb{MV}_\mu(\lambda_b)} I(\mathbb{Q})\backslash I(\mathbb{A}_f)/\mathrm{I}_p^{\mathbf{a}}\mathrm{I}^p,$$

where $\mathrm{I}^p \cong \mathrm{K}^p$ and $\mathrm{I}_p^{\mathbf{a}}$ is a very special parahoric subgroup of $J_b(\mathbb{Q}_p)$. Moreover the bijection is equivariant for prime-to-p Hecke operators.

In fact, $\underline{\mathrm{Sh}}_{\mathrm{K,bas}}$ is equidimensional by [12, Theorem 3.4], so we have obtained a description of the set of all irreducible components in this case. We also remark that for Theorem A, the assumption that G is unramified over F is not necessary, and we in fact obtain results for general quasi-split G over F. This allows us to obtain a generalization of Theorem B. The key input for this is a generalization of (†) for the integral models constructed by Kisin–Pappas [24], which we prove in §5.

Theorem B and its generalization reflect the general philosophy going back to Serre and Deuring that components of the basic locus are parameterized by class sets for an inner form of the structure group. We refer to [42], [20], and [28] for some special cases of this result.

The main contribution of this paper is the information that the compact open subgroups $\mathrm{I}_p^{\mathbf{a}} \subset I(\mathbb{Q}_p)$ are very special. For many applications this is a crucial piece of information. For example, in [28], the authors used the description of irreducible components in the supersingular locus of quaternionic Shimura varieties to prove an arithmetic level raising result on the way to proving cases of the Beilinson–Bloch–Kato conjecture. For this, they used the interpretation of functions on $\Sigma^{\mathrm{top}}(\underline{\mathrm{Sh}}_{\mathrm{K,bas}})$ as automorphic forms for I. Thus the knowledge of $\mathrm{I}_p^{\mathbf{a}}$ is needed to determine the level of these automorphic forms. In [27], the authors used a formula for the number of irreducible components in the supersingular locus of unitary Shimura varieties to prove results on the image of the Torelli map. This requires information on the volume of $\mathrm{I}_p^{\mathbf{a}}$.

1.3. The proof of Theorem A

Our proof of Theorem A makes use of techniques from p-adic harmonic analysis developed in [45], and the Deligne–Lusztig reduction method for affine Deligne–Lusztig varieties developed in [14]. For simplicity in the introduction, we assume that G has no factors of type A or E_6. After a series of reduction steps, we can assume that G is an unramified adjoint group over F, that F has characteristic 0, and that $b \in B(G, \mu)$ is basic. It is known that the stabilizer of every $Z \in \Sigma^{\text{top}}(X_\mu(b))$ is a parahoric subgroup of $J_b(F)$, so the question is to prove that such a parahoric subgroup must have maximal volume. The proof proceeds in two steps.

(1) Show that there exists $Z \in \Sigma^{\text{top}}(X_\mu(b))$ whose stabilizer is a parahoric subgroup of $J_b(F)$ of maximal volume.

(2) Show that all the stabilizers have the maximal volume.

The Deligne–Lusztig reduction method in [14] works for the affine Deligne–Lusztig varieties in the affine flag variety. It keeps track of geometric information such as the dimension and the number of irreducible components of top dimension. To keep track of the stabilizers of top-dimensional irreducible components under the action of $J_b(F)$, we introduce a refined reduction method in the context of motivic counting. Then we use the explicit dimension formula for $X_\mu(b)$ and a certain affine Deligne–Lusztig variety $X_{w_0 t^\mu}(b)$ in the affine flag variety to obtain a $J_b(F)$-equivariant bijection $\Sigma^{\text{top}}(X_{w_0 t^\mu}(b)) \xrightarrow{\sim} \Sigma^{\text{top}}(X_\mu(b))$. We combine the explicit reduction path constructed in [14] with a refinement of the argument in [19] to obtain an element of $\Sigma^{\text{top}}(X_{w_0 t^\mu}(b))$ whose stabilizer in $J_b(F)$ has maximal volume. This finishes step (1).

For step (2), consider the quantity

$$Q(\mu, b) := |J_b(F) \backslash \Sigma^{\text{top}}(X_\mu(b))|^{-1} \cdot \sum_Z \text{vol}(\text{Stab}_Z(J_b(F)))^{-1},$$

where the sum is over a set of representatives for the $J_b(F)$-orbits in $\Sigma^{\text{top}}(X_\mu(b))$. The results of [45] imply that $Q(\mu, b)$ depends only on b, not on μ. Moreover, for the given b there exists $\mu_1 \in X_*(T)^+$ such that $|J_b(F) \backslash \Sigma^{\text{top}}(X_{\mu_1}(b))| = 1$. By step (1) applied to (μ_1, b), we know that $Q(\mu_1, b)$ is equal to the inverse of the maximal volume attained by parahoric subgroups of $J_b(F)$. Since $Q(\mu, b) = Q(\mu_1, b)$, and since $\text{Stab}_Z(J_b(F))$ is a parahoric subgroup of $J_b(F)$ for each $Z \in \Sigma^{\text{top}}(X_\mu(b))$, we conclude that $\text{Stab}_Z(J_b(F))$ must be a parahoric subgroup of maximal volume for each Z. This finishes step (2).

For step (2), the assumption that F has characteristic 0 is crucial. This is due to the fact that the results we use from [45] rely on the Base Change Fundamental Lemma, a result only known for characteristic 0 local fields in general.

1.4. Outline of the paper

In §2 we introduce notations and some preliminary group theoretic results. In §2.2, we define very special parahoric subgroups and prove the equivalence of this condition with that of having maximal volume and maximal log volume. We then introduce affine Deligne–Lusztig varieties and establish the relation between components of $X_\mu(b)$ and $X_{w_0 t^\mu}(b)$ in §2.4. In §3, we give a reinterpretation of the Deligne–Lusztig reduction method in terms of motivic counting. We apply this in §3.4 to show the existence of a component in $X_{w_0 t^\mu}(b)$

whose stabilizer is a very special parahoric. In §4 , we prove Theorem A. In §4.2 and §4.3, we reduce the proof to the case where $\mathrm{char}(F) = 0$, G is adjoint, unramified over F, and F-simple, and b is basic. The proof then proceeds in §4.5 and §4.6 as outlined above, with some extra work needed to handle the case of type A and E_6, which is the content of §4.4. Finally in §5, we apply our results to study the basic locus of Shimura varieties and prove Theorem B. As mentioned, the key input is an analogue of the p-adic uniformization for the integral models of Shimura varieties constructed by Kisin–Pappas, which we prove following the method in [43, §7] using results of [44].

2. Affine Deligne–Lusztig varieties

2.1. *The Iwahori–Weyl group*

2.1.1. Let F be a non-archimedean local field with valuation ring O_F and residue field $k_F = \mathbb{F}_q$. We fix an algebraic closure $\overline{F}$ of F. Let F^{ur} be the maximal unramified extension of F inside $\overline{F}$, and let $\breve{F}$ be the completion of F^{ur}. We denote by $O_{\breve{F}}$ the valuation ring of $\breve{F}$, and denote by $\mathbf{k}$ the residue field of $\breve{F}$, which is an algebraic closure of k_F. Fix an algebraic closure $\overline{\breve{F}}$ of $\breve{F}$, and fix an F^{ur}-algebra embedding $\overline{F} \to \overline{\breve{F}}$. We write Γ for $\mathrm{Gal}(\overline{F}/F)$ and write Γ_0 for the inertia subgroup of Γ, which is identified with $\mathrm{Gal}(\overline{\breve{F}}/\breve{F})$. We let $\sigma \in \mathrm{Aut}(\breve{F}/F)$ denote the q-Frobenius.

Let G be a connected reductive group over F. We fix a maximal F^{ur}-split torus S in G defined over F, which exists by [4, Corollaire 5.1.12]. By [37, Proposition 2.3.9], S is also maximal $\breve{F}$-split. Let T be the centralizer of S in G. By Steinberg's theorem G is quasi-split over $\breve{F}$, so T is a maximal torus in G. Let N be the normalizer of T in G, and let

$$\breve{W}_0 := N(\breve{F})/T(\breve{F}).$$

In other words $\breve{W}_0$ is the relative Weyl group of $G_{\breve{F}}$.

The *Iwahori–Weyl group* is defined to be

$$\breve{W} := N(\breve{F})/T(\breve{F})_1$$

where $T(\breve{F})_1$ is the kernel of the Kottwitz homomorphism $T(\breve{F}) \to X_*(T)_{\Gamma_0}$. We have a natural short exact sequence

$$0 \to X_*(T)_{\Gamma_0} \to \breve{W} \to \breve{W}_0 \to 0. \tag{2.1.1.1}$$

For each $\lambda \in X_*(T)_{\Gamma_0}$, we write t^λ for the corresponding element of $\breve{W}$. Such elements of $\breve{W}$ are called *translation elements*.

2.1.2. Let $\breve{\mathscr{A}}$ be the apartment of $G_{\breve{F}}$ corresponding to $S_{\breve{F}}$. Thus $\breve{\mathscr{A}}$ is an affine $\mathbb{R}$-space under $X_*(T)_{\Gamma_0} \otimes_{\mathbb{Z}} \mathbb{R}$. The Frobenius σ and the Iwahori–Weyl group $\breve{W}$ act on $\breve{\mathscr{A}}$ via affine transformations. Since $\breve{\mathscr{A}}$ is naturally identified with the apartment of $G_{F^{\mathrm{ur}}}$ corresponding to $S_{F^{\mathrm{ur}}}$, there exists a σ-stable alcove in $\breve{\mathscr{A}}$ by [40, §1.10.3] as the residue field of F is finite.

We fix such a σ-stable alcove $\breve{\mathfrak{a}}$. Let $\breve{\mathcal{I}} \subset G(\breve{F})$ be the Iwahori subgroup corresponding to $\breve{\mathfrak{a}}$. Then $\breve{\mathcal{I}}$ is σ-stable and we write $\mathcal{I}$ for the corresponding Iwahori subgroup $\breve{\mathcal{I}}^\sigma$ of $G(F)$.

As explained in [10], the choice of $\breve{\mathfrak{a}}$ gives rise to a subgroup $\breve{W}_a$ of $\breve{W}$ called the *affine Weyl group*. This is by definition the subgroup generated by the set $\breve{\mathbb{S}}$ of simple reflections in the walls of $\breve{\mathfrak{a}}$. The pair $(\breve{W}_a, \breve{\mathbb{S}})$ is a Coxeter group.

Let Ω be the stabilizer of $\breve{\mathfrak{a}}$ in $\breve{W}$. Then by [10, Lemma 14], we have

$$\breve{W} = \breve{W}_a \rtimes \Omega,$$

and Ω is (canonically) isomorphic to $\pi_1(G)_{\Gamma_0}$. The length function on the Coxeter group $(\breve{W}_a, \breve{\mathbb{S}})$ extends to a function

$$\breve{\ell} : \breve{W} \longrightarrow \mathbb{Z}_{\geq 0}$$

with respect to which Ω is the set of length-zero elements of $\breve{W}$. The Frobenius σ naturally acts on $\breve{W}$, stabilizing the subset $\breve{\mathbb{S}} \subset \breve{W}$ (as $\breve{\mathfrak{a}}$ is σ-stable). In particular, σ induces an automorphism of the Coxeter group $(\breve{W}_a, \breve{\mathbb{S}})$.

By [10, p. 195], there exists a reduced root system Σ such that

$$\breve{W}_a \cong Q^\vee(\Sigma) \rtimes W(\Sigma),$$

where $Q^\vee(\Sigma)$ and $W(\Sigma)$ denote the coroot lattice and Weyl group of Σ respectively. The roots of Σ are proportional to the roots of the relative root system for $G_{\breve{F}}$. However the root systems themselves may not be isomorphic.

2.1.3. Let $\breve{K}$ be a subset of $\breve{\mathbb{S}}$. We write $\breve{W}_{\breve{K}} \subset \breve{W}$ for the subgroup generated by $\breve{K}$. We let $\breve{W}^{\breve{K}}$ (resp. ${}^{\breve{K}}\breve{W}$) denote the set of minimal length representatives for the cosets in $\breve{W}/\breve{W}_{\breve{K}}$ (resp. $\breve{W}_{\breve{K}}\backslash\breve{W}$).

For each $w \in \breve{W}$, we choose a lift $\dot{w} \in N(\breve{F})$ of w. We assume furthermore that $\sigma(\dot{w}) = \dot{w}$ if $\sigma(w) = w$. Indeed, to see that this can always be arranged, it suffices to see that the Lang map $T(\breve{F})_1 \to T(\breve{F})_1, t \mapsto t\sigma(t)^{-1}$ is surjective. Now $T(\breve{F})_1 = \mathcal{T}^0(O_{\breve{F}})$ where $\mathcal{T}^0$ is the connected Néron model of T over O_F, see [34, Remark 2.2 (iii)]. The desired surjectivity follows from Greenberg's theorem [9, Proposition 3] (whose proof holds regardless of the characteristic of F) applied to $\mathcal{T}^0$.

Let $\breve{K}$ be a subset of $\breve{\mathbb{S}}$ such that $\breve{W}_{\breve{K}}$ is finite. In this case $\breve{K}$ corresponds to a standard parahoric subgroup of $G(\breve{F})$ containing $\breve{\mathcal{I}}$, which we denote by $\breve{\mathcal{K}}$. By the Bruhat decomposition, the map $w \mapsto \dot{w}$ induces a bijection

$$\breve{W}_{\breve{K}}\backslash\breve{W}/\breve{W}_{\breve{K}} \xrightarrow{\sim} \breve{\mathcal{K}}\backslash G(\breve{F})/\breve{\mathcal{K}}.$$

If furthermore $\breve{K}$ is σ-stable, then so is $\breve{\mathcal{K}}$, and we write $\mathcal{K} = \breve{\mathcal{K}}^\sigma$ for the corresponding parahoric subgroup of $G(F)$. In what follows we will often abuse notation and write $\breve{\mathcal{K}}$ (resp. $\mathcal{K}$) for the parahoric group scheme over $O_{\breve{F}}$ (resp. O_F) when there is no risk of confusion. The same is applied to the notations $\breve{\mathcal{I}}$ and $\mathcal{I}$.

2.1.4. Let A denote the maximal F-split subtorus of S, which is also a maximal F-split torus in G. We write Z_A and N_A for the centralizer and normalizer of A in G respectively. Since Z_A is anisotropic modulo center over F, there is a unique parahoric subgroup $\mathcal{Z}_A$ of $Z_A(F)$. The *relative Iwahori–Weyl group* is defined to be

$$W := N_A(F)/\mathcal{Z}_A$$

It admits a natural map to the relative Weyl group $W_0 := N_A(F)/Z_A(F)$ of G over F.

We write $\mathscr{D}$ for the relative local Dynkin diagram of (G, A, F), and write Δ for the set of vertices of $\mathscr{D}$. Let $\mathcal{A}$ be the apartment associated to A, and let $\mathfrak{a}$ be the base alcove in $\mathcal{A}$ determined by the Iwahori subgroup $\mathcal{I}$ of $G(F)$. For each $v \in \Delta$, let α_v be the corresponding non-divisible simple affine root on $\mathcal{A}$. As explained in [40, 1.11], Δ is naturally identified with the set of σ-orbits C in $\breve{\mathbb{S}}$ such that $\breve{W}_C$ is finite. For $v \in \Delta$, we write $C_v \subset \breve{\mathbb{S}}$ for the corresponding σ-orbit, and write $s_v \in W$ for the reflection in $\mathcal{A}$ along α_v. By [36, Lemma 1.6], there is a natural isomorphism $W \cong \breve{W}^\sigma$ induced by the inclusion map $N_A(F) \to N(\breve{F})$. By [29, A.8], s_v corresponds to the longest element of $\breve{W}_{C_v}$ under this isomorphism. We set

$$\mathbb{S} = \{s_v \mid v \in \Delta\}.$$

We also note that if $w \in W$, then the lifting $\dot{w}$ in $N(\breve{F})$ chosen in §2.1.3 is contained in $N_A(F)$, which follows from our assumption that $\dot{w}$ is σ-invariant.

2.2. Parahoric subgroups of maximal volume

We keep the notations of §2.1. In this subsection we give a description of the parahoric subgroups of $G(F)$ that have maximal volume.

2.2.1. For a vertex $v \in \Delta$, we define $d(v) := \breve{\ell}(s_v)$. When G is simply connected and absolutely almost simple, this coincides with the integer attached to v in [40, 1.8], cf. [36, Remark 1.13 (ii)]. We say that a special vertex $v \in \Delta$ is *very special* if $d(v)$ is minimal among all special vertices v' lying in the connected component of $\mathscr{D}$ containing v.

Let $x \in \mathcal{A}$ be a point lying in the closure $\overline{\mathfrak{a}}$ of $\mathfrak{a}$. We associate to x a set of vertices

$$\Delta_x := \{v \in \Delta \mid s_v(x) \neq x\}.$$

It is easy to see that Δ_x has non-empty intersection with each connected component of $\mathscr{D}$.

Definition 2.2.2. A point x lying in the closure $\overline{\mathfrak{a}}$ of $\mathfrak{a}$ is said to be *very special* if Δ_x contains exactly one very special vertex in each connected component of $\mathscr{D}$. A parahoric subgroup of $G(F)$ is said to be *very special* if it is $G(F)$-conjugate to a standard parahoric subgroup associated to a very special $x \in \overline{\mathfrak{a}}$.

Remark 2.2.3. When G is simply connected and absolutely almost simple, our definition of a very special parahoric subgroup is the same as that in [2, A.4]. There is also a notion of a very special parahoric subgroup defined in [47, Definition 6.1]. When G is quasi-split, it can be shown that these two notions are equivalent. However, they differ for non-quasi-split G (cf. [47, Lemma 6.1]).

2.2.4. We now fix a choice of Haar measure on $G(F)$ such that all Iwahori subgroups of $G(F)$ have volume 1. Let $\mathcal{K}$ be a parahoric subgroup of $G(F)$ and $\breve{\mathcal{K}}$ the associated parahoric subgroup of $G(\breve{F})$. We define the *log-volume* of $\mathcal{K}$ by

$$\log\mathrm{vol}(\mathcal{K}) := \dim \overline{\mathcal{K}}/\overline{\mathcal{I}}, \tag{2.2.4.1}$$

where $\overline{\mathcal{K}}$ (resp. $\overline{\mathcal{I}}$) denotes the reductive quotient of the special fiber of $\breve{\mathcal{K}}$ (resp. the image of the special fiber of $\breve{\mathcal{I}}$ in $\overline{\mathcal{K}}$). If $\mathcal{K}$ is a standard parahoric corresponding to a σ-stable subset $\breve{K} \subset \breve{\mathbb{S}}$, then we have

$$\log\mathrm{vol}(\mathcal{K}) = \breve{\ell}(w_{\breve{K}}), \tag{2.2.4.2}$$

where $w_{\breve{K}}$ is the longest element of $\breve{W}_{\breve{K}}$.

We have the Bruhat decompositions

$$\breve{\mathcal{K}} = \coprod_{w \in \breve{W}_{\breve{K}}} \breve{\mathcal{I}}\dot{w}\breve{\mathcal{I}}$$

and

$$\mathcal{K} = \coprod_{w \in \breve{W}_{\breve{K}}^{\sigma}} \mathcal{I}\dot{w}\mathcal{I}.$$

By [36, Proposition 1.11], we have

$$\mathrm{vol}(\mathcal{K}) = \sum_{w \in \breve{W}_{\breve{K}}^{\sigma}} q^{\breve{\ell}(w)}. \tag{2.2.4.3}$$

Proposition 2.2.5. *Let $\mathcal{K}$ be a parahoric subgroup of $G(F)$. Then the following are equivalent:*

(1) *$\mathcal{K}$ is a very special parahoric;*

(2) *$\mathcal{K}$ is of maximal volume among all the parahoric subgroups of $G(F)$;*

(3) *$\mathcal{K}$ has maximal log-volume.*

Remark 2.2.6. When G is simply connected and absolutely almost simple, the equivalence between (1) and (2) is [2, Proposition A.5]. The equivalence between (3) and the other two conditions will be used in the proof of Corollary 4.2.4 below, especially when we alter the local field.

2.2.7. To prove Proposition 2.2.5 we follow the method in [2, A.4]. We begin with some preparation. Assume that G is almost simple over F and let Φ be the relative root system $\Phi(G, A)$. We let Φ^{nd} denote the system of non-divisible roots in Φ and we write $\mathbb{W}$ for the Weyl group of Φ^{nd}, which is identified with the relative Weyl group W_0 of G (see [38, §3.5]).

For an element $v \in \Delta$, we define $K(v) := \mathbb{S} \setminus \{s_v\} \subset \mathbb{S}$. We let $W_{K(v)}$ denote the subgroup of $W \cong \breve{W}^{\sigma}$ generated by $K(v)$. Then the natural map $\mathrm{Aff}(\mathcal{A}) \to \mathrm{GL}(X_*(A) \otimes \mathbb{R})$ (i.e., taking the linear part) induces an identification between $W_{K(v)}$ and a subgroup of $\mathbb{W}$, which we denote by $\mathbb{W}_v$. We denote the inverse isomorphism by $\iota_v : \mathbb{W}_v \xrightarrow{\sim} W_{K(v)}$. For $w \in \mathbb{W}_v$, we set

$$d(w, v) := \breve{\ell}(\iota_v(w)),$$

where we consider $W_{K(v)}$ as a subgroup of $\breve{W}$. For each $v' \in \Delta \setminus \{v\}$, we write $\overline{\alpha}_{v'}$ for the unique proportion of the vector part of $\alpha_{v'}$ that lies in Φ^{nd}. We let Φ_v denote the sub-root system of Φ^{nd} generated by $\overline{\alpha}_{v'}$ with $v' \in \Delta \setminus \{v\}$.

We define an ordering on Φ_v by specifying the positive simple roots to be given by $\overline{\alpha}_{v'}$ with $v' \in \Delta \setminus \{v\}$, and we write Φ_v^+ (resp. Φ_v^-) for the subset of positive (resp. negative) roots. Note that the ordering on Φ_v depends on v; it is possible that there exist $v_1, v_2 \in \Delta$ such that $\Phi_{v_1} = \Phi_{v_2}$ but $\Phi_{v_1}^+ \neq \Phi_{v_2}^+$.

For $\overline{\alpha} \in \Phi_v$, we define an integer $d(\overline{\alpha}, v)$ as follows. If $\overline{\alpha} = \overline{\alpha}_{v'}$ for some $v' \in \Delta \setminus \{v\}$, then we define $d(\overline{\alpha}, v) = d(v')$. In general, we define $d(\overline{\alpha}, v)$ by specifying that its dependence on $\overline{\alpha}$ is $\mathbb{W}_v$-invariant. This is well-defined since if $v_1, v_2, \in \Delta \setminus \{v\}$ are such that $\overline{\alpha}_{v_1}$ and $\overline{\alpha}_{v_2}$ are $\mathbb{W}_v$-conjugate, then $d(v_1) = d(v_2)$; cf. [2, A.4].

Lemma 2.2.8. *For each $w \in \mathbb{W}_v$, we have*

$$d(w, v) = \sum_{\overline{\alpha} \in \Phi_v^+, w\overline{\alpha} \in \Phi_v^-} d(\overline{\alpha}, v). \tag{2.2.8.1}$$

Proof. Let $s_1 \cdots s_n$ be a reduced word decomposition for $w \in \mathbb{W}_v$, where s_i is the simple reflection corresponding to $\overline{\alpha}_{v_i}$ for $v_i \in \Delta \setminus \{v\}$. For $i = 1, \ldots, n$, set $w_i = s_{i+1} \cdots s_n$. Then the association $s_i \mapsto w_i^{-1}\overline{\alpha}_{v_i}$ defines a bijection

$$\{s_1, \ldots, s_n\} \xrightarrow{\sim} \{\overline{\alpha} \in \Phi_v^+ \mid w\overline{\alpha} \in \Phi_v^-\};$$

see [**?**, §10.3, Lemma A]. By $\mathbb{W}_v$-invariance, for $i = 1, \ldots, n$ we have

$$d(w_i^{-1}\overline{\alpha}_{v_i}, v) = d(\overline{\alpha}_{v_i}, v) = \breve{\ell}(s_i).$$

By [36, Sublemma 1.12] and induction, we have

$$d(w, v) = \sum_{i=1}^{n} \breve{\ell}(s_i)$$

and the result follows. ∎

Now let $v_0 \in \Delta$ be a special vertex. By definition (see [40, §1.9]), this means that $\Phi_{v_0} = \Phi^{\mathrm{nd}}$, or equivalently, that we have an isomorphism $\mathbb{W} \cong \mathbb{W}_{v_0}$. Thus the integer, $d(\overline{\alpha}, v_0)$ is well-defined for any $\overline{\alpha} \in \Phi^{\mathrm{nd}}$.

Lemma 2.2.9. *Assume G is almost simple over F and let $v, v_0 \in \Delta$ with v_0 a very special vertex. Then for all $\overline{\alpha} \in \Phi_v$, we have $d(\overline{\alpha}, v) \leq d(\overline{\alpha}, v_0)$.*

Proof. Since $d(\overline{\alpha}, v)$ and $d(\overline{\alpha}, v_0)$ only depend on the $\mathbb{W}_v$-orbit of $\overline{\alpha}$, it suffices to prove this in the case that $\overline{\alpha} \in \Phi_v^+$ is a simple root, i.e. $\overline{\alpha} = \overline{\alpha}_{v'}$ with $v' \in \Delta \setminus \{v\}$. If $v' \neq v_0$, then

$v' \in \Delta\backslash\{v_0\}$ and hence $\overline{\alpha}$ is also a simple root for $\Phi^+_{v_0}$ by the definition of the ordering on $\Phi^+_{v_0}$. We therefore have $d(\overline{\alpha}, v) = d(\overline{\alpha}, v_0) = d(v')$.

If $v' = v_0$, by inspection of Tits' table [40, §4], we find that

$$d(\overline{\alpha}, v) := d(v') = \min_{v''\in\Delta} d(v'')$$

unless G is of type ${}^2A''_{2m-1}$, ${}^2D'_n$, ${}^2D''_{2m}$, ${}^4D_{2m+1}$ or ${}^3E_6$. In these cases, one computes explicitly that $d(\overline{\alpha}, v) \leq d(\overline{\alpha}, v_0)$. ∎

Proof of Proposition 2.2.5. It suffices to prove the result for $\mathcal{K}$ a standard parahoric. We first consider the case where G is adjoint and simple over F. Let $\breve{K}_0, \breve{K} \subset \breve{\mathbb{S}}$ be σ-stable subsets with corresponding parahoric subgroups $\mathcal{K}_0$ and $\mathcal{K}$ of $G(F)$, and corresponding subsets $K_0, K_0 \subset \mathbb{S}$. Assume that $\mathcal{K}_0$ is a very special parahoric. Then we need to show that

$$\mathrm{vol}(\mathcal{K}) \leq \mathrm{vol}(\mathcal{K}_0),$$
$$\log \mathrm{vol}(\mathcal{K}) \leq \log \mathrm{vol}(\mathcal{K}_0)$$

and that strict inequality holds in each case if $\mathcal{K}$ is not very special.

Since $\mathcal{K}_0$ is very special, we have $K_0 = K(v_0)$ for $v_0 \in \Delta$ a very special vertex. Moreover, since $\mathcal{K}$ is contained inside a parahoric corresponding to some $v \in \Delta$, we may assume $K = K(v)$.

Since v_0 is a very special vertex, $\mathbb{W}_{v_0} = \mathbb{W}$ and we have $\Phi^{\mathrm{nd}} = \Phi_{v_0}$. Let $u \in \mathbb{W}_v$ be the unique element such that $u(\Phi^+_v) \subset \Phi^+_{v_0}$. Then $u(\Phi^-_v) \subset \Phi^-_{v_0}$. It follows that the map $\overline{\alpha} \mapsto u(\overline{\alpha})$ induces a bijection

$$\{\overline{\alpha} \in \Phi^+_v, w\overline{\alpha} \in \Phi^-_v\} \xrightarrow{\sim} \{\overline{\alpha} \in u(\Phi^+_v), uwu^{-1}\overline{\alpha} \in \Phi^-_{v_0}\}. \tag{2.2.9.1}$$

Thus for $w \in \mathbb{W}_v$ we have

$$\begin{aligned} d(w, v) &= \sum_{\overline{\alpha}\in\Phi^+_v, w\overline{\alpha}\in\Phi^-_v} d(\overline{\alpha}, v) \\ &= \sum_{\overline{\alpha}\in u(\Phi^+_v), uwu^{-1}\overline{\alpha}\in\Phi^-_{v_0}} d(\overline{\alpha}, v) \\ &\leq \sum_{\overline{\alpha}\in u(\Phi^+_v), uwu^{-1}\overline{\alpha}\in\Phi^-_{v_0}} d(\overline{\alpha}, v_0) \\ &\leq \sum_{\overline{\alpha}\in\Phi^+_{v_0}, uwu^{-1}\overline{\alpha}\in\Phi^-_{v_0}} d(\overline{\alpha}, v_0) = d(uwu^{-1}, v_0). \end{aligned}$$

Here the first equality follows from Lemma 2.2.8. The second equality follows from the bijection (2.2.9.1) and the $\mathbb{W}_v$-invariance of $d(-, v)$. The first inequality follows from Lemma 2.2.9. Thus by (2.2.4.3), we have

$$\mathrm{vol}(\mathcal{K}) = \sum_{w\in\mathbb{W}_v} q^{d(w,v)} \leq \sum_{w\in\mathbb{W}_v} q^{d(w,v_0)} \leq \sum_{w\in\mathbb{W}} q^{d(w,v_0)} = \mathrm{vol}(\mathcal{K}_0).$$

If $\mathcal{K}$ is not special, then the second inequality is strict. If $\mathcal{K}$ is special but not very special, then the first inequality is strict. We thus obtain the equivalence $(1) \Leftrightarrow (2)$.

Similarly, if we let $w_v \in \mathbb{W}_v$ (resp. $w_{v_0} \in \mathbb{W}_{v_0}$) denote the image of $w_{\breve{K}}$ (resp. $w_{\breve{K}_0}$), then we have

$$\begin{aligned}\log \operatorname{vol}(\breve{W}_{\breve{K}}) = d(w_v, v) = \sum_{\overline{\alpha} \in u(\Phi_v^+)} d(\overline{\alpha}, v) \leq \sum_{\overline{\alpha} \in \Phi_{v_0}^+} d(\overline{\alpha}, v_0) \\ = d(w_{v_0}, v_0) = \log \operatorname{vol}(\breve{W}_{\breve{K}_0}).\end{aligned}$$

If $\mathcal{K}$ is not very special, then the inequality is strict. Thus we obtain $(1) \Leftrightarrow (3)$.

The case with general G is reduced to the above special case by considering the direct product decomposition of G_{ad} into F-simple factors. In fact, by (2.2.4.1) (resp. (2.2.4.3)), we know that the log-volume (resp. volume) of a parahoric subgroup of $G(F)$ is equal to the product of the log-volumes (resp. volumes) of corresponding parahoric subgroups of the F-simple factors of G_{ad}. ∎

2.3. σ-conjugacy classes

We keep the setting of §2.1, and assume in addition that G is quasi-split over F.

2.3.1. Under the assumption that G is quasi-split over F, we can fix a σ-stable special point $\breve{\mathfrak{s}}$ lying in the closure of $\breve{\mathfrak{a}}$ (cf. [46, Lemma 6.1]). For an abelian group X and a $\mathbb{Z}$-algebra R, we write X_R for $X \otimes_{\mathbb{Z}} R$. The choice of $\breve{\mathfrak{s}}$ gives rise to a σ-equivariant isomorphism

$$X_*(T)_{\Gamma_0,\mathbb{R}} \cong \breve{\mathscr{A}}, \tag{2.3.1.1}$$

which sends 0 to $\breve{\mathfrak{s}}$. We let $\breve{\mathbb{S}}_0 \subset \breve{\mathbb{S}}$ denote the subset of simple reflections fixing $\breve{\mathfrak{s}}$. Then $\breve{\mathbb{S}}_0$ is preserved by the action of σ. The identification (2.3.1.1) determines a chamber ${X_*(T)_{\Gamma_0,\mathbb{R}}}^+$ in $X_*(T)_{\Gamma_0,\mathbb{R}} \cong X_*(S)_{\mathbb{R}}$ (with respect to the relative roots of $(G_{\breve{F}}, S_{\breve{F}})$), namely the one whose image under (2.3.1.1) contains the alcove $\breve{\mathfrak{a}}$. We let ${X_*(T)_{\Gamma_0}}^+$ (resp. ${X_*(T)_{\Gamma_0,\mathbb{Q}}}^+$) denote the preimage of ${X_*(T)_{\Gamma_0,\mathbb{R}}}^+$ under the map $X_*(T)_{\Gamma_0} \to X_*(T)_{\Gamma_0,\mathbb{R}}$ (resp. under the map $X_*(T)_{\Gamma_0,\mathbb{Q}} \to X_*(T)_{\Gamma_0,\mathbb{R}}$).

Note that ${X_*(T)_{\Gamma_0,\mathbb{R}}}^+$ gives rise to an ordering of the relative roots of $(G_{\breve{F}}, S_{\breve{F}})$. Since G is quasi-split over $\breve{F}$, this uniquely determines an ordering of the absolute roots in $X^*(T)$, and determines a Borel subgroup of $G_{\breve{F}}$ containing $T_{\breve{F}}$. Since C is σ-stable, this Borel subgroup comes from a Borel subgroup B of G containing T.

2.3.2. For $b \in G(\breve{F})$, we let $[b]$ denote the σ-conjugacy class of b, namely

$$[b] = \{h^{-1} b \sigma(h) \mid h \in G(\breve{F})\}.$$

We shall sometimes write $[b]_G$ if we want to specify G. Let $B(G)$ be the set of σ-conjugacy classes in $G(\breve{F})$.

The elements of $B(G)$ have been classified by Kottwitz in [26]. For $b \in G(\breve{F})$, we write $\overline{\nu}_b \in (X_*(T)_{\Gamma_0,\mathbb{Q}}{}^+)^\sigma$ for its dominant Newton point. (Note that $(X_*(T)_{\Gamma_0,\mathbb{Q}}{}^+)^\sigma$ is canonically identified with $(X_*(T)_{\mathbb{Q}}{}^+)^\Gamma$, where $X_*(T)_{\mathbb{Q}}{}^+$ consists of the B-dominant elements of $X_*(T)_{\mathbb{Q}}$.) The map $b \mapsto \overline{\nu}_b$ induces a map $\overline{\nu} : B(G) \to (X_*(T)_{\Gamma_0,\mathbb{Q}}{}^+)^\sigma$.

We let $\tilde{\kappa} : G(\breve{F}) \to \pi_1(G)_{\Gamma_0}$ denote the Kottwitz homomorphism and we write

$$\kappa : G(\breve{F}) \longrightarrow \pi_1(G)_\Gamma$$

for the composition of $\tilde{\kappa}$ with the natural projection $\pi_1(G)_{\Gamma_0} \to \pi_1(G)_\Gamma$. This factors through a map $B(G) \to \pi_1(G)_\Gamma$, which we still denote by κ.

By [26, §4.13], the map

$$(\overline{\nu}, \kappa) : B(G) \longrightarrow (X_*(T)_{\Gamma_0,\mathbb{Q}}{}^+)^\sigma \times \pi_1(G)_\Gamma$$

is injective. We sometimes write $\overline{\nu}^G$ and κ_G for $\overline{\nu}$ and κ if we want to specify G.

An element $b \in G(\breve{F})$ is said to be *basic* if $\overline{\nu}_b$ is central. Similarly we define *basic* elements of $B(G)$.

2.3.3. For $b \in G(\breve{F})$, let J_b denote the σ-centralizer group of b. It is a reductive group over F such that

$$J_b(R) = \{g \in G(\breve{F} \otimes_F R) \mid g\text{ß}b\sigma(g) = b\}$$

for any F-algebra R. Let M be the centralizer of $\overline{\nu}_b$, where we consider $\overline{\nu}_b$ as an element of $(X_*(T)_{\mathbb{Q}}{}^+)^\Gamma \subset (X_*(T)^\Gamma)_{\mathbb{Q}}$ as explained in §2.3.2. Then M is a Levi subgroup of G defined over F and J_b is an inner form of M over F.

2.3.4. The maps $\overline{\nu}$ and κ on $B(G)$ can be described in a more explicit way as follows. Let $B(\breve{W}, \sigma)$ be the set of σ-conjugacy classes in $\breve{W}$. The map $\breve{W} \to G(\breve{F})$, $w \mapsto \dot{w}$ defined in §2.1 induces a well-defined map

$$B(\breve{W}, \sigma) \longrightarrow B(G). \tag{2.3.4.1}$$

For each $w \in \breve{W}$, there exists a positive integer n such that σ^n acts trivially on $\breve{W}$ and such that $w\sigma(w) \cdots \sigma^{n-1}(w) = t^\lambda$ for some $\lambda \in X_*(T)_{\Gamma_0}$. We set $\nu_w := \frac{\lambda}{n} \in X_*(T)_{\Gamma_0,\mathbb{Q}}$ and we let $\overline{\nu}_w$ denote the unique $\breve{W}_0$-conjugate of ν_w that lies in $X_*(T)_{\Gamma_0,\mathbb{Q}}{}^+$. Then $\overline{\nu}_w$ is necessarily fixed by σ. We let $\tilde{\kappa}(w) \in \pi_1(G)_{\Gamma_0}$ denote the image of w under the quotient map $\breve{W} \to \breve{W}/\breve{W}_a \cong \pi_1(G)_{\Gamma_0}$, and we let $\kappa(w)$ be the image of $\tilde{\kappa}(w)$ in $\pi_1(G)_\Gamma$. By [14, Proposition 3.6], we have a commutative diagram:

$$\begin{tikzcd}
B(\breve{W}, \sigma) \arrow[rr, "(2.3.4.1)"] \arrow[dr, "(\overline{\nu}{,}\kappa)"'] & & B(G) \arrow[dl, "(\overline{\nu}{,}\kappa)"] \\
& (X_*(T)_{\Gamma_0,\mathbb{Q}}{}^+)^\sigma \times \pi_1(G)_\Gamma &
\end{tikzcd}.$$

2.4. *Affine Deligne–Lusztig varieties*

We keep the setting and notation of §2.3. We assume in addition that G splits over a tamely ramified extension of F and that $\mathrm{char}(F)$ is either zero or coprime to the order of $\pi_1(G_{\mathrm{ad}})$.

2.4.1. Let $\breve{K} \subset \breve{\mathbb{S}}$ be a σ-stable subset that corresponds to a parahoric subgroup $\breve{\mathcal{K}} \subset G(\breve{F})$. For $w \in \breve{W}_{\breve{K}} \backslash \breve{W} / \breve{W}_{\breve{K}}$ and $b \in G(\breve{F})$, we set

$$X_{\breve{K},w}(b)(\mathbf{k}) = \{g\breve{\mathcal{K}} \in G(\breve{F})/\breve{\mathcal{K}} \mid g ß b\sigma(g) \in \breve{\mathcal{K}}\dot{w}\breve{\mathcal{K}}\}. \tag{2.4.1.1}$$

If $\mathrm{char}(F) > 0$, then $X_{\breve{K},w}(b)(\mathbf{k})$ is the set of $\mathbf{k}$-points of a locally closed sub-scheme $X_{\breve{K},w}(b)$ of the partial affine flag variety $\mathrm{Gr}_{\breve{\mathcal{K}}}$. In this case $X_{\breve{K},w}(b)$ is locally of finite type over $\mathbf{k}$ (cf. [33]). If $\mathrm{char}(F) = 0$, then $X_{\breve{K},w}(b)(\mathbf{k})$ is the set of $\mathbf{k}$-points of a locally closed sub-scheme $X_{\breve{K},w}(b)$ of the Witt vector partial affine flag variety $\mathrm{Gr}_{\breve{\mathcal{K}}}$ constructed by X. Zhu [47] and Bhatt–Scholze [1]. In this case $X_{\breve{K},w}(b)$ is locally of perfectly finite type over $\mathbf{k}$ (see [13, Theorem 1.1]). In both cases, we call $X_{\breve{K},w}(b)$ the *affine Deligne–Lusztig variety* associated to b, w, and $\breve{K}$.

The group $J_b(F)$ (see §2.3.3) acts on $X_{\breve{K},w}(b)$ via $\mathbf{k}$-scheme automorphisms. By [13, Theorem 1.1], the induced $J_b(F)$-action on the set of irreducible components of $X_{\breve{K},w}(b)$ has finitely many orbits. The results in [13] also have the following easy consequence.

Lemma 2.4.2. *Every irreducible component of $X_{\breve{K},w}(b)$ is quasi-compact.*

Proof. Let Z be an irreducible component of $X_{\breve{K},w}(b)$. By [13, Proposition 5.4], there is a dense open subset $U \subset Z$ which is contained in a finite union $\bigcup_i S_i$ of Schubert varieties in $\mathrm{Gr}_{\breve{\mathcal{K}}}$. Since the Schubert varieties are closed in $\mathrm{Gr}_{\breve{\mathcal{K}}}$, we have $Z \subset \bigcup_i S_i$. Moreover, since Z is closed in $X_{\breve{K},w}(b)$, it is locally closed in $\bigcup_i S_i$. Now the Schubert varieties are of finite type over $\mathbf{k}$ when $\mathrm{char}(F) > 0$ and of perfectly finite type over $\mathbf{k}$ when $\mathrm{char}(F) = 0$ (cf. [13, §4]), so the underlying topological space of $\bigcup_i S_i$ is Noetherian. It follows that Z is quasi-compact. ∎

2.4.3. We are mainly interested in $X_{\breve{K},w}(b)$ in the following two cases:

- (Iwahori level.) We have $\breve{K} = \emptyset$, i.e., $\breve{\mathcal{K}} = \breve{\mathcal{I}}$.
- (Maximal special level.) We have $\breve{K} = \breve{\mathbb{S}}_0$, i.e., $\breve{\mathcal{K}}$ is the maximal special parahoric subgroup corresponding to the special point $\breve{\mathfrak{s}}$.

When $\breve{K} = \emptyset$, we simply write $X_w(b)$ for $X_{\emptyset,w}(b)$. When $\breve{K} = \breve{\mathbb{S}}_0$, the restriction of the natural map $\breve{W} \to \breve{W}_0$ to $\breve{W}_{\breve{K}} \subset \breve{W}$ induces an isomorphism $\breve{W}_{\breve{K}} \xrightarrow{\sim} \breve{W}_0$. In other words, our choice of $\breve{\mathfrak{s}}$ determines a splitting of the exact sequence (2.1.1.1). In this case we shall identify $\breve{W}_0$ with $\breve{W}_{\breve{K}}$, viewed as a subgroup of $\breve{W}$. We have natural bijections

$$X_*(T)_{\Gamma_0}{}^+ \cong X_*(T)_{\Gamma_0}/\breve{W}_0 \cong \breve{W}_0 \backslash \breve{W} / \breve{W}_0.$$

Here the first bijection follows from the fact that $X_*(T)_{\Gamma_0,\mathbb{R}}{}^+$ is naturally isomorphic to $X_*(T)_{\Gamma_0,\mathbb{R}}/W_0$, together with the observation that the torsion part $X_*(T)_{\Gamma_0,\mathrm{tors}}$ of $X_*(T)_{\Gamma_0}$

injects into ${X_*(T)_{\Gamma_0}}^+$ (which follows from the definitions) and injects into $X_*(T)_{\Gamma_0}/\breve{W}_0$ (because $X_*(T)_{\Gamma_0}/\breve{W}_0$ injects into $\pi_1(G)_{\Gamma_0}$, whereas the kernel of $X_*(T)_{\Gamma_0} \to \pi_1(G)_{\Gamma_0}$ is torsion-free, being the free abelian group generated by the Γ_0-orbits of absolute coroots). The second bijection is induced by the inclusion $X_*(T)_{\Gamma_0} \hookrightarrow \breve{W}, \mu \mapsto t^\mu$ (see (2.1.1.1)). For $\mu \in {X_*(T)_{\Gamma_0}}^+$, we write $X_\mu(b)$ for $X_{\breve{\mathbb{S}}_0, t^\mu}(b)$. We sometimes write $X_\mu^G(b)$ for $X_\mu(b)$ if we need to specify the group G. If G is unramified over F, then every cocharacter μ' of $G_{\overline{F}}$ is conjugate to a unique element $\mu \in {X_*(T)_{\Gamma_0}}^+ = X_*(T)_{\Gamma_0}$. In this case we also write $X_{\mu'}(b)$ for $X_\mu(b)$.

2.4.4. For $\lambda, \lambda' \in X_*(T)_{\Gamma_0,\mathbb{Q}} \cong X_*(S)_\mathbb{Q}$, we write $\lambda \leq \lambda'$ if $\lambda' - \lambda$ is a non-negative rational linear combination of the positive coroots in $X_*(S)$ (with respect to $(G_{\breve{F}}, S_{\breve{F}})$ and the ordering defined in §2.3.1).

For $\mu \in {X_*(T)_{\Gamma_0}}^+$, we define

$$B(G,\mu) := \{[b] \in B(G) \mid \overline{\nu}_b \leq \mu^\diamond, \kappa(b) = \mu^\natural\}.$$

Here $\mu^\natural$ is the image of μ in $\pi_1(G)_\Gamma$, and $\mu^\diamond \in {X_*(T)_{\Gamma_0,\mathbb{Q}}}^+$ denotes the average of the σ-orbit of the image of μ in ${X_*(T)_{\Gamma_0,\mathbb{Q}}}^+$. The set $B(G,\mu)$ has a unique basic element, which is also the unique minimal element with respect to the natural partial order on $B(G,\mu)$ (see [18, §2]).

The following criterion for the non-emptiness of $X_\mu(b)$, originally conjectured by Kottwitz and Rapoport, was proved by Gashi [6] for unramified groups and by the first-named author [14, Theorem 7.1] in general.

Theorem 2.4.5. *For $\mu \in {X_*(T)_{\Gamma_0}}^+$, we have $X_\mu(b) \neq \emptyset$ if and only if $[b] \in B(G,\mu)$.* ∎

2.4.6. Now we recall the dimension formula for $X_\mu(b)$. For $b \in G(\breve{F})$, the *defect* of b is defined as

$$\mathrm{def}_G(b) := \mathrm{rank}_F\, G - \mathrm{rank}_F\, J_b.$$

We let ρ denote the half sum of positive roots in the root system Σ (see §2.1). The following theorem was proved by Görtz–Haines–Kottwitz–Reumann [7] and [41] for split G, and by Hamacher [11] and X. Zhu [47] independently for unramified groups. The result in general was proved by the first-named author [16, Theorem 2.29].

Theorem 2.4.7. *Assume $[b] \in B(G,\mu)$. Then we have*

$$\dim X_\mu(b) = \langle \mu - \overline{\nu}_b, \rho\rangle - \frac{1}{2}\mathrm{def}_G(b).$$

∎

Definition 2.4.8. For a scheme X of finite Krull dimension and each non-negative integer d, we write $\Sigma^d(X)$ for the set of irreducible components of X of dimension d (which is allowed to be empty). We write $\Sigma^{\mathrm{top}}(X)$ for $\Sigma^{\dim(X)}(X)$. We write $\Sigma(X)$ for the set of all irreducible components of X.

2.4.9. The main object of interest in this paper is the set $\Sigma^{\mathrm{top}}(X_\mu(b))$. To study this set it will be useful to relate $X_\mu(b)$ to a certain affine Deligne–Lusztig variety with Iwahori level.

We have a natural projection map

$$\pi : \mathrm{Gr}_{\breve{\mathcal{I}}} \longrightarrow \mathrm{Gr}_{\breve{\mathcal{K}}}$$

between the partial affine flag varieties, which exhibits $\mathrm{Gr}_{\breve{\mathcal{I}}}$ as an étale fibration over $\mathrm{Gr}_{\breve{\mathcal{K}}}$ with fibers isomorphic to $\overline{\mathcal{K}}/\overline{\mathcal{I}}$ when $\mathrm{char}(F) > 0$ (resp. the perfection of $\overline{\mathcal{K}}/\overline{\mathcal{I}}$ when $\mathrm{char}(F) = 0$). See §2.2.4 for $\overline{\mathcal{K}}/\overline{\mathcal{I}}$.

As in §2.4.3, we identify $\breve{W}_0$ with the subgroup $\breve{W}_{\breve{\mathbb{S}}_0}$ of $\breve{W}$. For $\mu \in X_*(T)_{\Gamma_0}{}^+$, the map π induces a $J_b(F)$-equivariant map

$$\bigcup_{w \in \breve{W}_0 t^\mu \breve{W}_0} X_w(b) \longrightarrow X_\mu(b). \tag{2.4.9.1}$$

In fact, the left hand side is equal to $\pi^{-1}(X_\mu(b))$.

Proposition 2.4.10. *Let w_0 denote the longest element of $\breve{W}_0$. The map $X_{w_0 t^\mu}(b) \to X_\mu(b)$ induces a $J_b(F)$-equivariant bijection*

$$\Sigma^{\mathrm{top}}(X_{w_0 t^\mu}(b)) \xrightarrow{\sim} \Sigma^{\mathrm{top}}(X_\mu(b)).$$

Proof. Write Y for the left hand side of (2.4.9.1). Since the map (2.4.9.1) is a $J_b(F)$-equivariant fibration with irreducible fibers, it induces a $J_b(F)$-equivariant bijection

$$\Sigma^{\mathrm{top}}(Y) \xrightarrow{\sim} \Sigma^{\mathrm{top}}(X_\mu(b)).$$

Note that the $J_b(F)$-action on Y stabilizes $X_w(b)$ for each $w \in \breve{W}_0 t^\mu \breve{W}_0$. Moreover, each $X_w(b)$ is locally closed in Y. By [14, Theorem 9.1], for $w \in \breve{W}_0 t^\mu \breve{W}_0$, we have

$$\dim X_w(b) \leq \dim X_{w_0 t^\mu}(b)$$

with equality if and only if $w = w_0 t^\mu$. Thus the inclusion map $X_{w_0 t^\mu}(b) \hookrightarrow Y$ induces a $J_b(F)$-equivariant bijection

$$\Sigma^{\mathrm{top}}(X_{w_0 t^\mu}(b)) \xrightarrow{\sim} \Sigma^{\mathrm{top}}(Y).$$

The statement is proved. ∎

3. Deligne–Lusztig reduction method and motivic counting

3.1. The Grothendieck–Deligne–Lusztig monoid

Recall that $\mathbf{k}$ is a fixed algebraic closure of $\mathbb{F}_q$. Let H be an abstract group. We retain the notations introduced in Definition 2.4.8.

Definition 3.1.1. Let $\mathcal{S}^H$ be the category of perfect $\mathbf{k}$-schemes V that are equipped with an H-action and satisfy the following conditions:

(1) The scheme V is locally of perfectly finite type over $\mathbf{k}$.

(2) Each irreducible component of V is quasi-compact.

(3) The H-action on $\Sigma(V)$ has finitely many orbits.

We define morphisms in $\mathcal{S}^H$ to be the $\mathbf{k}$-scheme morphisms that are H-equivariant.

3.1.2. It is a simple exercise to check that the category $\mathcal{S}^H$ is essentially small. Thus the isomorphism classes in $\mathcal{S}^H$ form a set. Let $\mathbb{N}[\mathcal{S}^H]$ be the free commutative monoid generated by this set. For any object V in $\mathcal{S}^H$, we denote by $[V]$ the element of $\mathbb{N}[\mathcal{S}^H]$ given by the isomorphism class of V.

For any $\mathbf{k}$-scheme Q, we write Q^{pfn} for the perfection of Q, which is a perfect $\mathbf{k}$-scheme. We write $\mathbb{A}^1$ for $\mathbb{A}^1_{\mathbf{k}}$, and write $\mathbb{G}_m$ for $\mathbb{A}^1_{\mathbf{k}} - \{0\}$. Then $\mathbb{G}_m^{\mathrm{pfn}}$ equipped with the trivial H-action is an object in $\mathcal{S}^H$. Moreover, if V is in $\mathcal{S}^H$, then $V \times_{\mathbf{k}} \mathbb{G}_m^{\mathrm{pfn}}$ equipped with the product H-action is also in $\mathcal{S}^H$. We thus define an endomorphism $\mathbb{T}$ of $\mathbb{N}[\mathcal{S}^H]$ by

$$[V] \longmapsto [V \times_{\mathbf{k}} \mathbb{G}_m^{\mathrm{pfn}}], \quad \text{for any object } V \text{ in } \mathcal{S}^H.$$

Lemma 3.1.3. *Let V be an object in $\mathcal{S}^H$, and let U be an H-stable open subscheme of V. Then U equipped with the induced H-action is an object in $\mathcal{S}^H$.*

Proof. Clearly U satisfies condition (1) in Definition 3.1.1. We verify the other two conditions. For each $Z \in \Sigma(U)$, the closure $\overline{Z}$ of Z in V is an element of $\Sigma(V)$. Conversely, for each $Z' \in \Sigma(V)$, either $Z' \cap U = \emptyset$, or $Z' \cap U$ is an element of $\Sigma(U)$. Hence we have a bijection

$$\Sigma(U) \xrightarrow{\sim} \{Z' \in \Sigma(V) \mid Z' \cap U \neq \emptyset\}, \quad Z \mapsto \overline{Z}.$$

The right hand side is an H-stable subset of $\Sigma(V)$, and the bijection is H-equivariant. Since V satisfies condition (3), so does U.

Since V satisfies conditions (1) and (2), each $Z' \in \Sigma(V)$ is Noetherian as a topological space. For an arbitrary $Z \in \Sigma(V)$, we know that Z is open in $\overline{Z}$ (since $Z = \overline{Z} \cap U$), and that $\overline{Z}$ is noetherian (since $\overline{Z} \in \Sigma(U)$). Hence Z is quasi-compact. Thus U satisfies condition (2). ∎

Definition 3.1.4. Let $\sim$ be the minimal equivalence relation on $\mathbb{N}[\mathcal{S}^H]$ generated by the following rules.

(1) If there is a morphism $V_1 \to V_2$ in $\mathcal{S}^H$ such that forgetting the H-actions this is a Zariski-locally trivial $\mathbb{G}_m^{\mathrm{pfn}}$-bundle, then $[V_1] \sim \mathbb{T}[V_2]$.

(2) If there is a morphism $V_1 \to V_2$ in $\mathcal{S}^H$ such that forgetting the H-actions this is a Zariski-locally trivial $\mathbb{A}^{1,\mathrm{pfn}}$-bundle, then $[V_1] \sim \mathbb{T}[V_2] + [V_2]$.

(3) Suppose there is a morphism $V' \to V$ in $\mathcal{S}^H$ that is a closed embedding. By Lemma 3.1.3, the open subscheme $V \setminus V'$ of V is an object in $\mathcal{S}^H$. We require that $[V] \sim [V'] + [V \setminus V']$.

3.1.5. We recall the general notion of a quotient monoid. Let $(M,+)$ be a commutative monoid. An equivalence relation $\equiv$ on M is called a *congruence*, if for all $x,x',y,y'\in M$ such that $x\equiv x'$ and $y\equiv y'$, we have $x+y\equiv x'+y'$. If $\equiv$ is a congruence, then the quotient set $M/\!\equiv$ inherits from M the structure of a commutative monoid. This is called the *quotient monoid* of M by $\equiv$. Starting with an arbitrary equivalence relation $\sim$ on M, we obtain a congruence $\equiv$ on M by declaring $x\equiv y$ if and only if we can write $x=\sum_{i=1}^n x_i$ and $y=\sum_{i=1}^n y_i$ for some $x_i,y_i\in M$ such that $x_i\sim y_i$ for each i.

Definition 3.1.6. Let $\equiv$ be congruence on $\mathbb{N}[\mathcal{S}^H]$ associated to $\sim$, and let GDL^H be the quotient monoid $\mathbb{N}[\mathcal{S}^H]/\!\equiv$. We call GDL^H the *Grothendieck–Deligne–Lusztig monoid*. For any object V in $\mathcal{S}^H$, we denote the image of $[V]$ under $\mathbb{N}[\mathcal{S}^H]\to\mathrm{GDL}^H$ by $[\![V]\!]$.

3.1.7. One easily checks that the endomorphism $\mathbb{T}$ of $\mathbb{N}[\mathcal{S}^H]$ descends to an endomorphism of GDL^H, which we still denote by $\mathbb{T}$. We write $\mathbb{L}$ for $\mathbb{T}+1\in\mathrm{End}(\mathrm{GDL}^H)$.

3.2. Calculus of top irreducible components

3.2.1. Let H be an abstract group as before. One can formally calculate "top-dimensional irreducible components" of elements of GDL^H. To this end we first introduce a commutative monoid TIC^H which is much simpler than GDL^H and serves to record information about top-dimensional irreducible components. Let $\mathcal{S}et_f^H$ be the category of H-sets which contain only finitely many H-orbits. This is an essentially small category. We let TIC^H be the set of pairs (Σ,d), where Σ is an isomorphism class in $\mathcal{S}et_f^H$, and $d\in\mathbb{Z}_{\geq 0}$. Given two elements $(\Sigma_1,d_1),(\Sigma_2,d_2)\in\mathrm{TIC}^H$, we define their sum to be

$$(\Sigma_1,d_1)+(\Sigma_2,d_2):=\begin{cases}(\Sigma_1,d_1), & \text{if } d_1>d_2,\\ (\Sigma_2,d_2), & \text{if } d_2>d_1,\\ (\Sigma_1\sqcup\Sigma_2,d_1), & \text{if } d_1=d_2.\end{cases}$$

This makes TIC^H a commutative monoid. In the above definition of the sum, if $d_1\geq d_2$, then we say that (Σ_1,d_1) *makes non-trivial contribution to the sum.*

Define an endomorphism $\mathbb{T}$ of TIC^H by

$$\mathbb{T}:(\Sigma,d)\longmapsto(\Sigma,d+1).$$

We write $\mathbb{L}$ for $\mathbb{T}+1\in\mathrm{End}(\mathrm{TIC}^H)$; it is easy to see that in fact $\mathbb{L}=\mathbb{T}$ in $\mathrm{End}(\mathrm{TIC}^H)$.

Note that every object V in $\mathcal{S}^H$ has finite Krull dimension. The sets $\Sigma(V)$ and $\Sigma^d(V)$ for all $d\in\mathbb{Z}_{\geq 0}$ (Definition 2.4.8) equipped with the natural H-actions are all objects in $\mathcal{S}et_f^H$.

Definition 3.2.2. For any $X=\sum_{i=1}^n[V_i]\in\mathbb{N}[\mathcal{S}^H]$, we define

$$\dim X:=\max_{1\leq i\leq n}\dim V_i\in\mathbb{Z}_{\geq 0},$$

and define

$$\Sigma^{\mathrm{top}}(X) := \coprod_{1\le i\le n} \Sigma^{\dim X}(V_i),$$

which is an object in $\mathcal{S}et_f^H$. The pair consisting of the isomorphism class of $\Sigma^{\mathrm{top}}(X)$ and the integer $\dim X$ is thus an element of TIC^H, which we denote by $\tilde{\mathfrak{C}}(X) \in \mathrm{TIC}^H$.

Lemma 3.2.3. *The map $\tilde{\mathfrak{C}} : \mathbb{N}[\mathcal{S}^H] \to \mathrm{TIC}^H$ is a monoid homomorphism, and descends to a monoid homomorphism $\mathfrak{C} : \mathrm{GDL}^H \to \mathrm{TIC}^H$. Moreover, $\mathfrak{C}$ is equivariant with respect to the endomorphisms $\mathbb{T}$ on GDL^H and $\mathbb{T}$ on TIC^H (see §3.1.7 and §3.2.1).*

Proof. It follows from the definitions that $\tilde{\mathfrak{C}}$ is a monoid homomorphism. To show that $\tilde{\mathfrak{C}}$ descends to GDL^H, it suffices to check that any $X, X' \in \mathbb{N}[\mathcal{S}^H]$ with $X \sim X'$ satisfies $\tilde{\mathfrak{C}}(X) = \tilde{\mathfrak{C}}(X')$. For this, we only need to analyze the three situations in Definition 3.1.4. Namely, we may assume that X and X' are the two sides of $\sim$ in those situations.

In situation (1), we have $X = [V_1]$ and $X' = \mathbb{T}[V_2]$. We have $\dim V_1 = \dim V_2 + 1$, and taking the inverse image along $V_1 \to V_2$ induces an H-equivariant bijection $\Sigma^{\mathrm{top}}(V_2) \xrightarrow{\sim} \Sigma^{\mathrm{top}}(V_1)$. (In fact we have an H-equivariant bijection $\Sigma^d(V_2) \xrightarrow{\sim} \Sigma^{d+1}(V_1)$ for arbitrary d.) Thus we have $\tilde{\mathfrak{C}}([V_1]) = \mathbb{T}\tilde{\mathfrak{C}}([V_2])$. For the same reason, we also have $\tilde{\mathfrak{C}}(\mathbb{T}[V_2]) = \mathbb{T}\tilde{\mathfrak{C}}([V_2])$. Thus we have $\tilde{\mathfrak{C}}([V_1]) = \tilde{\mathfrak{C}}(\mathbb{T}[V_2])$ as desired.

One treats situation (2) similarly, noting that $\tilde{\mathfrak{C}}(\mathbb{T}[V_2] + [V_2]) = \tilde{\mathfrak{C}}(\mathbb{T}[V_2])$.

Now consider situation (3). We have $X = [V]$ and $X' = [V'] + [V \setminus V']$. Observe that for each $Z \in \Sigma(V)$, precisely one of the following two statements holds:

- We have $Z \subset V'$, and $Z \in \Sigma(V')$.
- The intersection $Z_1 := Z \cap (V\backslash V')$ is dense in Z. Moreover, $Z_1 \in \Sigma(V\backslash V')$, and $\dim Z_1 = \dim Z$ (cf. the proof of Lemma 3.1.3).

It follows that for each $d \in \mathbb{Z}_{\ge 0}$ we have an H-equivariant bijection

$$\begin{aligned} \Sigma^d(V') \sqcup \Sigma^d(V\backslash V') &\xrightarrow{\sim} \Sigma^d(V) \\ Z &\longmapsto \bar{Z}. \end{aligned}$$

Therefore we have $\tilde{\mathfrak{C}}([V]) = \tilde{\mathfrak{C}}([V'] + [V\backslash V'])$, as desired. We have proved that $\tilde{\mathfrak{C}}$ descends to GDL^H.

For any V in $\mathcal{S}^H$, we have $\dim(V \times_{\mathbf{k}} \mathbb{G}_m^{\mathrm{pfn}}) = \dim(V) + 1$, and we have a natural H-equivariant bijection $\Sigma^{\mathrm{top}}(V \times_{\mathbf{k}} \mathbb{G}_m^{\mathrm{pfn}}) \xrightarrow{\sim} \Sigma^{\mathrm{top}}(V)$. It follows that $\tilde{\mathfrak{C}}$ is equivariant with respect to $\mathbb{T}$ on the two sides. Since $\mathfrak{C}$ is induced by $\tilde{\mathfrak{C}}$, it is also equivariant with respect to $\mathbb{T}$ on the two sides. ■

3.3. *Class polynomials and motivic counting*

We assume that G is as in §2.4, i.e., G is quasi-split, tamely ramified, and $\mathrm{char}(F) \nmid |\pi_1(G_{\mathrm{ad}})|$ if $\mathrm{char}(F) > 0$. Then we have the affine Deligne–Lusztig variety $X_w(b)$ associated to $w \in \breve{W}$ and $b \in G(\breve{F})$. The motivation behind the definition of the Grothendieck–Deligne–Lusztig monoid is that it gives a natural setting to apply the Deligne–Lusztig reduction method for $X_w(b)$. We recall the reduction method in the proposition below.

Proposition 3.3.1. *Let $w \in \breve{W}$, $s \in \breve{\mathbb{S}}$, and $b \in G(\breve{F})$. If* $\mathrm{char}(F) > 0$*, then the following two statements hold.*

(1) *If $\breve{\ell}(sw\sigma(s)) = \breve{\ell}(w)$, then there exists a $J_b(F)$-equivariant morphism*

$$X_w(b) \longrightarrow X_{sw\sigma(s)}(b)$$

which is a universal homeomorphism.

(2) *If $\breve{\ell}(sw\sigma(s)) = \breve{\ell}(w) - 2$, then $X_w(b)$ has a $J_b(F)$-stable closed subscheme X_1 satisfying the following conditions:*
   - *There exist a* **k***-scheme Y_1 with a $J_b(F)$-action, and $J_b(F)$-equivariant morphisms $f_1 : X_1 \to Y_1$ and $g_1 : Y_1 \to X_{sw\sigma(s)}(b)$, where f_1 is a Zariski-locally trivial $\mathbb{A}^1$-bundle and g_1 is a universal homeomorphism.*
   - *Let X_2 be the open subscheme of $X_w(b)$ complement to X_1, which is $J_b(F)$-stable. There exist a* **k***-scheme Y_2 with a $J_b(F)$-action, and $J_b(F)$-equivariant morphisms $f_2 : X_2 \to Y_2$ and $g_2 : Y_2 \to X_{sw}(b)$, where f_2 is a Zariski-locally trivial $\mathbb{G}_m$-bundle and g_2 is a universal homeomorphism.*

If $\mathrm{char}(F) = 0$*, then the above two statements still hold, but with "$\mathbb{A}^1$-bundle" and "$\mathbb{G}_m$-bundle" replaced by "$\mathbb{A}^{1,\mathrm{pfn}}$-bundle" and "$\mathbb{G}_m^{\mathrm{pfn}}$-bundle" respectively.*

Proof. The equal characteristic case is proved in [8, §2.5]. The mixed characteristic case follows from the same proof. ∎

3.3.2. Let $w \in \breve{W}$ and $b \in G(\breve{F})$. By the discussion in §2.4.1 and Lemma 2.4.2, we know that the perfection $X_w(b)^{\mathrm{pfn}}$ of $X_w(b)$ is an object in $\mathcal{S}^{J_b(F)}$. (Of course $X_w(b) = X_w(b)^{\mathrm{pfn}}$ if $\mathrm{char}(F) = 0$.) To simplify the notation, we write $[[X_w(b)]]$ for the element $[[X_w(b)^{\mathrm{pfn}}]] \in \mathrm{GDL}^{J_b(F)}$.

Using the formalism in §3.1, we can reformulate Proposition 3.3.1 in the following proposition (which is weaker, but more convenient for applications).

Proposition 3.3.3. *Let $w \in \breve{W}$, $s \in \breve{\mathbb{S}}$, and $b \in G(\breve{F})$. The following statements hold.*

(1) *If $\breve{\ell}(sw\sigma(s)) = \breve{\ell}(w)$, then*

$$[[X_w(b)]] = [[X_{sw\sigma(s)}(b)]] \in \mathrm{GDL}^{J_b(F)}.$$

(2) *If $\breve{\ell}(sw\sigma(s)) = \breve{\ell}(w) - 2$, then*

$$[[X_w(b)]] = (\mathbb{L} - 1)[[X_{sw}(b)]] + \mathbb{L}\,[[X_{sw\sigma(s)}(b)]] \in \mathrm{GDL}^{J_b(F)}.$$

Proof. This follows from Proposition 3.3.1 and the following three observations. Firstly, if a morphism of **k**-schemes is universally homeomorphic, then the perfection of this morphism is an isomorphism, by [1, Lemma 3.8]. Secondly, if a morphism of **k**-schemes is a Zariski-locally trivial $\mathbb{A}^1$-bundle (resp. $\mathbb{G}_m$-bundle), then the perfection of this morphism is a Zariski-locally trivial $\mathbb{A}^{1,\mathrm{pfn}}$-bundle (resp. $\mathbb{G}_m^{\mathrm{pfn}}$-bundle). For this one uses that perfection does not change the Zariski topology, and commutes with fiber products over **k** (by the universal property of the perfection functor). Thirdly, the perfections of the **k**-schemes X_1, X_2, Y_1, Y_2 in Proposition 3.3.1 (2), equipped with the natural $J_b(F)$-actions, are all objects in $\mathcal{S}^{J_b(F)}$. Indeed, the assertion for X_2 follows from the fact that $X_w(b)^{\mathrm{pfn}}$ is in $\mathcal{S}^{J_b(F)}$ and Lemma 3.1.3. The assertion for Y_2 follows from the fact that $X_{sx}(b)^{\mathrm{pfn}}$ is in $\mathcal{S}^{J_b(F)}$, and the fact that the perfection of g_2 is a $J_b(F)$-equivariant isomorphism. The assertion for Y_1 follows from the fact that $X_{sx\sigma(x)}(b)^{\mathrm{pfn}}$ is in $\mathcal{S}^{J_b(F)}$, and the fact that the perfection of g_1 is a $J_b(F)$-equivariant isomorphism. The assertion for X_1 follows from the assertion for Y_1, the fact that the perfection of f_1 is locally of perfectly finite type, and the fact that pulling back along the perfection of f_1 induces a $J_b(F)$-equivariant bijection $\Sigma(Y_1^{\mathrm{pfn}}) \xrightarrow{\sim} \Sigma(X_1^{\mathrm{pfn}})$. ∎

3.3.4. We will use Proposition 3.3.3 to define a refinement of the class polynomials for affine Hecke algebras, which are more suited for the study of the $J_b(F)$-action on $\Sigma^{\mathrm{top}}(X_w(b))$. We first recall the definition of the usual class polynomials. Here we use the convention of [16, §2.8.2], which differs from that in [14].

Let $\mathbf{q}$ be an indeterminate, and let $\mathbb{Z}[\mathbf{q}^{\pm 1}]$ be the Laurent polynomial ring. Let $\mathbb{H}$ be the affine Hecke algebra over $\mathbb{Z}[\mathbf{q}^{\pm 1}]$ attached to $\breve{W}$. Thus $\mathbb{H}$ is the associative $\mathbb{Z}[\mathbf{q}^{\pm 1}]$-algebra generated by symbols $\{T_w \mid w \in \breve{W}\}$ subject to the following relations:

- $T_w T_{w'} = T_{ww'}$ if $\breve{\ell}(ww') = \breve{\ell}(w) + \breve{\ell}(w')$;
- $(T_s + 1)(T_s - \mathbf{q}) = 0$ for all $s \in \breve{\mathbb{S}}$.

The action of σ on $\breve{W}$ induces an automorphism σ of $\mathbb{H}$ characterized by $\sigma(T_w) = T_{\sigma(w)}$ for all $w \in \breve{W}$. Define $[\mathbb{H}, \mathbb{H}]_\sigma$ to be the $\mathbb{Z}[\mathbf{q}^{\pm 1}]$-submodule of $\mathbb{H}$ generated by $h\sigma(h') - h'\sigma(h)$, where h and h' run over elements of $\mathbb{H}$. Define the *σ-cocenter* (or simply *cocenter*) to be the quotient module $\bar{\mathbb{H}}_\sigma := \mathbb{H}/[\mathbb{H}, \mathbb{H}]_\sigma$.

For any $\mathcal{O} \in B(\breve{W}, \sigma)$, let $\mathcal{O}_{\min}$ be the set of minimal length elements of $\mathcal{O}$. By [17, Theorem 5.3, Theorem 6.7], the cocenter $\bar{\mathbb{H}}_\sigma$ is a free $\mathbb{Z}[\mathbf{q}^{\pm 1}]$-module with a basis given by $\{T_{\mathcal{O}} \mid \mathcal{O} \in B(\breve{W}, \sigma)\}$. Here $T_{\mathcal{O}}$ is the image of T_w in $\bar{\mathbb{H}}_\sigma$ for some (or equivalently, any) $w \in \mathcal{O}_{\min}$. Moreover, for any $w \in \breve{W}$, we have

$$T_w \equiv \sum_{\mathcal{O} \in B(\breve{W}, \sigma)} F_{w,\mathcal{O}} T_{\mathcal{O}} \mod [\mathbb{H}, \mathbb{H}]_\sigma,$$

where $F_{w,\mathcal{O}} \in \mathbb{Z}[\mathbf{q}]$ is the *class polynomial*, uniquely determined by the above identity.

3.3.5. As indicated above, we now refine the polynomials $F_{w,\mathcal{O}}$ where $(w, \mathcal{O}) \in \breve{W} \times B(\breve{W}, \sigma)$. The refined polynomials will be indexed by pairs $(w, C) \in \breve{W} \times \mathscr{C}(\breve{W})$, where $\mathscr{C}(\breve{W})$ is a set more refined than $B(\breve{W}, \sigma)$. For the definition of $\mathscr{C}(\breve{W})$ we first recall some notations. For $w, w' \in \breve{W}$ and $s \in \breve{\mathbb{S}}$, we write

$$w \xrightarrow{s}_\sigma w'$$

if $w' = sw\sigma(s)$ and $\breve{\ell}(w') \leq \breve{\ell}(w)$. We write

$$w \rightarrow_\sigma w'$$

if there is a sequence $w = w_1, w_2, \dots, w_n = w'$ in $\breve{W}$ such that for each $2 \le k \le n$ we have $w_{k-1} \xrightarrow{s_k}_\sigma w_k$ for some $s_k \in \breve{\mathbb{S}}$. We write

$$w \approx_\sigma w'$$

if $w \to_\sigma w'$ and $w' \to_\sigma w$. We write $w \tilde{\approx}_\sigma w'$ if there exists $\tau \in \Omega$ such that $w \approx_\sigma \tau w' \sigma(\tau)^{-1}$. The following theorem is proved in [17, Theorem 2.9].

Theorem 3.3.6. *Let $\mathcal{O}$ be a σ-conjugacy class in $\breve{W}$. Then for each $w \in \mathcal{O}$, there exists $w' \in \mathcal{O}_{\min}$ such that $w \to_\sigma w'$.* ■

Definition 3.3.7. Let $\breve{W}_{\sigma,\min}$ be the set of $w \in \breve{W}$ such that w has minimal length in its own σ-conjugacy class. We write $\mathscr{C}(\breve{W})$ for the set $\breve{W}_{\sigma,\min}/\tilde{\approx}_\sigma$, and we view each element of $\mathscr{C}(\breve{W})$ as a subset of $\breve{W}$. We denote by π the natural map $\mathscr{C}(\breve{W}) \to B(\breve{W}, \sigma)$ sending $C \in \mathscr{C}(\breve{W})$ to the unique σ-conjugacy class in $\breve{W}$ containing C. We denote the composition of the map (2.3.4.1) with π by $\Psi : \mathscr{C}(\breve{W}) \to B(G)$.

3.3.8. For any $C \in \mathscr{C}(\breve{W})$ and $b \in G(\breve{F})$, we write $[[X_C(b)]]$ for $[[X_w(b)]] \in \mathrm{GDL}^{J_b(F)}$ for arbitrary $w \in C$. By Proposition 3.3.3 (1) and the fact that right multiplication by $\dot{\tau}$ induces a $J_b(F)$-equivariant isomorphism $X_w(b) \xrightarrow{\sim} X_{\tau^{-1}w\sigma(\tau)}(b)$ for all $w \in \breve{W}$ and $\tau \in \Omega$, the definition of $[[X_C(b)]]$ is independent of the choice of $w \in C$.

We now construct the refined polynomials in the following theorem. Let $\mathbb{N}[\mathbf{q}-1]$ denote the set of polynomials in the variable $\mathbf{q}-1$ with positive integral coefficients. The second statement in the theorem can be viewed as a “motivic counting” result.

Theorem 3.3.9. *Fix $w \in \breve{W}$. There exists a map*

$$\begin{aligned} \mathscr{C}(\breve{W}) &\longrightarrow \mathbb{N}[\mathbf{q}-1] \\ C &\longmapsto F_{w,C}(\mathbf{q}-1) \end{aligned}$$

satisfying the following conditions.

(1) *For each $\mathcal{O} \in B(\breve{W}, \sigma)$, we have*

$$F_{w,\mathcal{O}}(\mathbf{q}) = \sum_{C \in \mathscr{C}(\breve{W}), \pi(C) = \mathcal{O}} F_{w,C}(\mathbf{q}-1) \in \mathbb{Z}[\mathbf{q}].$$

In particular, we have $F_{w,\mathcal{O}}(\mathbf{q}) \in \mathbb{N}[\mathbf{q}-1]$.

(2) *For each $b \in G(\breve{F})$, we have*

$$[[X_w(b)]] = \sum_{C \in \mathscr{C}(\breve{W}), \Psi(C) = [b]} F_{w,C}(\mathbb{L}-1) \cdot [[X_C(b)]] \in \mathrm{GDL}^{J_b(F)}. \quad (3.3.9.1)$$

Proof. We prove the statement by induction on $\ell(w)$.

If $w \in \breve{W}_{\sigma,\min}$, then by [16, §2.8.2], for any $\mathcal{O} \in B(\breve{W}, \sigma)$, we have

$$F_{w,\mathcal{O}} = \begin{cases} 1, & \text{if } w \in \mathcal{O}; \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, for $C \in \mathscr{C}(\breve{W})$, we set

$$F_{w,C} := \begin{cases} 1, & \text{if } w \in C; \\ 0, & \text{otherwise.} \end{cases}$$

In this case, the map $C \mapsto F_{w,C}$ satisfies conditions (1) and (2).

Now assume that $w \notin \breve{W}_{\sigma,\min}$. Then by Theorem 3.3.6, there exists $w_1 \in \breve{W}$ and $s \in \breve{\mathbb{S}}$ such that $w \approx_\sigma w_1$ and $sw_1\sigma(s) < w_1$. By [16, §2.8.2], for any $O \in B(\breve{W}, \sigma)$, we have

$$F_{w,O}(\mathbf{q}) = (\mathbf{q} - 1)F_{sw_1,O}(\mathbf{q}) + \mathbf{q}F_{sw_1\sigma(s),O}(\mathbf{q}).$$

For $C \in \mathscr{C}(\breve{W})$, we set

$$F_{w,C}(\mathbf{q} - 1) := (\mathbf{q} - 1)F_{sw_1,C}(\mathbf{q} - 1) + \mathbf{q}F_{sw_1\sigma(s),C}(\mathbf{q} - 1),$$

where $F_{sw_1,C}(\mathbf{q} - 1)$ and $F_{sw_1\sigma(s),C}(\mathbf{q} - 1)$ are defined by the induction hypothesis. Since condition (1) holds for sw_1 and $sw_1\sigma(s)$, it also holds for w.

By Proposition 3.3.3, for any $b \in G(\breve{F})$ we have

$$[[X_w(b)]] = [[X_{w_1}(b)]] = (\mathbb{L} - 1)[[X_{sw_1}(b)]] + \mathbb{L}[[X_{sw_1\sigma(s)}(b)]].$$

By the induction hypothesis, we have the following identities in $\mathrm{GDL}^{J_b(F)}$:

$$[[X_{sw_1}(b)]] = \sum_{C \in \mathscr{C}(\breve{W}), \Psi(C) = [b]} F_{sw_1,C}(\mathbb{L} - 1) \cdot [[X_C(b)]],$$

$$[[X_{sw_1\sigma(s)}(b)]] = \sum_{C \in \mathscr{C}(\breve{W}), \Psi(C) = [b]} F_{sw_1\sigma(s),C}(\mathbb{L} - 1) \cdot [[X_C(b)]].$$

Then

$$\begin{aligned} [[X_w(b)]] &= (\mathbb{L} - 1)[[X_{sw_1}(b)]] + \mathbb{L}[[X_{sw_1\sigma(s)}(b)]] \\ &= \sum_C \Big((\mathbb{L} - 1) \cdot F_{sw_1,C}(\mathbb{L} - 1) + \mathbb{L} \cdot F_{sw_1\sigma(s),C}(\mathbb{L} - 1)\Big) \cdot [[X_C(b)]] \\ &= \sum_C F_{w,C}(\mathbb{L} - 1) \cdot [[X_C(b)]] \in \mathrm{GDL}^{J_b(F)}, \end{aligned}$$

where both summations are over $C \in \mathscr{C}(\breve{W})$ with $\Psi(C) = [b]$. Thus (2) holds for w. ■

Remark 3.3.10. (1) The polynomials $F_{w,C}$ are not uniquely characterized by condition (1) in Theorem 3.3.9. This is because the cocenter of the affine Hecke algebra over $\mathbb{Z}[\mathbf{q}]$ has a torsion part, cf. [15, §5.2]. (In contrast, as we have mentioned above, the cocenter of the affine Hecke algebra over $\mathbb{Z}[\mathbf{q}^{\pm 1}]$ is free.)

(2) Fix $b \in G(\breve{F})$, and let $K_0^{J_b(F)}$ be the Grothendieck group of the monoid $\mathrm{GDL}^{J_b(F)}$. The endomorphism $\mathbb{L}$ of $\mathrm{GDL}^{J_b(F)}$ gives rise to a $\mathbb{Z}[\mathbf{q}]$-module structure on $K_0^{J_b(F)}$ via the specialization $\mathbf{q} \mapsto \mathbb{L}$. The $\mathbb{Z}[\mathbf{q}]$-submodule of $K_0^{J_b(F)}$ generated by $\{[[X_w(b)]] \mid w \in \breve{W}\}$ is not necessarily torsion-free as a $\mathbb{Z}[\mathbf{q}]$-module. It would be interesting to compare the torsion phenomenon here with the cocenter of the affine Hecke algebra over $\mathbb{Z}[\mathbf{q}]$.

(3) As we have seen in the proof of Theorem 3.3.9, the construction of $F_{w,C}$ depends on G only via the triple $(\breve{W}, \breve{\ell} : \breve{W} \longrightarrow \mathbb{Z}_{\geq 0}, \sigma \in \mathrm{Aut}(\breve{W}))$. In §4.2 below, we will use the same idea to reduce the study of general G to unramified groups.

Corollary 3.3.11. *Let $w \in \breve{W}$ and $b \in G(\breve{F})$. For each $C \in \mathscr{C}(\breve{W})$, choose an element $w_C \in C$. The isomorphism class of the $J_b(F)$-set $\Sigma^{\mathrm{top}}(X_w(b))$ (resp. the integer $\dim X_w(b)$) is given by the first (resp. second) coordinate of the element*

$$\sum_{C \in \mathscr{C}(\breve{W}), \Psi(C)=[b]} F_{w,C}(\mathbb{L}-1) \cdot \big(\Sigma^{\mathrm{top}}(X_{w_C}(b)), \dim X_{w_C}(b)\big) \in \mathrm{TIC}^{J_b(F)}.$$

Proof. Note that the isomorphism class of the $J_b(F)$-set $\Sigma^{\mathrm{top}}(X_w(b))$ and the integer $\dim X_w(b)$ do not change if we replace $X_w(b)$ by its perfection. The corollary then follows from applying the $\mathbb{T}$-equivariant homomorphism $\mathfrak{C}$ in Lemma 3.2.3 to the two sides of (3.3.9.1). ∎

Remark 3.3.12. Fix $b \in G(\breve{F})$. For $x \in \breve{W}_{\sigma,\min}$, by [14, Theorem 4.8] we know that $X_x(b) \neq \emptyset$ if and only if $\Psi(x) = [b]$, that $X_x(b)$ is equidimensional, and that the $J_b(F)$-action on $\Sigma^{\mathrm{top}}(X_x(b))$ is transitive. Moreover, when $X_x(b) \neq \emptyset$, we have an explicit formula for $\dim X_x(b)$ (see [14, Theorem 4.8]), and we know that the stabilizer of each irreducible component of $X_x(b)$ in $J_b(F)$ is a parahoric subgroup of $J_b(F)$ with an explicit description (see the proof of [45, Proposition 3.1.4]). The upshot is that we explicitly understand the elements $(\Sigma^{\mathrm{top}}(X_{w_C}(b)), \dim X_{w_C}(b)) \in \mathrm{TIC}^{J_b(F)}$ for all $C \in \mathscr{C}(\breve{W})$ and $w_C \in C$. Thus by Corollary 3.3.11, the determination of the $J_b(F)$-set $\Sigma^{\mathrm{top}}(X_w(b))$ and $\dim X_w(b)$ for general $w \in \breve{W}$ reduces to the computation of the polynomials $F_{w,C}$. It also follows that for general w, the stabilizer of each element of $\Sigma^{\mathrm{top}}(X_w(b))$ in $J_b(F)$ is a parahoric subgroup, cf. [45, Proposition 3.1.4].

3.4. Stabilizer of one irreducible component

We keep the setting and notation of §3.3. In this subsection we assume in addition that G is F-simple and adjoint. We will apply the results in §3.3 to study the stabilizers for the $J_b(F)$-action on $\Sigma^{\mathrm{top}}(X_{w_0 t^\mu}(b))$, where $\mu \in X_*(T)_{\Gamma_0}{}^+$ and w_0 is the longest element of $\breve{W}_0$.

3.4.1. Recall that for δ an automorphism of $(\breve{W}_a, \breve{\mathbb{S}})$ and $\breve{K} \subset \breve{\mathbb{S}}$ a δ-stable subset, a *δ-twisted Coxeter element* of $\breve{W}_{\breve{K}}$ is an element which can be written as $s_1 \cdots s_n$, where $s_1, \ldots, s_n \in \breve{W}_{\breve{K}}$ are distinct and form a set of representatives of the δ-orbits in $\breve{K}$. For $w \in \breve{W}_a$ we write $\mathrm{supp}_\delta(w)$ for the smallest δ-stable subset $\breve{K}$ of $\breve{\mathbb{S}}$ such that $w \in \breve{W}_{\breve{K}}$. As explained in §2.4.3, we identify $\breve{W}_0$ with the subgroup $\breve{W}_{\breve{\mathbb{S}}_0}$ of $\breve{W}$. Note that every $w \in \breve{W}$ can be written in a unique way as $w = xt^\mu y$, where $\mu \in X_*(T)_{\Gamma_0}{}^+$, $x, y \in \breve{W}_0$, and $t^\mu y \in {}^{\breve{\mathbb{S}}_0}\breve{W}$. Moreover, $\breve{\ell}(w) = \breve{\ell}(x) + \breve{\ell}(t^\mu y) = \breve{\ell}(x) + \breve{\ell}(t^\mu) - \breve{\ell}(y)$.

The following result gives a refinement of [14, Proposition 11.6].

Proposition 3.4.2. *Assume G is F-simple and adjoint. Let $\breve{K}$ be a σ-stable subset of $\breve{\mathbb{S}}_0$. Let $w = xt^\mu y \in \breve{W}$, with $\mu \in X_*(T)^+_{\Gamma_0}$, $x, y \in \breve{W}_0$, and $t^\mu y \in {}^{\breve{\mathbb{S}}_0}\breve{W}$. Assume that $\mu \neq 0$, that*

$\operatorname{supp}_\sigma(x) = \breve{K}$, *and that* y *is a* σ*-twisted Coxeter element of* $\breve{W}_{\breve{\mathbb{S}}_0\setminus\breve{K}}$. *Then there exists a* σ*-twisted Coxeter element* c *of* $\breve{W}_0$ *with* $t^\mu c \in {}^{\breve{\mathbb{S}}_0}\breve{W}$ *such that for each* $b \in G(\breve{F})$, *we have*

$$[[X_w(b)]] = (\mathbb{L}-1)^{\breve{\ell}(x)}[[X_{t^\mu c}(b)]] + P \in \mathrm{GDL}^{J_b(F)}$$

for some $P \in \mathrm{GDL}^{J_b(F)}$.

Proof. We follow the method in [14, Proposition 11.6].

We proceed by induction on $|\breve{K}|$. The case $|\breve{K}| = 0$ is clear, as we can take $c = y$. We thus assume that the result is true for all $\breve{K}' \subsetneq \breve{K}$. We may also assume that the result is true for all $x' \in \breve{W}_{\breve{K}}$ with $\operatorname{supp}_\sigma(x') = \breve{K}$ and $\breve{\ell}(x') < \breve{\ell}(x)$. We set $\breve{K}_1 := \{s \in \breve{K} \mid t^\mu ys \notin {}^{\breve{\mathbb{S}}_0}\breve{W}\}$. Then as in [14, Proposition 11.6], $\breve{K}_1$ is a proper subset of $\breve{K}$, and every $s \in \breve{K}_1$ commutes with y and with $t^\mu y$.

We write $x = ux'$ where $u \in \breve{W}_{\sigma^{-1}(\breve{K}_1)}$ and $x' \in {}^{\sigma^{-1}(\breve{K}_1)}\breve{W}$. We let $u = s_1 \cdots s_n$ be a reduced word decomposition for u. We write $u_i = s_1 \cdots s_i$ and set $x_i = u_i^{-1} x\sigma(u_i)$ for $i = 0, \ldots, n$. In particular, $x_0 = x$. Then $\breve{\ell}(x_i) \leq \breve{\ell}(u_i ßx) + \breve{\ell}(\sigma(u_i)) = (\breve{\ell}(x) - \breve{\ell}(u_i)) + \breve{\ell}(u_i) = \breve{\ell}(x)$ for all $i = 1, \ldots, n$. There are two possibilities:

Case (i): There exists k such that $\breve{\ell}(x) = \breve{\ell}(x_i)$ for $i = 1, \ldots, k-1$ and $\breve{\ell}(x_k) < \breve{\ell}(x)$.

Case (ii): $\breve{\ell}(x_i) = \breve{\ell}(x)$ for all $i = 1, \ldots, n$.

In Case (i), we have $\breve{\ell}(x_{k-1}) = \breve{\ell}(x)$ and $\breve{\ell}(x_k) = \breve{\ell}(x_{k-1}) - 2 = \breve{\ell}(x) - 2$. Therefore $\breve{\ell}(sx_{k-1}) = \breve{\ell}(x_{k-1}) - 1$. Moreover, we have $\breve{\ell}(w) = \breve{\ell}(x) + \breve{\ell}(t^\mu y) = \breve{\ell}(x_{k-1}) + \breve{\ell}(t^\mu y) = \breve{\ell}(x_{k-1}t^\mu y)$. By Proposition 3.3.3, we have

$$[[X_w(b)]] = [[X_{x_{k-1}t^\mu y}(b)]] = (\mathbb{L}-1)[[X_{s_k x_{k-1} t^\mu y}(b)]] + \mathbb{L}[[X_{x_k t^\mu y}(b)]].$$

Since $\breve{\ell}(s_k x_{k-1}\sigma(s_k)) < \breve{\ell}(x_{k-1})$, we have $\operatorname{supp}_\sigma(s_k x_{k-1}) = \operatorname{supp}_\sigma(x) = \breve{K}$. Thus by induction hypothesis, we have

$$[[X_{s_k x_{k-1} t^\mu y}(b)]] = (\mathbb{L}-1)^{\breve{\ell}(s_k x_{k-1})}[[X_{t^\mu c}(b)]] + P'$$

for some σ-twisted Coxeter element c of $\breve{W}_0$ with $t^\mu c \in {}^{\breve{\mathbb{S}}_0}\breve{W}$ and $P' \in \mathrm{GDL}^{J_b(F)}$. Since $\breve{\ell}(s_k x_{k-1}) = \breve{\ell}(x) - 1$, we have

$$[[X_w(b)]] = (\mathbb{L}-1)^{\breve{\ell}(x)}[[X_{t^\mu c}(b)]] + P$$

with $P \in \mathrm{GDL}^{J_b(F)}$ as desired.

In Case (ii), we have $x \approx_\sigma x_n = x'\sigma(u)$. It follows that $w \approx_\sigma x_n t^\mu y$ and hence

$$[[X_w(b)]] = [[X_{x_n t^\mu y}(b)]] \in \mathrm{GDL}^{J_b(F)}$$

by Proposition 3.3.3. We first consider the case where $x \notin \breve{W}_{\sigma^{-1}(\breve{K}_1)}$. Then $x' \neq 1$ and there exists $s \in \sigma^{-1}(\breve{K}) \setminus \sigma^{-1}(\breve{K}_1)$ such that $sx_n < x_n$. Moreover $\breve{\ell}(y\sigma(s)) = \breve{\ell}(y) + 1$ and $t^\mu y\sigma(s) \in {}^{\breve{\mathbb{S}}_0}\breve{W}$ and we have

$$\breve{\ell}(sx_n t^\mu y\sigma(s)) = \breve{\ell}(sx_n) + \breve{\ell}(t^\mu) - \breve{\ell}(y\sigma(s)) = \breve{\ell}(x_n t^\mu y) - 2.$$

It follows that

$$\begin{aligned}
[[X_w(b)]] &= [[X_{x_n t^\mu y}(b)]]\\
&= (\mathbb{L}-1)[[X_{sx_n t^\mu y}(b)]] + \mathbb{L}[[X_{sx_n t^\mu y\sigma(s)}(b)]]\\
&= (\mathbb{L}-1)[[X_{sx_n t^\mu y}(b)]] + (\mathbb{L}-1)[[X_{sx_n t^\mu y\sigma(s)}(b)]] + [[X_{sx_n t^\mu y\sigma(s)}(b)]].
\end{aligned}$$

Note that $sx_n < x_n$ and $\mathrm{supp}(sx_n) \subset \mathrm{supp}(x_n)$. Thus $\mathrm{supp}_\sigma(sx_n) \subset \breve{K}$.

If $\mathrm{supp}_\sigma(sx_n) = \breve{K}$, the induction hypothesis applied to $X_{sx_n t^\mu y}(b)$ gives

$$[[X_{sx_n t^\mu y}(b)]] = (\mathbb{L}-1)^{\breve{\ell}(sx_n)}[[X_{t^\mu c}(b)]] + P'$$

for some σ-twisted Coxeter element c of $\breve{W}_0$ and $P' \in \mathrm{GDL}^{J_b(F)}$. It follows that

$$[[X_w(b)]] = (\mathbb{L}-1)^{\breve{\ell}(x)}[[X_{t^\mu c}(b)]] + P$$

with $P \in \mathrm{GDL}^{J_b(F)}$.

Similarly, if $\mathrm{supp}_\sigma(sx_n) \neq \breve{K}$, the induction hypothesis applied to the pair

$$\left(X_{sx_n t^\mu y\sigma(s)}(b), \mathrm{supp}_\sigma(sx_n)\right)$$

gives

$$[[X_{sx_n t^\mu y\sigma(s)}(b)]] = (\mathbb{L}-1)^{\breve{\ell}(sx_n)}[[X_{t^\mu c}(b)]] + P'$$

for some $P' \in \mathrm{GDL}^{J_b(F)}$, and hence

$$[[X_w(b)]] = (\mathbb{L}-1)^{\breve{\ell}(x)}[[X_{t^\mu c}(b)]] + P$$

with $P \in \mathrm{GDL}^{J_b(F)}$.

Finally we consider the case $x \in \breve{W}_{\sigma^{-1}(\breve{K}_1)}$. Since $\breve{K}_1$ is a proper subset of $\breve{K}$ and $\mathrm{supp}_\sigma(x) = \breve{K}$, there exists $m \in \mathbb{N}$ such that $x, \sigma(x), \dots, \sigma^{m-1}(x) \in \breve{W}_{\sigma^{-1}(\breve{K}_1)}$ and $\sigma^m(x) \notin \breve{W}_{\sigma^{-1}(\breve{K}_1)}$. We have

$$[[X_{xt^\mu y}(b)]] = [[X_{\sigma(x)t^\mu y}(b)]] = \ldots = [[X_{\sigma^{m-1}(x)t^\mu y}(b)]].$$

The argument above applied to $\sigma^m(x)$ shows that

$$[[X_{\sigma(m)t^\mu y}(b)]] = (\mathbb{L}-1)^{\breve{\ell}(x)}[[X_{t^\mu c}(b)]] + P,$$

for some σ-twisted Coxeter element $c \in \breve{W}_0$ and $P \in \mathrm{GDL}^{J_b(F)}$ as desired. ■

3.4.3. For an element $\tau \in \Omega$, the Iwahori–Weyl group and affine Weyl group of $J_{\dot{\tau}}$ are isomorphic to $\breve{W}$ and $\breve{W}_a$ respectively, and the Frobenius actions are both given by $\mathrm{Ad}(\tau) \circ \sigma$.

We need the following result which is proved in [19]. Set $V := X_*(T)_{\Gamma_0} \otimes_{\mathbb{Z}} \mathbb{R}$.

Proposition 3.4.4. *Let $p : \breve{W} \subset \mathrm{Aff}(V) \to \mathrm{GL}(V)$ be the natural map. Consider the σ-twisted conjugation action of $\breve{W}_a$ on $\breve{W}$. Let O be a $\breve{W}_a$-orbit in $\breve{W}$ with $O \subset \breve{W}_a\tau$ for some $\tau \in \Omega$. If $p(O) \subset \breve{W}_0$ contains a σ-twisted Coxeter element of $\breve{W}_0$, then there exists a unique $\mathrm{Ad}(\tau) \circ \sigma$-stable subset $\breve{K}$ of $\breve{\mathbb{S}}$ such that $W_{\breve{K}}$ is finite and the set $O_{\min}$ of minimal length elements of O is precisely the set of $\mathrm{Ad}(\tau) \circ \sigma$-twisted Coxeter elements of $\breve{W}_{\breve{K}}$. Moreover, the standard parahoric subgroup of $J_{\dot{\tau}}(F)$ corresponding to $\breve{K}$ is very special.*

Remark 3.4.5. In Proposition 3.4.4, the unique $\breve{K}$ is explicitly computed in each case in [19]. The "moreover" part of the proposition immediately follows from the explicit description.

The main result of this subsection is the following proposition.

Proposition 3.4.6. *Assume that G is F-simple and adjoint. Let $[b] \in B(G, \mu)$ be the unique basic element. Then there exists $Z \in \Sigma^{\mathrm{top}}(X_{w_0 t^\mu}(b))$ such that $\mathrm{Stab}_Z(J_b(F))$ is a very special parahoric subgroup of $J_b(F)$.*

Proof. Since μ is dominant, $t^\mu \in {}^{\breve{\mathbb{S}}_0}\breve{W}$. If $\mu = 0$, then we may take $b = 1$. In this case, $J_b(F) = G(F)$ and $X_\mu(b) = G(F)/\mathcal{K}$ is discrete; here $\mathcal{K} \subset G(F)$ is the parahoric subgroup corresponding to $\breve{\mathbb{S}}_0$ which is very special (cf. Remark 2.2.3). For any $Z \in X_\mu(b)$, the stabilizer $\mathrm{Stab}_Z(J_b(F))$ is conjugate to $\mathcal{K}$ and thus is a very special parahoric subgroup of $G(F)$. Now the statement on $X_{w_0 t^\mu}(b)$ follows from Proposition 2.4.10.

Now assume that $\mu \neq 0$. By Proposition 3.4.2 applied to $\breve{K} = \breve{\mathbb{S}}_0$ and $w = w_0 t^\mu$, there exists a σ-twisted Coxeter element c of $\breve{W}_0$ such that

$$[[X_{w_0 t^\mu}(b)]] = (\mathbb{L} - 1)^{\breve{\ell}(w_0)} [[X_{t^\mu c}(b)]] + P',$$

where $P' \in \mathrm{GDL}^{J_b(F)}$.

Let $\tau \in \Omega$ be the unique element such that $\kappa(\tau) = \mu^\natural \in \pi_1(G)_{\Gamma_0}$. Upon replacing b by another representative in $[b]$, we may assume $b = \dot{\tau}$. By Proposition 3.4.4 and Theorem 3.3.6, there exists an $\mathrm{Ad}(\tau) \circ \sigma$-stable subset $\breve{K} \subset \breve{\mathbb{S}}$ and an $\mathrm{Ad}(\tau) \circ \sigma$-twisted Coxeter element c' of $\breve{W}_{\breve{K}}$ such that the associated parahoric $\mathcal{J}$ of $J_{\dot{\tau}}(F)$ is very special, $c'\tau$ is of minimal length in its σ-conjugacy class, and $t^\mu c \to_\sigma c'\tau$.

By Proposition 3.3.3, we have

$$[[X_{t^\mu c}(\dot{\tau})]] = (\mathbb{L} - 1)^{\frac{\breve{\ell}(t^\mu c) - \breve{\ell}(c')}{2}} [[X_{c'\tau}(\dot{\tau})]] + Q$$

for some $Q \in \mathrm{GDL}^{J_{\dot{\tau}}(F)}$ and hence

$$[[X_{w_0 t^\mu}(\dot{\tau})]] = (\mathbb{L} - 1)^{\breve{\ell}(w_0) + \frac{\breve{\ell}(t^\mu c) - \breve{\ell}(c')}{2}} [[X_{c'\tau}(\dot{\tau})]] + P$$

for some $P \in \mathrm{GDL}^{J_{\dot{\tau}}(F)}$. By Lemma 3.2.3, the above equality implies that

$$\mathfrak{C}([[X_{w_0 t^\mu}(\dot{\tau})]]) = \mathbb{T}^{\breve{\ell}(w_0) + \frac{\breve{\ell}(t^\mu c) - \breve{\ell}(c')}{2}} \mathfrak{C}([[X_{c'\tau}(\dot{\tau})]]) + H \tag{3.4.6.1}$$

for some $H \in \mathrm{TIC}^{J_{\dot{\tau}}(F)}$. Here on the right side, the addition is in the monoid $\mathrm{TIC}^{J_{\dot{\tau}}(F)}$.

By definition, $\breve{\ell}(c)$ is the semisimple F-rank of G and $\breve{\ell}(c')$ is the semisimple F-rank of J_b. Hence $\breve{\ell}(c) - \breve{\ell}(c') = \mathrm{def}_G(\dot{\tau})$. Since $\tau \in \Omega$, $\bar{\nu}_{\dot{\tau}}$ is central in G. Thus $\langle \bar{\nu}_{\dot{\tau}}, \rho \rangle = 0$. By Theorem 2.4.7 and [14, Theorem 10.1],

$$\begin{aligned}
\dim X_{w_0 t^\mu}(\dot{\tau}) &= \breve{\ell}(w_0) + \dim X_\mu(\dot{\tau}) \\
&= \breve{\ell}(w_0) + \langle \mu, \rho \rangle + \frac{\breve{\ell}(c') - \breve{\ell}(c)}{2} \\
&= \breve{\ell}(w_0) + \frac{\breve{\ell}(t^\mu) - \breve{\ell}(c) + \breve{\ell}(c')}{2} \\
&= \breve{\ell}(c') + \breve{\ell}(w_0) + \frac{\breve{\ell}(t^\mu c) - \breve{\ell}(c')}{2} \\
&= \dim(X_{c'\tau}(\dot{\tau})) + \breve{\ell}(w_0) + \frac{\breve{\ell}(t^\mu c) - \breve{\ell}(c')}{2},
\end{aligned}$$

where the fourth equality follows from the fact that $t^\mu c \in {}^{\breve{\mathbb{S}}_0}\breve{W}$.

By the above computation, the first term in the sum

$$\mathbb{T}^{\breve{\ell}(w_0) + \frac{\breve{\ell}(t^\mu c) - \breve{\ell}(c')}{2}} \mathfrak{C}([[X_{c'\tau}(\dot{\tau})]]) + H$$

makes a non-trivial contribution to the sum in the sense of §3.2.1. Thus we have a $J_{\dot{\tau}}(F)$-equivariant embedding $\Sigma^{\mathrm{top}}(X_{c'\tau}(\dot{\tau})) \longrightarrow \Sigma^{\mathrm{top}}(X_{w_0 t^\mu}(\dot{\tau}))$. It remains to find an element of $\Sigma^{\mathrm{top}}(X_{c'\tau}(\dot{\tau}))$ whose stabilizer in $J_{\dot{\tau}}(F)$ is a very special parahoric subgroup.

By [14, Page 383, line 3], $X_{c'\tau}(\dot{\tau}) \cong J_{\dot{\tau}}(F) \times^{\mathcal{J}} X^{\breve{\mathcal{K}}}_{c'\tau}(\dot{\tau})$, where $X^{\breve{\mathcal{K}}}_{c'\tau}(\dot{\tau})$ is a classical Deligne–Lusztig variety (resp. perfection of a classical Deligne–Lusztig variety) if $\mathrm{char}(F) > 0$ (resp. $\mathrm{char}(F) = 0$) defined by

$$X^{\breve{\mathcal{K}}}_{c'\tau}(\dot{\tau}) \cong \{g\overline{\mathcal{I}} \in \overline{\mathcal{K}}/\overline{\mathcal{I}} \mid g ß\sigma'(g) \in \overline{\mathcal{I}} c' \overline{\mathcal{I}}\}.$$

Here $\sigma' = \mathrm{Ad}(\tau) \circ \sigma$, and note that $\overline{\mathcal{I}}$ is a σ'-stable Borel subgroup of $\overline{\mathcal{K}}$.

Since c' is a σ'-Coxeter element of $\breve{W}_{\breve{K}}$, $X^{\breve{\mathcal{K}}}_{c'\tau}(\dot{\tau})$ is irreducible. Hence

$$\Sigma^{\mathrm{top}}(X_{c'\tau}(\dot{\tau})) \cong J_{\dot{\tau}}(F)/\mathcal{J}$$

as $J_{\dot{\tau}}(F)$-sets and the stabilizer of the elements are isomorphic to $\mathcal{J}$. ∎

4. Component stabilizers for $X_\mu(b)$

4.1. The main theorem and some consequences

4.1.1. We keep the notation and assumptions of §2.4. In particular, G is a quasi-split tamely ramified reductive group over F, and $\mathrm{char}(F) \nmid |\pi_1(G_{\mathrm{ad}})|$ if $\mathrm{char}(F) > 0$.

We now state our main theorem, which confirms conjectures made by X. Zhu and Rapoport.

Theorem 4.1.2. *Let $\mu \in X_*(T)^+_{\Gamma_0}$ and $[b] \in B(G,\mu)$. Each stabilizer for the $J_b(F)$-action on $\Sigma^{\mathrm{top}}(X_\mu(b))$ is a very special parahoric subgroup of $J_b(F)$.*

By [45, Proposition 3.1.4], we already know that each stabilizer for the $J_b(F)$-action on $\Sigma^{\mathrm{top}}(X_\mu(b))$ is a parahoric subgroup of $J_b(F)$. In the proof of Theorem 4.1.2 below we shall freely use this fact.

We now deduce an immediate consequence of Theorem 4.1.2.

Definition 4.1.3. Fix μ and b as in Theorem 4.1.2. We write $\mathscr{N}(\mu,b)$ for the number of $J_b(F)$-orbits in $\Sigma^{\mathrm{top}}(X_\mu(b))$.

Corollary 4.1.4. *There is an identification of $J_b(F)$-sets*

$$\Sigma^{\mathrm{top}}(X_\mu(b)) \cong \coprod_{i=1}^{\mathscr{N}(\mu,b)} J_b(F)/\mathcal{J}_i,$$

where $\mathcal{J}_i \subset J_b(F)$ is a very special parahoric subgroup for each i. ■

4.1.5. When G is unramified, an explicit formula for $\mathscr{N}(\mu,b)$ was first conjectured by M. Chen and X. Zhu, and was proved independently by the second and third named authors in [45] and by S. Nie in [32]. In the appendix of [45], a generalization of this formula for ramified G is given. We now recall this formula when G is unramified, as this will be needed in §4.3 below.

Consider the dual group $\widehat{G}$ of G over $\mathbb{C}$. We fix a pinning $(\widehat{B},\widehat{T},\widehat{\mathbb{X}}_+)$ of $\widehat{G}$, and fix an isomorphism between the based root datum of $(\widehat{G},\widehat{B},\widehat{T})$ and the dual of the based root datum of (G,B,T). (See §2.3.1 for B.) We then have a unique Γ-action on $\widehat{G}$ via automorphisms preserving $(\widehat{B},\widehat{T},\widehat{\mathbb{X}}_+)$ such that the induced Γ-action on the based root datum of $(\widehat{G},\widehat{B},\widehat{T})$ is compatible with the natural Γ-action on the based root datum of (G,B,T), see for instance [45, §5.1]. Now Γ_0 acts trivially on $X_*(T)$, so the element $\mu \in X_*(T)^+_{\Gamma_0}$ can be viewed as a $\widehat{B}$-dominant character of $\widehat{T}$. Let V_μ be the irreducible representation of $\widehat{G}$ of highest weight μ. Let $\widehat{\mathcal{S}}$ be the identity component of the Γ-fixed points of $\widehat{T}$. Then $X^*(\widehat{\mathcal{S}})$ is identified with the maximal torsion-free quotient of $X_*(T)_\Gamma = X_*(T)_\sigma$. As in [45, Definition 2.6.4], b determines an element ${}_b \in X^*(\widehat{\mathcal{S}})$. We omit the explicit definition of λ_b here. Let $V_\mu(\lambda_b)$ be the weight space in the $\widehat{\mathcal{S}}$-representation V_μ of weight λ_b. The geometric Satake provides us with a canonical basis $\mathbb{MV}_\mu(\lambda_b)$ of $V_\mu(\lambda_b)$.

In the theorem below, the numerical identity is proved independently by the second and third named authors [45, Theorem A] and Nie [32, Theorem 0.5]. The second statement is due to Nie [32, Theorem 0.5].

Theorem 4.1.6. *Keep the assumptions in §2.4, and assume that G is unramified over F. We have*

$$\mathscr{N}(\mu,b) = \dim V_\mu(\lambda_b).$$

Moreover, there is a natural bijection between $J_b(F)\backslash\Sigma^{\mathrm{top}}(X_\mu(b))$ and $\mathbb{MV}_\mu(\lambda_b)$. ■

4.2. Reduction to adjoint unramified F-simple groups in characteristic 0

In this subsection, we show that to prove Theorem 4.1.2, it suffices to prove it in the case where $\mathrm{char}(F) = 0$, and G is an adjoint F-simple unramified group over F.

4.2.1. Let $w \in \breve{W}$ and $[b] \in B(G)$. We first construct some combinatorial data involving only the affine Weyl group $\breve{W}_a$ together with the length function $\breve{\ell}$ and the action of σ on $\breve{W}_a$, but not the reductive group G. This allows us to connect different reductive groups over different local fields.

Let $\mathrm{Aut}^0(\breve{W}_a)$ be the group of length-preserving automorphisms of $\breve{W}_a$. We may regard σ as an element of $\mathrm{Aut}^0(\breve{W}_a)$. Let $\widehat{W}_a = \breve{W}_a \rtimes \mathrm{Aut}^0(\breve{W}_a)$. We have a natural group homomorphism

$$i : \breve{W} \longrightarrow \widehat{W}_a, \qquad w\tau \longmapsto (w, \mathrm{Ad}(\tau)) \text{ for } w \in \breve{W}_a, \tau \in \Omega.$$

Moreover, the map i is compatible with the actions of σ. (Here the action of σ on $\widehat{W}_a$ is given by $(w, f) \mapsto (\sigma(w), \sigma \circ f \circ \sigma^{-1})$.)

4.2.2. By [14, Theorem 3.7], the set $B(G)$ is in natural bijection with a certain subset of σ-conjugacy classes in $\breve{W}$. By composing with the map i, we may associate to any $[b] \in B(G)$ a σ-conjugacy class $C_{[b]}$ in $\widehat{W}_a$. Let G' be a connected reductive group over a (possibly different) local field F', let $b' \in G'(\breve{F}')$, and let w' be an element of the Iwahori–Weyl group $\breve{W}'$ of $G_{\breve{F}'}$. Note that any length-preserving isomorphism of $\breve{W}_a$ to $\breve{W}'_a$ extends in a unique way to a group isomorphism $\widehat{W}_a \to \widehat{W}'_a$. Write σ for the Frobenius in $\mathrm{Aut}(\breve{F}'/F')$, and write $[b']$ for the σ'-conjugacy class of b' in $G'(\breve{F}')$. Then $[b']$ determines a σ'-conjugacy class $C_{[b']}$ in $\widehat{W}'_a$. We say that the triples (G, b, w) and (G', b', w') are *associated* if the following conditions are satisfied:

- We have $\kappa_G(w) = \kappa_G(b)$ and $\kappa_{G'}(w') = \kappa_{G'}(b')$.
- There exists a length-preserving isomorphism $f : \breve{W}_a \xrightarrow{\sim} \breve{W}'_a$ such that the diagram

$$\begin{tikzcd} \breve{W}_a \arrow[r, "f"] \arrow[d, "\sigma"'] & \breve{W}'_a \arrow[d, "\sigma'"] \\ \breve{W}_a \arrow[r, "f"] & \breve{W}'_a \end{tikzcd}$$

  commutes, and we have $f(C_{[b]}) = C_{[b']}$ and $f(i(w)) = i'(w')$. Here $i' : \breve{W}' \to \widehat{W}'_a$ is the natural homomorphism analogous to i.

In this case, f induces an isomorphism from the affine Weyl group of J_b to the affine Weyl group of $J_{b'}$. We thus obtain a bijection between the standard parahoric subgroups of $J_b(F)$ and those of $J_{b'}(F)$, cf. [45, Lemma 3.2.2]. Let $\mathcal{J} \subset J_b(F)$ and $\mathcal{J}' \subset J_{b'}(F')$ be parahoric subgroups. We say that $\mathcal{J}$ and $\mathcal{J}'$ are *associated with respect to* f, if there exist $j \in J_b(F)$ and $j' \in J_{b'}(F')$ such that $j\mathcal{J}j^{-1}$ and $j'\mathcal{J}'j'^{-1}$ are standard parahoric subgroups which correspond to each other under the above-mentioned bijection.

Now suppose that (G, b, w) and (G', b', w') are associated, and fix $f : \breve{W}_a \xrightarrow{\sim} \breve{W}'_a$ as above. By [14, Theorem 4.8], for all $w \in \breve{W}_{\sigma,\min}$ we have $\dim X_w(b) = \breve{\ell}(w) - \gamma_b$, where γ_b is a constant depending only on b, not on w. Similarly we define $\gamma_{b'}$.

Proposition 4.2.3. *We have* $\dim X_w(b) - \dim X_{w'}(b') = \gamma_{b'} - \gamma_b$*, and there is a bijection*

$$\Theta : J_b(F)\backslash\Sigma^{\mathrm{top}}(X_w(b)) \xrightarrow{\sim} J_{b'}(F')\backslash\Sigma^{\mathrm{top}}(X_{w'}(b'))$$

satisfying the following condition:

For $Z \in \Sigma^{\mathrm{top}}(X_w(b))$ *and* $Z' \in \Sigma^{\mathrm{top}}(X_{w'}(b'))$ *such that* $\Theta(J_b(F)Z) = J_{b'}(F)Z'$*, the parahoric subgroups* $\mathrm{Stab}_Z(J_b(F))$ *and* $\mathrm{Stab}_{Z'}(J_{b'}(F'))$ *are associated with respect to* f*.*

Proof. We first claim that for all $x \in \breve{W}_a$, $xw\sigma(x)^{-1}$ and $x'w'\sigma'(x')^{-1}$ have the same length, where $x' = f(x) \in \breve{W}'_a$. Indeed, $xw\sigma(x)^{-1} = x\mathrm{Ad}(w)(\sigma(x)^{-1})w$ by definition has the same length as its component in $\breve{W}_a$ (under $\breve{W} = \breve{W}_a \rtimes \Omega$), which is $x\mathrm{Ad}(w)(\sigma(x)^{-1})w_a \in \breve{W}_a$. Here w_a denotes the component of w in $\breve{W}_a$. The assumption that (G, b, w) and (G', b', w') are associated implies that

$$f(x\mathrm{Ad}(w)(\sigma(x)^{-1})w_a) = x'\mathrm{Ad}(w')(\sigma'(x')^{-1})w'_a, \tag{4.2.3.1}$$

where w'_a is the coordinate of w' in $\breve{W}'_a$. Since f preserves length, and since the right hand side of the above equality is the component in $\breve{W}'_a$ of $x'w'\sigma'(x')^{-1}$, the claim is proved.

We now reduce the proposition to the case where $w \in \breve{W}_{\sigma,\min}$. Assume that $w \notin \breve{W}_{\sigma,\min}$. As in the proof of Theorem 3.3.9, we can find $w_1 \in \breve{W}$ and $s \in \breve{\mathbb{S}}$ such that $w_1 \approx_\sigma w$ and $sw_1\sigma(s) < w_1$. Thus w_1 is obtained from w by consecutively σ-conjugating by simple reflections in $\breve{\mathbb{S}}$ in a way that the length is preserved in each step. If we consecutively σ'-conjugate w' by the images of these simple reflections under $f : \breve{W}_a \xrightarrow{\sim} \breve{W}'_a$, then we obtain an element $w'_1 \in \breve{W}'$, which satisfies $w'_1 \approx_{\sigma'} w'$ by the claim above. Moreover, by (4.2.3.1), the component in $\breve{W}_a$ of $sw_1\sigma(s)$ (resp. of w_1) is related to the component in $\breve{W}'_a$ of $s'w'_1\sigma'(s')$ (resp. of w'_1) under f, where $s' = f(s)$. Hence $s'w'_1\sigma'(s') < w'_1$. By Proposition 3.3.3, we have

$$[[X_w(b)]] = [[X_{w_1}(b)]] = (\mathbb{L} - 1)[[X_{sw_1}(b)]] + \mathbb{L}\,[[X_{sw_1\sigma(s)}(b)]]$$

and

$$[[X_{w'}(b')]] = [[X_{w'_1}(b')]] = (\mathbb{L} - 1)[[X_{s'w'_1}(b')]] + \mathbb{L}\,[[X_{s'w'_1\sigma'(s')}(b')]].$$

By construction (G, b, sw_1) is associated with $(G', b', s'w'_1)$, and $(G, b, sw_1\sigma(s))$ is associated with $(G', b', s'w'_1\sigma'(s'))$. By induction on the length of w, we may assume that the proposition holds for these two pairs of associated triples. It then follows from the above two identities that the proposition also holds for the associated triples (G, b, w), (G', b', w'). (Here the statement about dimensions in the induction hypothesis implies that the non-empty subset $U \subset \{sw_1, sw_1\sigma(s)\}$ such that $\Sigma^{\mathrm{top}}(X_w(b)) = \coprod_{u\in U} \Sigma^{\mathrm{top}}(X_u(b))$ matches with the analogous subset of $\{s'w'_1, s'w'_1\sigma'(s')\}$ in the obvious sense.)

It remains to prove the proposition assuming that $w \in \breve{W}_{\sigma,\min}$. By the claim at the beginning of the proof and the fact that conjugating by length-zero elements does not change the length, we necessarily have $w' \in \breve{W}'_{\sigma',\min}$. By [14, Theorem 4.8], the relation between the dimensions holds, and $J_b(F)$ (resp. $J_{b'}(F')$) acts transitively on $\Sigma^{\mathrm{top}}(X_w(b))$ (resp. $\Sigma^{\mathrm{top}}(X_{w'}(b'))$). In particular we have a unique bijection Θ (between two singletons). The condition about association of stabilizers follows from the explicit description of $X_w(b)$ and $X_{w'}(b')$ in terms of finite Deligne–Lusztig varieties given in the proof [14, Theorem 4.8], cf. the proof of [45, Proposition 3.1.4]. ∎

Corollary 4.2.4. *To prove Theorem 4.1.2, it suffices to prove it when* $\mathrm{char}(F) = 0$ *and* G *is an adjoint* F*-simple unramified group over* F.

Proof. We first assume Theorem 4.1.2 is true for unramified adjoint groups over local fields of characteristic 0. Let G be an arbitrary (i.e., quasi-split, tamely ramified, reductive) group over an arbitrary local field F. By Proposition 2.4.10, it suffices to show that the stabilizer in $J_b(F)$ of every element of $\Sigma^{\mathrm{top}}(X_{w_0 t^\mu}(b))$ is a very special parahoric of $J_b(F)$.

Since $X_\mu(b) \neq \emptyset$, we have $X_{w_0 t^\mu}(b) \neq \emptyset$ and thus $\kappa_G(w_0 t^\mu) = \kappa_G(b)$. By [14, Theorem 3.7], there exists $w \in \breve{W}$ such that $[b] = [\dot{w}]$. In particular, we have $\kappa_G(w_0 t^\mu) = \kappa_G(w)$. By replacing w by a suitable element in the σ-orbit of w, we may assume furthermore that $w_0 t^\mu \breve{W}_a = w \breve{W}_a$.

We choose F' a local field of characteristic 0 and G' an adjoint unramified group over F' such that there is a length-preserving isomorphism $f : \breve{W}_a \to \breve{W}'_a$ such that $f \circ \sigma = \sigma' \circ f$; see [14, §6.1, §6.2] for the construction of such a group. Since G' is adjoint, we have $f(i(\breve{W})) \subset i'(\breve{W}')$. Let $\mu' \in X_*(T')$ with $f(i(t^\mu)) = i'(t^{\mu'})$. Then $f(i(w_0 t^\mu)) = i'(w'_0 t^{\mu'})$. Let $w' \in \breve{W}'$ with $f(i(w)) = i'(w')$ and $[b'] \in B(G')$ with $[b'] = [\dot{w}']$. Since $w_0 t^\mu \breve{W}_a = w \breve{W}_a$, we have $w'_0 t^{\mu'} \breve{W}'_a = w' \breve{W}'_a$. Therefore $\kappa_{G'}(b') = \kappa_{G'}(w') = \kappa_{G'}(w'_0 t^{\mu'})$.

Hence $(G, b, w_0 t^\mu)$ and $(G', b', w'_0 t^{\mu'})$ are associated.

By Proposition 4.2.3, for any $Z \in \Sigma^{\mathrm{top}}(X_{w_0 t^\mu}(b))$, its stabilizer $\mathrm{Stab}_Z(J_b(F))$ is associated to $\mathrm{Stab}_{Z'}(J_{b'}(F'))$ for some $Z' \in \Sigma^{\mathrm{top}}(X_{w'_0 t^{\mu'}}(b'))$. By assumption, $\mathrm{Stab}_{Z'}(J_{b'}(F'))$ is a very special parahoric subgroup of $J_{b'}(F')$. By the equivalence (1) $\Leftrightarrow$ (3) in Proposition 2.2.5 and by the formula (2.2.4.2) for the log-volume, we know that $\mathrm{Stab}_Z(J_b(F))$ is a very special parahoric subgroup of $J_b(F)$.

Now the reduction from the adjoint unramified case to the adjoint unramified F-simple case follows from the fact that any adjoint unramified group over F is a direct product of adjoint unramified F-simple groups. ∎

4.3. *Reduction to the basic case*

We assume that $\mathrm{char}(F) = 0$ and that G is an adjoint F-simple unramified group over F. By Corollary 4.2.4, we can reduce the proof of Theorem 4.1.2 to this case. In this subsection we show that we can further reduce the proof to the case where b is basic. We follow the strategy of [12, §5].

4.3.1. Let $\breve{K} = \breve{\mathbb{S}}_0$, and let $\mathcal{K}$ and $\breve{\mathcal{K}}$ be the corresponding parahoric subgroups of $G(F)$ and $G(\breve{F})$ respectively, as in §2.4.3. In our current setting, $\mathcal{K}$ is in fact a hyperspecial subgroup of $G(F)$.

Let $M \subset G$ denote the standard Levi subgroup of G given by the centralizer of $\overline{\nu}_b^G$. We view $\breve{\mathscr{A}}$ as an apartment for M and let $\breve{\mathfrak{a}}_M \subset \breve{\mathscr{A}}$ be the (unique) alcove with respect to M such that $\breve{\mathfrak{a}} \subset \breve{\mathfrak{a}}_M$. We denote by $\breve{W}_M$ the Iwahori–Weyl group for M and denote by Ω_M the subgroup of length zero elements determined by $\breve{\mathfrak{a}}_M$. Upon replacing b by an element of its σ-conjugacy class in $G(\breve{F})$, we may assume that $b \in M(\breve{F})$ and that $\overline{\nu}_b^M = \overline{\nu}_b^G$ (see e.g. [5, Lemma 2.5.1]). Then b is basic in M. Upon further replacing b by an element of its σ-conjugacy class in $M(\breve{F})$, we may assume that $b = \dot{\tau}$ for some $\tau \in \Omega_M$.

Let P be the standard parabolic subgroup of G with Levi subgroup M. Let N be the unipotent radical of P. Let $\breve{\mathcal{K}}_M$ (resp. $\breve{\mathcal{K}}_P$) denote the intersection $M(\breve{F}) \cap \breve{\mathcal{K}}$ (resp $P(\breve{F}) \cap \breve{\mathcal{K}}$). These arise from group schemes $\mathcal{K}_M$ and $\mathcal{K}_P$ defined over O_F, and $\mathcal{K}_M(O_F)$ is a hyperspecial subgroup of $M(F)$. As in [12, §5], we define

$$X_\mu^{M\subset G}(b)(\mathbf{k}) := \{g \in M(\breve{F})/\breve{\mathcal{K}}_M \mid g^{-1}b\sigma(g) \in \breve{\mathcal{K}}\dot{t}^\mu\breve{\mathcal{K}}\},$$
$$X_\mu^{P\subset G}(b)(\mathbf{k}) := \{g \in P(\breve{F})/\breve{\mathcal{K}}_P \mid g^{-1}b\sigma(g) \in \breve{\mathcal{K}}\dot{t}^\mu\breve{\mathcal{K}}\}.$$

These can be identified with the sets of $\mathbf{k}$-points of perfect subschemes $X_\mu^{M\subset G}(b)$ and $X_\mu^{P\subset G}(b)$ of $\mathrm{Gr}_{\breve{\mathcal{K}}_M}$ and $\mathrm{Gr}_{\breve{\mathcal{K}}_P}$ respectively.

The natural maps $M \leftarrow P \rightarrow G$ induce maps

$$X_\mu^{M\subset G}(b) \xleftarrow{p} X_\mu^{P\subset G}(b) \xrightarrow{q} X_\mu(b) ,$$

which are easily seen to be $J_b(F)$-equivariant. The same argument as [11, Lemma 2.2] shows that the map q is an immersion. By the Iwasawa decomposition, q is also surjective, and hence gives a decomposition of $X_\mu(b)$ into locally closed subschemes (cf. [12, Lemma 5.2]). Hence we obtain a $J_b(F)$-equivariant bijection

$$\Sigma^{\mathrm{top}}(X_\mu^{P\subset G}(b)) \xrightarrow{\sim} \Sigma^{\mathrm{top}}(X_\mu(b)). \tag{4.3.1.1}$$

4.3.2. Let X and $\mathcal{X}$ be smooth finite-type affine group schemes over $\breve{F}$ and $O_{\breve{F}}$ respectively. The loop group LX and the positive loop group $L\mathcal{X}$ are defined to be the functors on perfect $\mathbf{k}$-algebras R given by

$$LX(R) = X(W(R) \otimes_{W(\mathbf{k})} \breve{F}), \qquad \text{and} \qquad L^+\mathcal{X}(R) = \mathcal{X}(W(R) \otimes_{W(\mathbf{k})} O_{\breve{F}}).$$

Then LX is representable by an ind-perfect ind-group-scheme, and $L^+\mathcal{X}$ is representable by the perfection of an affine group scheme over $\mathbf{k}$. We also define the n^{th} jet-group $L^n\mathcal{X}$ to be the functor on perfect $\mathbf{k}$-algebras R given by

$$L^n\mathcal{X}(R) = \mathcal{X}(W(R) \otimes_{W(\mathbf{k})} O_{\breve{F}}/(\pi^n)),$$

where π is a uniformizer in $\breve{F}$. Then $L^n\mathcal{X}$ is representable by the perfection of an algebraic group over $\mathbf{k}$.

Lemma 4.3.3. *The map $f_b : LN \rightarrow LN$ sending n to $n^{-1}b\sigma(n)b^{-1}$ is an isomorphism.*

Proof. Recall we have assumed $b = \dot{\tau}$ for $\tau \in \Omega_M$. Choose s sufficiently divisible such that $\tau\sigma(\tau)\ldots\sigma^{s-1}(\tau) = t^{\lambda_s}$ where $\lambda_s := s\bar{\nu}_b \in X_*(T)^+$. (Note that since we have assumed G is unramified, Γ_0 acts trivially on $X_*(T)$.) We set $b_s := b\sigma(b)\cdots\sigma^{s-1}\sigma(b)$. Then we have $b_s \in \dot{t}^{\lambda_s}T(\breve{F})_1$ and it suffices to show that the map

$$f_b^s = f_b \circ \ldots \circ f_b : n \longmapsto n^{-1}b_s\sigma^s(n)b_s^{-1}$$

is an isomorphism $LN \to LN$.

For $r \geq 0$, we define $N_r := N(F) \cap \mathcal{I}_r$ where $\mathcal{I}_r$ is the r^{th}-subgroup in the Moy–Prasad filtration of $\mathcal{I}$. Then $N_r = \mathcal{N}_r(\mathcal{O}_F)$ for an $\mathcal{O}_F$-group scheme $\mathcal{N}_r$ and we have

$$L^+\mathcal{N}_r = \ker(L^+\mathcal{N}_0 \to L^r\mathcal{N}_0).$$

Since $\lambda_s \in X_*(T)^+$, we have $\dot{t}^{\lambda_s}\sigma^s(N_r)\dot{t}^{-\lambda_s} \subset N_r$ for all r. It follows that f_b^s induces a morphism

$$f_{b,r}^s : L^r\mathcal{N}_0 \longrightarrow L^r\mathcal{N}_0$$

for each r. In fact $f_{b,r}^s$ is naturally defined before taking perfections and induces bijections tangent spaces. Indeed, let $\phi : L^r\mathcal{N}_0 \to L^r\mathcal{N}_0$ be the morphism $n \mapsto \dot{t}^{\lambda_s}\sigma(n)\dot{t}^{-\lambda_s}$. Then ϕ has derivative 0 (before taking perfections), and hence $f_{b,r}^s$ induces multiplication by -1 on tangent spaces at the identity by [39, Lemma 4.4.13]. The usual argument as in the proof of Lang's Theorem implies that $f_{b,r}^s$ is a bijection on tangent spaces and hence $f_{b,r}^s$ is an étale morphism.

Let F_s be the degree s unramified extension of F and let $J_{b_s}^{(s)}(F_s)$ denote the σ-centralizer group $J_{b_s}^{(s)}(F_s) := \{g \in G(\breve{F}) : g^{-1}b_s\sigma^s(g) = b_s\}$; then $J_{b_s}^{(s)}(F_s) \subset M(\breve{F})$. Let $n_1, n_2 \in LN(\mathbf{k})$ with $f_b^s(n_1) = f_b^s(n_2)$. Then we have

$$n_1n_2^{-1} = b_s\sigma^s(n_1n_2^{-1})b_s^{-1}$$

and hence $n_1n_2^{-1} \in J_{b_1}^{(s)}(F_s) \cap N(\breve{F}) = \{1\}$. Therefore the fibers of the map $f_b^s : LN \to LN$ are torsors for the trivial group, and hence the "pro-étale" covering $f_b^s|_{L^+\mathcal{N}_0} : L^+\mathcal{N}_0 \to L^+\mathcal{N}_0$ obtained by taking the inverse limit of the $f_{b,r}^s$ is trivial. It follows that $f_b^s|_{L^+\mathcal{N}_0}$ is an isomorphism (cf. [43, Lemma 4.3.4]).

Now fix an element $\chi \in X_*(T)^{+,\sigma} \cap X_*(Z_M)$, where Z_M is the center of M. Using the fact that $\mathrm{Ad}\dot{t}^\chi \circ f_b^s = f_b^s \circ \mathrm{Ad}\dot{t}^\chi$, we find that $f_b^s : \dot{t}^{-\chi}L^+\mathcal{N}_0\dot{t}^\chi \to \dot{t}^{-\chi}L^+\mathcal{N}_0\dot{t}^\chi$ is an isomorphism. Taking an inductive limit over χ, we find that $f_b^s : LN \to LN$ is an isomorphism. ■

4.3.4. We identify $\mathrm{Gr}_{\breve{\mathcal{K}}}$ with the fpqc quotient $LG/L^+\breve{\mathcal{K}}$. For $\lambda \in X_*(T)$, recall the semi-infinite orbit

$$S_{N,\lambda} := LN\dot{t}^\lambda L^+\breve{\mathcal{K}}/L^+\breve{\mathcal{K}} \subset \mathrm{Gr}_{\breve{\mathcal{K}}}.$$

We let $\mathrm{Gr}_{\breve{\mathcal{K}},\mu}$ denote the Schubert cell $L^+\breve{\mathcal{K}}\dot{t}^\mu L^+\breve{\mathcal{K}}/L^+\breve{\mathcal{K}}$ and $\mathrm{Gr}_{\breve{\mathcal{K}},\preccurlyeq\mu}$ the corresponding Schubert variety which is defined to be the closure of $\mathrm{Gr}_{\breve{\mathcal{K}},\mu}$ inside $\mathrm{Gr}_{\breve{\mathcal{K}}}$.

Let $\widehat{M} \subset \widehat{G}$ denote the Levi subgroup determined by M and the fixed pinning from §4.1.5. For an M-dominant element $\lambda \in X_*(T)$, we may consider λ as element of $X^*(\widehat{T})$

which is $\widehat{M}$-dominant with respect to the ordering determined by $\widehat{B} \cap \widehat{M}$. We write $V_\lambda^{\widehat{M}}$ for the irreducible representation of $\widehat{M}$ of highest weight λ.

We let $a_{\lambda,\mu}$ denote the multiplicity of $V_\lambda^{\widehat{M}}$ appearing in the $\widehat{M}$-representation $V_\mu|_{\widehat{M}}$, and we write ρ_M (resp. ρ_N) for the half sum of positive roots in M (resp. roots in N). The same argument as [7, Proposition 5.4.2] shows that

$$\dim S_{N,\lambda} \cap \mathrm{Gr}_{\breve{\mathcal{K}},\mu} = \langle \mu + \lambda, \rho \rangle - 2\langle \lambda, \rho_M \rangle,$$

and that we have $|\Sigma^{\mathrm{top}}(S_{N,\lambda} \cap \mathrm{Gr}_{\breve{\mathcal{K}},\mu})| = a_{\lambda,\mu}$.

Lemma 4.3.5. *Let $k_M \in L^+\breve{\mathcal{K}}_M(\mathbf{k})$ be an element such that $\dot{t}^{-\lambda} k_M \dot{t}^{\lambda} \in L^+\breve{\mathcal{K}}_M$. Then left multiplication by k_M induces an automorphism of $S_{N,\lambda} \cap \mathrm{Gr}_{\breve{\mathcal{K}},\mu}$, and we have $k_M(Z) = Z$ for all $Z \in \Sigma^{\mathrm{top}}(S_{N,\lambda} \cap \mathrm{Gr}_{\breve{\mathcal{K}},\mu})$.*

Proof. Let $n \in LN(R)$ where R is a $\mathbf{k}$-algebra. Then

$$k_M n \dot{t}_\lambda = (k_M n k_M^{-1}) k_M \dot{t}^\lambda = (k_M n k_M^{-1}) \dot{t}^\lambda (\dot{t}^{-\lambda} k_M \dot{t}^\lambda) \in LN \dot{t}^\lambda L^+\breve{\mathcal{K}}.$$

It follows that multiplication by k_M induces an automorphism of $S_{N,\lambda}$ with inverse given by multiplication by k_M^{-1}, and hence an automorphism of $S_{N,\lambda} \cap \mathrm{Gr}_{\breve{\mathcal{K}},\mu}$.

The group $\dot{t}^{-\lambda} \breve{\mathcal{K}}_M \dot{t}^\lambda \cap \breve{\mathcal{K}}_M$ arises as the $O_{\breve{F}}$-points of a smooth connected $O_{\breve{F}}$-scheme $\breve{\mathcal{K}}_\lambda$. Then as above, left multiplication induces a map

$$L^+\breve{\mathcal{K}}_\lambda \times (S_{N,\lambda} \cap \mathrm{Gr}_{\breve{\mathcal{K}},\mu}) \to S_{N,\lambda} \cap \mathrm{Gr}_{\breve{\mathcal{K}},\mu}.$$

Let $Z \in \Sigma^{\mathrm{top}}(S_{N,\lambda} \cap \mathrm{Gr}_{\breve{\mathcal{K}},\mu})$. Then $k_M(Z) \in \Sigma^{\mathrm{top}}(S_{N,\lambda} \cap \mathrm{Gr}_{\breve{\mathcal{K}},\mu})$ is contained in the image of $L^+\breve{\mathcal{K}}_\lambda \times Z \to S_{N,\lambda} \cap \mathrm{Gr}_{\breve{\mathcal{K}},\mu}$. The image of this map is an irreducible subscheme of $S_{N,\lambda} \cap \mathrm{Gr}_{\breve{\mathcal{K}},\mu}$ containing Z, hence is equal to Z. It follows that $k_M(Z) = Z$. ∎

4.3.6. We define the sets

$$I_{\mu,M} := \{\lambda \in X_*(T) \mid \lambda \in X_*(T) \text{ is } M\text{-dominant}, S_{N,\lambda} \cap \mathrm{Gr}_{\breve{\mathcal{K}},\mu} \neq \emptyset\},$$
$$I_{\mu,b,M} := \{\lambda \in I_{\mu,M} \mid \kappa_M(b) = \lambda^\natural \in \pi_1(M)_\Gamma\}.$$

Then there is a decomposition

$$X_\mu^{M \subset G}(b) = \coprod_{\lambda \in I_{\mu,b,M}} X_\lambda^M(b), \tag{4.3.6.1}$$

where each $X_\lambda^M(b)$ is locally closed inside $X_\mu^{M \subset G}(b)$. In the equal characteristic setting, this is proved in [11, Proposition 2.9 (1), (2)] and the same proof works in general.

Proposition 4.3.7. (1) *Let $\lambda \in I_{\mu,b,M}$ and $Z \in \Sigma^{\mathrm{top}}(X_\lambda^M(b))$. Then*

$$\dim p^{-1}(Z) \leq \dim X_\mu(b)$$

with equality if and only if $a_{\lambda,\mu} \neq 0$.

(2) *Let* $U \in \Sigma^{\mathrm{top}}(X_\mu(b))$ *and* $U^P \in \Sigma^{\mathrm{top}}(X_\mu^{P\subset G}(b))$ *the corresponding element. Then there exists* $\lambda \in I_{\mu,b,M}$ *with* $a_{\lambda,\mu} \neq 0$ *and* $Z \in \Sigma^{\mathrm{top}}(X_\lambda^M(b))$ *such that* $Z \cap p(U^P)$ *is open dense in* $p(U^P)$.

Proof. (1) By [11, Lemma 2.8, Proposition 2.9 (3)], which also holds in the mixed characteristic setting, we have

$$\begin{aligned}\dim p^{-1}(Z) &\leq \dim X_\lambda^M(b) + \langle \mu+\lambda, \rho\rangle - 2\langle\lambda, \rho_M\rangle - 2\langle\overline{\nu}_b, \rho_N\rangle \\ &= \langle\lambda, \rho_M\rangle - \frac{1}{2}\mathrm{def}_M(b) + \langle \mu+\lambda, \rho\rangle - 2\langle\lambda, \rho_M\rangle - 2\langle\overline{\nu}_b, \rho_N\rangle \\ &= \langle \mu - \overline{\nu}_b, \rho\rangle - \frac{1}{2}\mathrm{def}_G(b) \\ &= \dim X_\mu(b).\end{aligned}$$

The first and third equalities follow Theorem 2.4.7 and the second equality follows from the identities $\mathrm{def}_G(b) = \mathrm{def}_M(b)$ and $\langle\overline{\nu}_b, \rho_N\rangle = \langle\lambda, \rho_N\rangle$. For the last identity $\langle\overline{\nu}_b, \rho_N\rangle = \langle\lambda, \rho_N\rangle$, we use the fact that $\lambda - \overline{\nu}_b$ is a linear combination of coroots for M, and that $\langle\alpha^\vee, \rho_N\rangle = 0$ for any such coroot. By [7, Proposition 5.4.2], which again holds in mixed characteristic, the first inequality is an equality if and only if $a_{\lambda,\mu} \neq 0$.

(2) By [11, Proposition 2.9 (2)] and a similar calculation as in (1), for any $\lambda \in I_{\mu,b,M}$ and $x \in X_\lambda^M(b)$, we have

$$\dim p^{-1}(x) \leq \dim X_\mu(b) - \dim X_\lambda^M(b)$$

with equality if and only if $a_{\lambda,\mu} \neq 0$.

Since the $X_\lambda^M(b)$ are locally closed inside $X_\mu^{M\subset G}(b)$, there exists a unique $\lambda \in I_{\mu,b,M}$ such that $p(U^P) \cap X_\lambda^M(b)$ is open dense in $p(U^P)$. Since $p(U^P)$ is irreducible, we can further find a $Z \in \Sigma^{\mathrm{top}}(X_\lambda^M(b))$ such that $p(U^P) \cap Z$ is open dense in $p(U^P)$. Then we have

$$\dim p(U^P) \geq \dim X_\lambda^M(b).$$

It follows that these quantities are equal and we have $a_{\lambda,\mu} \neq 0$. ∎

Proposition 4.3.8. *Let* $\lambda \in I_{\mu,b,M}$ *with* $a_{\lambda,\mu} \neq 0$ *and let* $Z \in \Sigma^{\mathrm{top}}(X_\lambda^M(b))$. *Then the group* $\mathrm{Stab}_Z(J_b(F))$ *acts trivially on* $\Sigma^{\mathrm{top}}(p^{-1}(Z))$.

Proof. Let $Y' \to Z$ be an étale morphism such that the inclusion map $Z \to X_\lambda^M(b)$ lifts to a map $\iota : Y' \to LM$. The existence of Y' follows verbatim from the same argument as [33, Theorem 1.4], which shows that the morphism $LM \to \mathrm{Gr}_{\breve{\mathcal{K}}_M}$ admits sections locally for the étale topology (cf. [47, Proposition 1.20]). Upon replacing Y' with an irreducible component, we may assume that Y' is also irreducible. We let $Y \subset Z$ denote the image of Y', which is an open subscheme of Z.

We write $p^{-1}(Y')$ for the fiber product

$$\begin{tikzcd} p^{-1}(Y') \arrow[r] \arrow[d] & X_\mu^{P\subset G}(b) \arrow[d, "p"] \\ Y' \arrow[r] & X_\mu^{M\subset G}(b), \end{tikzcd}$$

and we write p' for the map $p^{-1}(Y') \to Y'$. The natural map $p^{-1}(Y') \to p^{-1}(Z)$ induces a bijection $\Sigma^{\mathrm{top}}(p^{-1}(Y')) \cong \Sigma^{\mathrm{top}}(p^{-1}(Z))$.

As in [12, Proposition 5.6], we set

$$\Phi := \{(m,n) \in \iota(Y') \times LN \mid mnL^+\breve{\mathcal{K}}_P \in X_\mu^{P\subset G}\} \subset LM \times LN.$$

We also set $\tilde{\Phi} := \{(y,n) \in Y' \times LN \mid \iota(y)nL^+\breve{\mathcal{K}}_P \in X_\mu^{P\subset G}\} \subset Y' \times LN$. Then Φ is the image of $\tilde{\Phi}$ under the map $(\iota, \mathrm{id}) : Y' \times LN \to LM \times LN$, and the morphism $\varphi : \tilde{\Phi} \to \Phi$ induces a bijection between irreducible components.

There is a natural morphism $\gamma : \tilde{\Phi} \to p^{-1}(Y')$ induced by $(y,n) \mapsto \iota(y)nL^+\breve{\mathcal{K}}_P$. The action of $L^+\breve{\mathcal{K}}_N$ on LN preserves $\tilde{\Phi}$ and the map γ factors through this action. We thus obtain a morphism

$$\tilde{\Phi}/L^+\breve{\mathcal{K}}_N \to p^{-1}(Y'). \tag{4.3.8.1}$$

We claim that (4.3.8.1) is an isomorphism. We define a map $\delta : p^{-1}(Y') \to \mathrm{Gr}_{\breve{\mathcal{K}}_P}$ by sending $x \in p^{-1}(Y')(R)$, for R a perfect $\mathbf{k}$-algebra, to $[\iota(p'(x))]^{-1}x$. Here we consider $[\iota(p'(x))]$ as an element of $LM(R)$ which acts on $\mathrm{Gr}_{\breve{\mathcal{K}}_P}$ via the inclusion $LM \to LP$. Then δ factors through the image of the immersion $\mathrm{Gr}_{\breve{\mathcal{K}}_N} \to \mathrm{Gr}_{\breve{\mathcal{K}}_P}$, and hence we obtain a morphism $\delta : p^{-1}(Y') \to \mathrm{Gr}_{\breve{\mathcal{K}}_N}$. The map $(p', \delta) : p^{-1}(Y') \to \tilde{\Phi}/L^+\breve{\mathcal{K}}_N$ is then an inverse for (4.3.8.1); this proves the claim.

Let $x \in \breve{W}$ such that $\breve{\mathcal{I}}_M\dot{x}\breve{\mathcal{I}}_M \subset \breve{\mathcal{K}}_M\dot{t}^\lambda\breve{\mathcal{K}}_M$ is the open cell. We replace Y' (and hence Y) by the open subscheme such that $m^{-1}b\sigma(m) \in \breve{\mathcal{I}}_M\dot{x}\breve{\mathcal{I}}_M$, for $m \in \iota(Y')$. Then the same argument as in [12, Proof of Proposition 5.6], shows that upon replacing ι if necessary, we may assume $m^{-1}b\sigma(m) \in \dot{t}^\lambda\breve{\mathcal{K}}_M$ for any $m \in \iota(Y')$. We then define

$$\mathscr{E} := \iota(Y') \times (LN \cap L^+\breve{\mathcal{K}}\dot{t}^\mu L^+\breve{\mathcal{K}}\dot{t}^{-\lambda}) \subset LM \times LN.$$

We write $\mathrm{Ad}_M : LM \times LN \to LM \times LN$ for the map $(m,n) \mapsto (m, mnm^{-1})$. This is easily seen to be an isomorphism with inverse given by $\mathrm{Ad}_M^{-1} : (m,n) \mapsto (m, m^{-1}nm)$. Define $\tilde{f}_b = \mathrm{Ad}_M ß \circ (\mathrm{id}, f_b) \circ \mathrm{Ad}_M : LM \times LN \to LM \times LN$. By Lemma 4.3.3, $\tilde{f}_b$ is an isomorphism. The restriction of $\tilde{f}_b$ to Φ gives an isomorphism $\tilde{f}_b : \Phi \to \mathscr{E}$ and we have a Cartesian diagram:

$$\begin{tikzcd} \Phi \arrow[r, "\tilde{f}_b"] \arrow[d] & \mathscr{E} \arrow[d] \\ \iota(Y') \times LN \arrow[r, "\tilde{f}_b"] & \iota(Y') \times LN \end{tikzcd}$$

We consider the projection

$$\mathrm{pr}_\lambda : LN \to LN\dot{t}^\lambda L^+\breve{\mathcal{K}}/L^+\breve{\mathcal{K}}$$

given by $n \mapsto n\dot{t}^\lambda\breve{\mathcal{K}}$. Then $LN \cap L^+\breve{\mathcal{K}}\dot{t}^\mu L^+\breve{\mathcal{K}}\dot{t}^{-\lambda}$ is the preimage of $S_{N,\lambda} \cap \mathrm{Gr}_{\breve{\mathcal{K}},\mu}$ under pr_λ. We write $\mathrm{pr} : \mathscr{E} \to S_{N,\lambda} \cap \mathrm{Gr}_{\breve{\mathcal{K}},\mu}$ for the composition of projection onto the second component $\mathrm{pr}_2 : \mathscr{E} \to LN \cap L^+\breve{\mathcal{K}}\dot{t}^\mu L^+\breve{\mathcal{K}}\dot{t}^{-\lambda}$ followed by pr_λ.

Let $Z' \in \Sigma(S_{N,\lambda} \cap \mathrm{Gr}_{\breve{\mathcal{K}},\mu})$. The same argument as [11, Proposition 2.9] shows that

$$\dim \gamma((\mathrm{pr} \circ \tilde{f}_b \circ \phi)^{-1}(Z')) \leq \dim X_\mu(b) = \dim p^{-1}(Y')$$

with equality if and only if $Z' \in \Sigma^{\mathrm{top}}(S_{N,\lambda} \cap \mathrm{Gr}_{\breve{\mathcal{K}},\mu})$. Indeed, if

$$y \in p'(\gamma((\mathrm{pr} \circ \tilde{f}_b)^{-1}(Z'))) \subset Y'$$

with $m = \iota(y)$, then we have

$$p^{-1}(y) \cap \gamma((\mathrm{pr} \circ \tilde{f}_b \circ \phi)^{-1}(Z')) = (\mathrm{pr}_\lambda \circ f_{m^{-1}b\sigma(m)})^{-1}(Z').$$

Then [11, Lemma 2.4] (cf. also [11, Proof of Proposition 2.9 (2)]) implies that

$$\dim p^{-1}(y) \cap \gamma((\mathrm{pr} \circ \tilde{f}_b \circ \phi)^{-1}(Z')) = \dim Z' - \langle \overline{\nu}_b, 2\rho_N \rangle$$

and hence

$$\begin{aligned}\dim \gamma((\mathrm{pr} \circ \tilde{f}_b \circ \phi)^{-1}(Z')) &= \dim X^M_\lambda(b) + \dim Z' - \langle \overline{\nu}_b, 2\rho_N \rangle \\ &\leq \dim X^M_\lambda(b) + \langle \mu + \lambda, \rho \rangle - 2\langle \lambda, \rho_M \rangle - \langle \overline{\nu}_b, 2\rho_N \rangle \\ &= \dim X_\mu(b)\end{aligned}$$

with equality if and only if $Z' \in \Sigma^{\mathrm{top}}(S_{N,\lambda} \cap \mathrm{Gr}_{\breve{\mathcal{K}},\mu})$. For the last equality, see the proof of Proposition 4.3.7.

It follows that the association $Z' \mapsto \gamma((\mathrm{pr} \circ \tilde{f}_b)^{-1}(Z'))$ induces a map $\theta : \Sigma^{\mathrm{top}}(S_{N,\lambda} \cap \mathrm{Gr}_{\breve{\mathcal{K}},\mu}) \to \Sigma^{\mathrm{top}}(p^{-1}(Y'))$. We have the following diagram of morphisms

$$p^{-1}(Y') \xleftarrow{\gamma} \tilde{\Phi} \xrightarrow{\phi} \Phi \xrightarrow{\tilde{f}_b} \mathscr{E} \xrightarrow{\mathrm{pr}} S_{N,\lambda} \cap \mathrm{Gr}_{\breve{\mathcal{K}},\mu}.$$

These morphisms all induce bijections of irreducible components, and hence it follows that θ is a bijection.

Let $j \in \mathrm{Stab}_Z(J_b(F))$ and let $U \in \Sigma^{\mathrm{top}}(p^{-1}(Z))$. We let $\tilde{U}_1, \tilde{U}_2 \subset \tilde{\Phi}$ denote the preimages of U and jU respectively under the composite map $\tilde{\Phi} \xrightarrow{\gamma} p^{-1}(Y') \to p^{-1}(Z)$, and we let $U_1, U_2 \subset \Phi$ denote their respective images under ϕ. For $i = 1, 2$, let $Z'_i \in \Sigma^{\mathrm{top}}(S_{N,\lambda} \cap \mathrm{Gr}_{\breve{\mathcal{K}},\mu})$ be the unique component containing $\mathrm{pr} \circ \tilde{f}_b(U_i)$. Then it suffices to show that $Z'_1 = Z'_2$.

Let $x = (m, n) \in U_1(\mathbf{k})$ such that the image $y = m\breve{\mathcal{K}}_M$ of x in Z lies in $j^{-1}Y$. Note that the set of such x is dense in U_1. Then jy lies in Y and we let $y' \in Y'(\mathbf{k})$ be a lift of jy to

$Y'(\mathbf{k})$. Then the element $\iota(y') \in LM(\mathbf{k})$ is of the form jmk_M for some $k_M \in L^+\breve{\mathcal{K}}_M(\mathbf{k})$, since it is a lift of $jy = jm\breve{\mathcal{K}}_M$.

Consider the element $z = (jmk_M, k_M^{-1}nk_M) \in \Phi$. Then we have $z \in U_2(\mathbf{k})$, and one computes that

$$\mathrm{pr}_2(\tilde{f}_b(z)) = k_M^{-1}n^{-1}b_m\sigma(n)b_m^{-1}k_M = k_M^{-1}\mathrm{pr}_2(\tilde{f}_b(x))k_M,$$

where $b_m = m^{-1}b\sigma(m)$. By the assumption on ι, we have $b_m, k_M^{-1}b_mk_M \in \dot{t}^\lambda\breve{L}^+\mathcal{K}_M$, and hence $\dot{t}^{-\lambda}k_M\dot{t}^\lambda \in L^+\breve{\mathcal{K}}_M$. Then by Lemma 4.3.5, we have $\mathrm{pr} \circ \tilde{f}_b(x) \in Z_2'$. Since this is true for a dense set of x in U_1, it follows that $\mathrm{pr} \circ \tilde{f}_b(U_1) \subset Z_2'$, and hence $Z_1' = Z_2'$. ■

Corollary 4.3.9. *Let $U \in \Sigma^{\mathrm{top}}(X_\mu(b))$. Then there exists $\lambda \in I_{\mu,b,M}$ and $Z \in \Sigma^{\mathrm{top}}(X_\lambda^M(b))$ such that*

$$\mathrm{Stab}_U(J_b(F)) = \mathrm{Stab}_Z(J_b(F)).$$

Proof. Let $U^P \in \Sigma^{\mathrm{top}}(X_\mu^{P\subset G}(b))$ be the component corresponding to U and let $Z := p(U^P) \subset X_\mu^{M\subset G}(b)$. By Lemma 4.3.7, we have $Z \in \Sigma^{\mathrm{top}}(X_\lambda^M(b))$ for some $\lambda \in I_{\mu,b,M}$ with $a_{\lambda,\mu} \neq 0$.

By the $J_b(F)$-equivariance of p, we have

$$\mathrm{Stab}_{U^P}(J_b(F)) \subset \mathrm{Stab}_Z(J_b(F)).$$

Since $U^P \in \Sigma^{\mathrm{top}}(p^{-1}(Z))$, Proposition 4.3.8 implies

$$\mathrm{Stab}_U(J_b(F)) = \mathrm{Stab}_{U^P}(J_b(F)) = \mathrm{Stab}_Z(J_b(F)).$$

The statement is proved. ■

Proposition 4.3.10. *In order to prove Theorem 4.1.2, it suffices to prove it when* $\mathrm{char}(F) = 0$*, G is F-simple, adjoint, and unramified over F, and b is basic.*

Proof. This follows from Corollary 4.2.4 and Corollary 4.3.9. ■

4.4. The special case of a sum of dominant minuscule cocharacters

We assume that $\mathrm{char}(F) = 0$, that G is F-simple, adjoint, and unramified over F, and that b is basic. Our goal in this subsection is to prove a partial result towards Theorem 4.1.2 when μ is a sum of minuscule dominant cocharacters. We use the idea of X. Zhu (see [47, §3.1.3]) that one can "separate" the summands of μ by constructing a convolution map from the affine Deligne–Lusztig variety of a Weil-restriction group to the original affine Deligne–Lusztig variety. This idea was originally used in loc. cit. to establish the dimension formula, and it was S. Nie who first applied this idea to the study of irreducible components (see [31] and [32]).

4.4.1. Let F_r denote the unramified extension of F of degree r inside $\breve{F}$. Let H be an unramified reductive group over F_r and let $G' := \mathrm{Res}_{F_r/F} H$. We canonically identify $\breve{F}$ with $\breve{F}_r$. For $b \in H(\breve{F})$ and μ a geometric cocharacter of H, we have the affine Deligne–Lusztig variety $X_\mu^H(b)$ as in §2.4.3. In this subsection we denote this by $X_\mu^H(b\sigma^r)$ to emphasize that H is a group over F_r and the Frobenius relative to F_r is σ^r. We also write $J_b^{(r)}$ for J_b (defined with respect to H over F_r), and write $B^{(r)}(H)$ for the set of σ^r-conjugacy classes in $H(\breve{F})$.

Let $\tau_0 : F_r \hookrightarrow \breve{F}$ be the inclusion and write τ_i for $\sigma^i(\tau_0)$ for $i = 1, \dots, r-1$. Thus $\{\tau_0, \dots, \tau_{r-1}\}$ is the set of F-algebra embeddings $F_r \to \breve{F}$. There is a canonical identification

$$G' \otimes_F \breve{F} \cong \prod_{i=0}^{r-1} H \otimes_{F_r, \tau_i} \breve{F}.$$

Let T_H be the centralizer of a fixed maximal F_r-split torus in H. Let $T' = \mathrm{Res}_{F_r/F} T_H$, which we view as an F-subgroup of G'. Then T' is the centralizer of a maximal F-split torus in G'. A cocharacter of T' is the same as a sequence $\mu' = (\mu_0, \dots, \mu_{r-1})$, where $\mu_i \in X_*(T_H)$. Fix a Borel subgroup of H containing T_H and use it to define the dominant cocharacters $X_*(T_H)^+$. This also defines a Borel subgroup of G' containing T' and defines $X_*(T')^+$. We fix a hyperspecial subgroup of $H(F_r)$ that is compatible with our choice of the maximal $\breve{F}_r$-split F_r-rational torus of H. This also determines a hyperspecial subgroup of $G'(F)$. We use these hyperspecial subgroups to define affine Deligne–Lusztig varieties at hyperspecial level for H and G'. For $b' = (b_0, \dots, b_r) \in G'(\breve{F})$, we define

$$\mathrm{Nm}(b') := b_0 \sigma(b_1) \cdots \sigma^{r-1}(b_{i-1}) \in H(\breve{F}).$$

The association $b' \mapsto \mathrm{Nm}(b')$ defines a bijection $B(G') \xrightarrow{\sim} B^{(r)}(H)$, and there is a natural isomorphism $J_{b'}(F) \cong J^{(r)}_{\mathrm{Nm}(b')}(F_r)$.

Lemma 4.4.2. *Let $\mu' = (\mu_0, \dots, \mu_{r-1}) \in X_*(T')^+$ and $[b'] \in B(G', \mu')$. We write $|\mu'|$ for $\sum_{i=0}^{r-1} \sigma^i(\mu_i) \in X_*(T_H)^+$. Then there is a natural morphism*

$$\theta : X^{G'}_{\preccurlyeq \mu'}(b') \longrightarrow X^H_{\preccurlyeq |\mu'|}(\mathrm{Nm}(b')\sigma^r)$$

which is $J_{b'}(F) \cong J^{(r)}_{\mathrm{Nm}(b')}(F_r)$-equivariant. Moreover, for each

$$U \in \Sigma^{\mathrm{top}}(X^H_{\preccurlyeq |\mu'|}(\mathrm{Nm}(b')\sigma^r)),$$

there exists $Z \in \Sigma^{\mathrm{top}}(X^{G'}_{\preccurlyeq \mu'}(b'))$ such that

$$\mathrm{Stab}_Z(J_{b'}(F)) = \mathrm{Stab}_U(J^{(r)}_{\mathrm{Nm}(b')}(F_r)).$$

Proof. The morphism θ is given by the isomorphism in [47, Lemma 3.5] and the left vertical map in the diagram on p. 459 of [47]. The $J_{b'}(F) \cong J^{(r)}_{\mathrm{Nm}(b')}(F_r)$-equivariance is clear from the construction. Let $U \in \Sigma^{\mathrm{top}}(X^H_{\preccurlyeq |\mu'|}(\mathrm{Nm}(b')\sigma^r))$. We claim that $\mathcal{J} := \mathrm{Stab}_U(J_{\mathrm{Nm}(b')}(F_r))$ acts trivially on $\Sigma^{\mathrm{top}}(\theta^{-1}(U))$. In fact, by the diagram on p. 459 of

[47], there exists $m \in \mathbb{N}$ and an $L^m H$-torsor U' over U equipped with a $\mathcal{J}$-action such that $U' \to U$ is $\mathcal{J}$-equivariant and such that there exists a $\mathcal{J}$-equivariant U'-scheme isomorphism $\theta^{-1}(U) \times_U U' \xrightarrow{\sim} F \times_{\mathbf{k}} U'$, where $\mathcal{J}$ acts trivially on F. Our claim follows. By the claim, we have $\mathrm{Stab}_Z(J_{b'}(F)) = \mathrm{Stab}_U(J^{(r)}_{\mathrm{Nm}(b')}(F_r))$ for arbitrary $Z \in \Sigma^{\mathrm{top}}(\theta^{-1}(U))$. By [31, Lemma 1.8], we have $\Sigma^{\mathrm{top}}(\theta^{-1}(U)) \subset \Sigma^{\mathrm{top}}(X^{G'}_{\preccurlyeq\mu'}(b'))$. The lemma follows. ■

Proposition 4.4.3. *Assume that μ is a sum of dominant minuscule cocharacters and $[b] \in B(G,\mu)$ is basic. Then for any $Z \in \Sigma^{\mathrm{top}}(X_\mu(b))$, $\mathrm{Stab}_Z(J_b(F))$ is a special parahoric subgroup of $J_b(F)$.*

Proof. We first consider the case where μ is minuscule. Let $M \subset G$ be a standard Levi subgroup such that there exists $b \in [b] \cap M(\breve{F})$ which is superbasic in M. We use the same notations as in §4.3.1 with respect to M. We choose $b \in [b] \cap M(\breve{F})$ that is superbasic in M, and upon σ-conjugating b in $M(\breve{F})$ we may assume that $b = \dot{\tau}$ for some $\tau \in \Omega_M$.

Let $Z \in \Sigma^{\mathrm{top}}(X_\mu(b))$ and we let $\mathcal{J} \subset J_b(F)$ denote the stabilizer of Z. Let P be the standard parabolic subgroup of G with Levi factor M. By [45, Theorem 3.1.1], $\mathcal{J}$ is a parahoric subgroup of $J_b(F)$. By Theorem 4.1.6, the map ϕ in [12, Theorem 5.12] is a bijection. Indeed, as explained in [12, Remark 1.5 (a)], the cardinality of the domain of ϕ is equal to $\dim V_\mu(\lambda_b)$, and the cardinality of the codomain is equal to $\mathscr{N}(\mu, b)$. Thus by [12, Theorem 5.12] and Theorem 4.1.6, ϕ is a surjective map between finite sets of equal cardinality and hence is a bijection. It follows from the "only if" part of [12, Theorem 5.12] that $J_b(F) \cap P(\breve{F})$ acts transitively on each $J_b(F)$-orbit in $\Sigma^{\mathrm{top}}(X_\mu(b))$. Hence we have

$$J_b(F) = (J_b(F) \cap P(\breve{F})) \cdot \mathcal{J}.$$

Note that $J_b(F) \cap P(\breve{F}) = Q(F)$ where Q is a minimal parabolic subgroup of J_b, since b is superbasic in $M(\breve{F})$. Thus

$$J_b(F) = Q(F) \cdot \mathcal{J}. \tag{4.4.3.1}$$

Recall that $\mathcal{J}$ is a parahoric subgroup of $J_b(F)$. In the following we show that this fact together with (4.4.3.1) implies that $\mathcal{J}$ is a special parahoric subgroup. By [3, Proposition 4.4.2], the equality (4.4.3.1) implies that $\mathcal{J}$ is contained in a special parahoric subgroup $\mathcal{J}_1$ of $J_b(F)$. (Indeed, (4.4.3.1) implies that $\mathcal{J}$ is contained in a "bon sous-groupe borné maximal" $\mathcal{J}_1'$, and the equivalence of (i) and (ii) in [3, Proposition 4.4.2] implies that $\mathcal{J}_1'$ is special; cf. [3, Proposition 4.4.6]. Since $\mathcal{J}$ is parahoric, it must be contained in a special parahoric $\mathcal{J}_1$ that is contained in $\mathcal{J}_1'$.) We are left to check that $\mathcal{J} = \mathcal{J}_1$, for which we need to use (4.4.3.1) again. Fix a maximal split torus A' in J_b whose centralizer is a Levi subgroup of Q. By (4.4.3.1), there exists $j \in Q(F)$ such that $j\mathcal{J}j^{-1}$ is associated with a facet in the apartment $\mathcal{A}'$ corresponding to A'. Thus up to conjugating $\mathcal{J}$ and $\mathcal{J}_1$ by j, we may assume that both $\mathcal{J}$ and $\mathcal{J}_1$ are associated with facets in $\mathcal{A}'$, and that (4.4.3.1) still holds. Then from (4.4.3.1) we get

$$\mathcal{J}_1 = (\mathcal{J}_1 \cap Q(F)) \cdot \mathcal{J}. \tag{4.4.3.2}$$

Let $\overline{\mathcal{J}_1}$ denote the reductive quotient of the special fiber of $\mathcal{J}_1$. Then the images of $\mathcal{J}_1 \cap Q(F)$ and $\mathcal{J}$ in $\overline{\mathcal{J}_1}$ are (k_F-points of) parabolic subgroups $\mathbb{B}$ and $\mathbb{P}$ respectively, and $\mathbb{B} \cap \mathbb{P}$ contains a maximal split torus in $\overline{\mathcal{J}_1}$, namely the reduction of A' (still denoted by A'). More precisely, $\mathbb{B}$ is the parabolic subgroup of $\overline{\mathcal{J}_1}$ containing (the reduction of) A' such that the roots of A' on $\mathrm{Lie}\,\mathbb{B}$ are those $\alpha \in X^*(A')$ appearing as vector parts of the affine roots of G vanishing at the special vertex in $\mathcal{A}'$ corresponding to $\mathcal{J}_1$ and which are also roots of A' on $\mathrm{Lie}(Q)$. Since Q is minimal parabolic, there does not exist a pair of opposite roots $\alpha, -\alpha$ as above. From this we see that the parabolic $\mathbb{B}$ is in fact minimal, i.e., a Borel subgroup (since every reductive group over k_F is quasi-split). By (4.4.3.2) we have $\overline{\mathcal{J}_1} = \mathbb{B}\mathbb{P}$, and by the Bruhat decomposition this is possible only when $\mathbb{P} = \overline{\mathcal{J}_1}$, or equivalently $\mathcal{J} = \mathcal{J}_1$. We have thus proved that $\mathcal{J}$ is a special parahoric subgroup of $J_b(F)$.

We now consider the case when μ is a sum of r dominant minuscule cocharacters. Let H be the pinned unramified reductive group over F_r such that its based root datum with the σ^r-action is identified with the based root datum of (G, B, T) with the σ-action. Let T_H be the maximal torus in the pinning of H. Then we have a canonical identification $X_*(T)^+ \cong X_*(T_H)^+$, and the image of μ in $X_*(T_H)^+$, denoted by μ_H, is also a sum of r dominant minuscule cocharacters. We have canonical identifications $G(\breve{F}) \cong H(\breve{F})$ and $(\breve{W}, \sigma) \cong (\breve{W}_H, \sigma^r)$. Let $b_H \in H(\breve{F})$ correspond to $b \in G(\breve{F})$, and let $w_{0,H}$ denote the longest element of $\breve{W}_H$. Then $(G, b, w_0 t^{\mu})$ and $(H, b_H, w_{0,H} t^{\mu_H})$ are associated as in §4.2. By Proposition 4.2.3, Proposition 2.4.10, and the fact that association of parahoric subgroups preserves being very special (see the proof of Corollary 4.2.4), it suffices to prove the result for $X^H_{\mu_H}(b_H\sigma^r)$.

Since μ_H is a sum of r dominant minuscule cocharacters, we can decompose μ_H as $\sum_{i=0}^{r-1} \sigma^i(\mu_i)$, where each μ_i is a dominant minuscule cocharacter in $X_*(T_H)^+$. Let $G' = \mathrm{Res}_{F_r/F} H$, and let $\mu' = (\mu_0, \cdots, \mu_{r-1})$, viewed as a cocharacter of a maximal torus in G' as in §4.4.1. Choose $b' \in G'(\breve{F})$ such that its image under $G'(\breve{F}) \to B(G') \xrightarrow{\sim} B^{(r)}(H)$ is the class of b_H. By Lemma 4.4.2 applied to the current situation, for every $U \in \Sigma^{\mathrm{top}}(X^H_{\preccurlyeq \mu_H}(b_H\sigma^r))$, there exists $Z \in \Sigma^{\mathrm{top}}(X^{G'}_{\preccurlyeq \mu'}(b'))$ such that $\mathrm{Stab}_U(J^{(r)}_{b_H}(F_r)) = \mathrm{Stab}_Z(J_{b'}(F))$. Note that μ' is minuscule, so $X^{G'}_{\preccurlyeq \mu'}(b') = X^{G'}_{\mu'}(b')$, and by the previous part of the proof, we know that $\mathrm{Stab}_Z(J_{b'}(F))$ is a special parahoric. The desired result for $X^H_{\mu_H}(b_H\sigma^r)$ follows by noting that the natural map $X^H_{\mu_H}(b_H\sigma^r) \to X^H_{\preccurlyeq \mu_H}(b_H\sigma^r)$ induces a $J^{(r)}_{b_H}(F_r)$-equivariant bijection between the sets of top-dimensional irreducible components. ■

4.5. Numerical relations

Another key ingredient in our proof of Theorem 4.1.2 is a set of numerical relations deduced from results in [45], which we discuss here.

4.5.1. We assume that $\mathrm{char}(F) = 0$, that G is F-simple, adjoint, and unramified over F, and that b is basic. We also assume that $[b]$ is not unramified, i.e., we assume that $\mathrm{def}_G(b) \neq 0$.

Since b is basic, J_b is an inner form of G. Thus we can transfer Haar measures on $G(F)$ to Haar measures on $J_b(F)$, as in [25, §1]. We fix the Haar measure on $G(F)$ giving volume 1 to hyperspecial subgroups, and transfer it to a Haar measure on $J_b(F)$. (This Haar measure on $J_b(F)$ may not give volume 1 to Iwahori subgroups.) For each $Z \in \Sigma^{\mathrm{top}}(X_\mu(b))$, the volume of the parahoric subgroup $\mathrm{Stab}_Z(J_b(F)) \subset J_b(F)$ depends on Z only via the $J_b(F)$-orbit $[Z]$ of Z. We denote this volume by $\mathrm{vol}([Z])$.

Let $S_{\mu,b}(t) \in \mathbb{Q}(t)$ be the rational function in [45, Theorem 6.1.3]. We have

$$S_{\mu,b}(0) = \mathscr{N}(\mu, b),$$
$$S_{\mu,b}(q) = e(J_b) \sum_{[Z]\in J_b(F)\backslash\Sigma^{\mathrm{top}}(X_\mu(b))} \mathrm{vol}([Z])^{-1}.$$

Here q denotes the cardinality of the residue field of F, and $e(J_b) \in \{\pm 1\}$ is the Kottwitz sign of J_b. (Recall $\mathscr{N}(\mu, b)$ from Definition 4.1.3.) We set

$$Q(\mu, b) := e(J_b)S_{\mu,b}(q)\mathscr{N}(\mu, b)^{-1} = \mathscr{N}(\mu, b)^{-1} \sum_{[Z]\in J_b(F)\backslash\Sigma^{\mathrm{top}}(X_\mu(b))} \mathrm{vol}([Z])^{-1}. \tag{4.5.1.1}$$

Proposition 4.5.2. *Keep the assumptions on F, G, and $[b]$ in §4.5.1. Assume that none of the simple factors of $G_{\overline{F}}$ is of type A. The following statements hold.*

(1) *Assume that G is not a Weil restriction of the split adjoint group of type E_6. Then there exists a minuscule $\mu_1 \in X_*(T)^+$ such that $\mathscr{N}(\mu_1, b) = 1$, and such that for all $\mu \in X_*(T)^+$ we have*
$$Q(\mu, b) = Q(\mu_1, b).$$

(2) *Assume that G is a Weil restriction of the split adjoint group of type E_6. (The Weil restriction is necessarily along an unramified extension of F since G is unramified). Then there exist $\mu_1, \mu_2 \in X_*(T)^+$, where μ_1 is minuscule and μ_2 is a sum of dominant minuscule cocharacters, such that $\mathscr{N}(\mu_1, b) = 1$ and such that for all $\mu \in X_*(T)^+$ we have*

$$Q(\mu, b) = Q(\mu_1, b) + C(\mu)(Q(\mu_2, b) - Q(\mu_1, b)), \tag{4.5.2.1}$$

for some $C(\mu) \in \mathbb{Q}$.

Proof. The proposition follows from the main result of [45] (i.e., the Chen–Zhu Conjecture), and the proof of [45, Theorem 6.3.2]. More precisely, part (1) follows from the equation below [45, (6.3.3)] and the main result [45, Theorem A] asserting that the numbers $\mathscr{M}(\mu, b)$ and $\mathscr{M}(\mu_1, b)$ in that equation are equal to $\mathscr{N}(\mu, b)$ and $\mathscr{N}(\mu_1, b)$ respectively . Part (2) follows from the equation below [45, (6.3.7)], the equation below [45, (6.3.8)], and [45, Theorem A] asserting that $\mathscr{M}(\mu, b) = \mathscr{N}(\mu, b)$. ∎

Remark 4.5.3. In Proposition 4.5.2, the conclusion in case (2) is weaker than that in case (1), and this originates from the dichotomy in [45, Proposition 6.3.2]. It turns out that in case (2), there is extra difficulty in trying to establish the key estimate [45, (6.3.1)], and in fact only the weaker statement [45, Proposition 6.3.2 (2)] is proved. If G is a Weil restriction of PGL_n, there seems to be even more serious difficulty in trying to establish [45, (6.3.1)]. As a result the type A case is not considered in [45, Proposition 6.3.2]. After the publishing of [45], the authors have realized that one can actually prove [45, (6.3.1)] when G is a Weil restriction of an adjoint unramified unitary group. We will not need this for the purposes of the current paper.

4.6. Proof of Theorem 4.1.2

By Proposition 4.3.10, we may assume without loss of generality that $\mathrm{char}(F) = 0$, that G is F-simple, adjoint, and unramified over F, and that b is basic. If $[b]$ is unramified, then Theorem 4.1.2 is already proved in [43, Theorem 4.4.14 (1)], cf. [45, Theorem 6.2.2]. We hence assume that $[b]$ is not unramified. Thus we are in the same setting as §4.5.1.

Let $\mathrm{vol}_{\max}$ be the volume of a very special parahoric subgroup of $J_b(F)$, where the Haar measure on $J_b(F)$ is as in §4.5.1. We know that every stabilizer for the $J_b(F)$-action on $\Sigma^{\mathrm{top}}(X_\mu(b))$ is a parahoric subgroup of $J_b(F)$, see Remark 3.3.12 and [45, Proposition 3.1.4]. As a result, the volume of such a stabilizer will be at most $\mathrm{vol}_{\max}$, and equality holds if and only if the stabilizer is very special. Since the quantity $Q(\mu, b)$ defined in (4.5.1.1) is the average of the volumes of these stabilizers, we see that Theorem 4.1.2 for (μ, b) is equivalent to the relation

$$Q(\mu, b) = \mathrm{vol}_{\max}^{-1} . \tag{4.6.0.1}$$

Since G is F-simple, the simple factors of $G_{\overline{F}}$ are isomorphic to each other. If they are of type A, then μ is necessarily a sum of dominant minuscule cocharacters in $X_*(T)$. In this case, Theorem 4.1.2 follows from Proposition 4.4.3 if we know that every special parahoric subgroup of $J_b(F)$ is automatically very special. Since J_b is an inner form of G and hence also of type A, it is indeed the case that special parahoric subgroups of $J_b(F)$ are automatically very special, by inspecting the tables in [40, §4].

Assume that G is as in Proposition 4.5.2 (1), and let μ_1 be as in that part of that proposition. Since $\mathscr{N}(\mu_1, b) = 1$, it follows from Proposition 3.4.6 that $Q(\mu_1, b) = \mathrm{vol}_{\max}^{-1}$. (Here Proposition 3.4.6 is indeed applicable since G is F-simple and adjoint.) But $Q(\mu, b) = Q(\mu_1, b)$, so (4.6.0.1) holds for (μ, b), and this implies that Theorem 4.1.2 holds for (μ, b).

We are left with the case where G is a Weil restriction of the split adjoint group of type E_6. In this case, let μ_1 and μ_2 be as in Proposition 4.5.2 (2). Since J_b is also of type E_6, by inspecting the tables in [40, §4] we see that every special parahoric subgroup of $J_b(F)$ is automatically very special. Thus by Proposition 4.4.3 we know that Theorem 4.1.2 holds for (μ_1, b) and (μ_2, b). It follows that

$$Q(\mu_1, b) = Q(\mu_2, b) = \mathrm{vol}_{\max}^{-1} .$$

Substituting this back to (4.5.2.1), we obtain (4.6.0.1) for (μ, b), and this implies that Theorem 4.1.2 holds for (μ, b).

The proof of Theorem 4.1.2 is complete.

5. Irreducible components of basic loci

5.1. Shimura varieties

5.1.1. We use the previous section to describe the irreducible components in the basic locus of certain Hodge type Shimura varieties constructed in [24]. Let $\mathbf{G}$ be a connected reductive group over $\mathbb{Q}$ and X a conjugacy class of homomorphisms

$$h : \mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \longrightarrow \mathbf{G}_{\mathbb{R}}$$

such that $(\mathbf{G}, X)$ is a Shimura datum. For any $\mathbb{C}$-algebra R we have $R \otimes_{\mathbb{R}} \mathbb{C} \cong R \times c^*(R)$, where c is the complex conjugation. For $h \in X$ we let μ_h denote the cocharacter of $\mathbf{G}_{\mathbb{C}}$ given by

$$R^\times \to R^\times \times c^*(R)^\times \xrightarrow{h} \mathbf{G}(R),$$

where R is an arbitrary $\mathbb{C}$-algebra and the first map is $z \mapsto (z, 1)$. The conjugacy class of μ_h^{-1} is defined over a number field $\mathbf{E} := \mathbf{E}(\mathbf{G}, X) \subset \mathbb{C}$ and we write $\{\mu\}$ for the corresponding geometric conjugacy class of cocharacters over $\overline{\mathbf{E}}$.

Let p be an odd prime and we write $G := \mathbf{G}_{\mathbb{Q}_p}$ for the base change of $\mathbf{G}$ to $\mathbb{Q}_p$. We let $\mathbb{A}_f$ denote the ring of finite adeles and $\mathbb{A}_f^p$ the finite adeles with trivial component at p. Let $\mathrm{K} = \mathrm{K}_p\mathrm{K}^p \subset \mathbf{G}(\mathbb{A}_f)$ where $\mathrm{K}_p \subset \mathbf{G}(\mathbb{Q}_p)$ and $\mathrm{K}^p \subset \mathbf{G}(\mathbb{A}_f^p)$ are compact open subgroups. Then for K^p sufficiently small

$$\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)(\mathbb{C}) = \mathbf{G}(\mathbb{Q})\backslash X \times \mathbf{G}(\mathbb{A}_f)/\mathrm{K}$$

arises as the complex points of an algebraic variety $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)$ defined over $\mathbf{E}$.

5.1.2. From now on, we will assume the datum $(\mathbf{G}, X)$ is of Hodge type. This means that there exists an embedding of Shimura data

$$\rho : (\mathbf{G}, X) \longrightarrow (\mathbf{GSp}(V, \psi), S^\pm)$$

where (V, ψ) is a symplectic space over $\mathbb{Q}$ and $(\mathbf{GSp}(V, \psi), S^\pm)$ is the standard Siegel Shimura datum. We will also make the following assumptions.

(†) The group $G := \mathbf{G}_{\mathbb{Q}_p}$ is quasi-split and splits over a tamely ramified extension of $\mathbb{Q}_p$. Moreover $p \nmid |\pi_1(G_{\mathrm{der}})|$, and K_p is a connected very special parahoric subgroup of $G(\mathbb{Q}_p)$.

Here we say a parahoric K_p is connected if it is the same as the stabilizer of a facet in the building for G. When G is unramified, every parahoric which is contained in a hyperspecial parahoric is connected. In the sequel we let $\mathcal{G}$ be the group scheme over $\mathbb{Z}_p$ corresponding to the parahoric K_p.

Let v be a prime of $\mathbf{E}$ lying above p with residue field $k_v = \mathbb{F}_q$. We write O for the ring of integers of $\mathbf{E}$ and $O_{(v)}$ for the localization of O at v. Under the assumptions above, Kisin–Pappas [24] have constructed an integral model $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ for $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)$ over $O_{(v)}$. We briefly recall the construction below.

By the discussion in [24, §2.3.15], upon replacing ρ with a different Hodge embedding, we may assume that there exists a $\mathbb{Z}_p$-lattice $V_{\mathbb{Z}_p} \subset V_{\mathbb{Q}_p}$ such that ρ induces a closed immersion $\mathcal{G} \to \mathrm{GL}(V_{\mathbb{Z}_p})$. From now on we fix ρ such that this condition is satisfied. We let $\mathrm{K}' = \mathrm{K}'_p\mathrm{K}'^p \subset \mathbf{GSp}(V_{\mathbb{A}_f})$ with $\mathrm{K}'_p \subset \mathbf{GSp}(V_{\mathbb{Q}_p})$ the stabilizer of $V_{\mathbb{Z}_p}$ and $\mathrm{K}'^p \subset \mathbf{GSp}(\mathbb{A}_f^p)$ a sufficiently small compact open subgroup. By [21, Lemma 2.1.2], up to shrinking K^p we may choose a sufficiently small K'^p such that the Hodge embedding ρ defines a closed immersion

$$\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X) \longrightarrow \mathrm{Sh}_{\mathrm{K}'}(\mathbf{GSp}(V), S^{\pm}) \otimes_{\mathbb{Q}} \mathbf{E}$$

of Shimura varieties. We let $V_{\mathbb{Z}_{(p)}} = V_{\mathbb{Z}_p} \cap V$ and we let $G_{\mathbb{Z}_{(p)}}$ denote the Zariski closure of $\mathbf{G}$ in $\mathbf{GSp}(V_{\mathbb{Z}_{(p)}})$. The choice of $V_{\mathbb{Z}_{(p)}}$ gives rise to an interpretation of $\mathrm{Sh}_{\mathrm{K}'}(\mathbf{GSp}(V), S^{\pm})$ as a moduli space of abelian varieties and hence to an integral model $\mathscr{S}_{\mathrm{K}'}(\mathbf{GSp}(V), S^{\pm})$ over $\mathbb{Z}_{(p)}$; see [24, §4] and [44, §6]. The integral model $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ is defined to be the normalization of the closure of $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)$ in $\mathscr{S}_{\mathrm{K}'}(\mathbf{GSp}(V), S^{\pm}) \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{(v)}$. We will write $\mathcal{A}$ for the pullback of the universal abelian scheme on $\mathscr{S}_{\mathrm{K}'}(\mathbf{GSp}(V), S^{\pm}) \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{(v)}$ to $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$.

5.2. Rapoport–Zink Uniformization

5.2.1. We fix a maximal $\breve{\mathbb{Q}}_p$-split $\mathbb{Q}_p$-rational torus S in G (cf. §2.1.1) such that K_p corresponds to a σ-stable special point $\breve{\mathfrak{s}}$ in the apartment corresponding to S. We let T denote the centralizer of S and we fix B a Borel subgroup of G containing T (which exists as we have assumed that G is quasi-split). We let $\mu \in X_*(T)^+_{\Gamma_0}$ denote the image of a dominant representative $\widetilde{\mu} \in X_*(T)^+$ of $\{\mu\}$. (Here Γ_0 is as in §2.1.1 with respect to $F = \mathbb{Q}_p$.) Then for $b \in B(G, \mu)$ we have the associated affine Deligne–Lusztig variety $X_\mu(b)$ as in §2.4 corresponding to the very special parahoric K_p.

To ease notation we write $\underline{\mathrm{Sh}}_{\mathrm{K}}$ for the geometric special fiber of $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$. By [44, §8], there exists a map

$$\mathcal{N} : \underline{\mathrm{Sh}}_{\mathrm{K}} \longrightarrow B(G, \mu)$$

which induces the *Newton stratification* on $\underline{\mathrm{Sh}}_{\mathrm{K}}$. We let $[b]_{\mathrm{basic}} \in B(G, \mu)$ denote the unique basic σ-conjugacy class in $B(G, \mu)$ and we write $\underline{\mathrm{Sh}}_{\mathrm{K,bas}}$ for the preimage of $[b]_{\mathrm{basic}}$ under $\mathcal{N}$. By [35, Theorem 3.6] this is a closed subscheme of $\underline{\mathrm{Sh}}_{\mathrm{K}}$, which is known as the *basic locus*.

Our goal is to understand the set $\Sigma^{\mathrm{top}}(\underline{\mathrm{Sh}}_{\mathrm{K,bas}})$ of top-dimensional irreducible components of $\underline{\mathrm{Sh}}_{\mathrm{K,bas}}$. This will follow from our study of $X_\mu(b)$ above and the following result, which is the analogue in our context of the Rapoport–Zink uniformization.

Proposition 5.2.2. *Let $b \in [b]_{\mathrm{basic}}$. There exists an isomorphism of perfect schemes*

$$I(\mathbb{Q}) \backslash X_\mu(b) \times \mathbf{G}(\mathbb{A}_f^p)/\mathrm{K}^p \cong \underline{\mathrm{Sh}}^{\mathrm{pfn}}_{\mathrm{K,bas}}$$

where I is a certain inner form of $\mathbf{G}$ with $I \otimes_{\mathbb{Q}} \mathbb{A}_f^p \cong \mathbf{G} \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ and $I \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong J_b$. Moreover this isomorphism is equivariant for prime-to-p Hecke operators.

Corollary 5.2.3. *There exists an identification*

$$\Sigma^{\mathrm{top}}(\underline{\mathrm{Sh}}_{\mathrm{K,bas}}) \cong \coprod_{i=1}^{\mathscr{N}(\mu,b)} I(\mathbb{Q})\backslash I(\mathbb{A}_f)/\mathrm{I}_p^i \mathrm{I}^p$$

where $\mathscr{N}(\mu, b)$ is as in Definition 4.1.3, I_p^i is a very special parahoric of $I(\mathbb{Q}_p)$ and $\mathrm{I}^p \cong \mathrm{K}^p$ under a fixed identification $I \otimes_{\mathbb{Q}} \mathbb{A}_f^p \cong G \otimes_{\mathbb{Q}} \mathbb{A}_f^p$. Moreover the following statements hold.

(1) *The identification is compatible with prime-to-p Hecke operators.*

(2) *If G is unramified, we may replace the indexing set with $\mathbb{MV}_\mu(\lambda_b)$.*

Proof. This follows from Proposition 5.2.2, Corollary 4.1.4, and the fact that the topology of a scheme is invariant under taking perfection. ∎

The rest of the section will be devoted to the proof of Proposition 5.2.2. The case when G is an unramified group is proved in [43, Corollary 7.2.6], a key input being the existence of a natural map

$$X_\mu(b)(\overline{\mathbb{F}}_p) \longrightarrow \underline{\mathrm{Sh}}_{\mathrm{K}}(\overline{\mathbb{F}}_p)$$

which was proved in [22, Proposition 1.4.4]. Our proposition follows similarly using results from [44]. We first recall some notations from [44, §6.2].

5.2.4. By construction, for a scheme T over $O_{(v)}$, a point $x \in \mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)(T)$ gives rise to a triple $(\mathcal{A}_x, \lambda, \epsilon^p_{\mathrm{K}'})$ where $\mathcal{A}_x$ is an abelian variety over T, λ is a weak polarization (cf. [44, §6.3]), and $\epsilon^p_{\mathrm{K}'}$ is a global section of the étale sheaf

$$\underline{\mathrm{Isom}}_{\lambda,\psi}(\widehat{V}(\mathcal{A}_x), V_{\mathbb{A}_f^p})/\mathrm{K}'^p.$$

Here $\widehat{V}(\mathcal{A}_x) = (\varprojlim_{p\nmid n} \mathcal{A}_x[n]) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the adelic prime-to-p Tate module of $\mathcal{A}_x$, and we refer the reader to loc. cit. for more details of the above étale sheaf.

For R a ring and M an R-module, we let $M^\otimes$ denote the direct sum of all R-modules obtained from M by taking duals, tensor products, and symmetric and exterior products. By [21, 1.3.2] and the assumption on ρ in §5.1.2, the subgroup $G_{\mathbb{Z}_{(p)}}$ of $\mathbf{GSp}(V, \psi)$ is the stabilizer of a collection of tensors $s_\alpha \in V^\otimes_{\mathbb{Z}_{(p)}}$. Let k be a finite field or $\overline{\mathbb{F}}_p$, and let $x \in \underline{\mathrm{Sh}}_{\mathrm{K}}(k)$. Then by the discussion in [44, §6], the abelian variety $\mathcal{A}_x$ is equipped with Frobenius-invariant tensors $s_{\alpha,\ell,x} \in T_\ell(\mathcal{A}_x)^\otimes$ for primes $\ell \neq p$ and φ-invariant tensors $s_{\alpha,0,x} \in \mathbb{D}(\mathscr{G}_x)^\otimes$. Here $T_\ell(\mathcal{A}_x)$ is the ℓ-adic Tate module of $\mathcal{A}_x$, $\mathscr{G}_x := \mathcal{A}_x[p^\infty]$ is the p-divisible group of $\mathcal{A}_x$, and $\mathbb{D}(\mathscr{G}_x)$ is its contravariant Dieudonné module. Upon fixing an isomorphism

$$\mathbb{D}(\mathscr{G}_x) \cong V^\vee_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} W(k)$$

taking $s_{\alpha,0,x}$ to s_α, which exists by [24, Proposition 3.3.8], the Frobenius φ is given by $\delta\sigma$ for an element $\delta \in G(W(k))[\frac{1}{p}]$ well defined up to σ-conjugation by $G(W(k))$.

We write $\mathbb{M}$ for the F-crystal of the p-divisible group associated to $\mathcal{A}$ over $\underline{\mathrm{Sh}}_{\mathrm{K}}$ and we let $\mathbb{M}[\frac{1}{p}]$ denote the associated isocrystal. By [23], there exists tensors $\mathbf{s}_{\alpha,0} \in \mathbb{M}[\frac{1}{p}]$ which specialize to $s_{\alpha,0,x}$ for all $x \in \underline{\mathrm{Sh}}_{\mathrm{K}}(\overline{\mathbb{F}})$.

5.2.5. Now let $k = \mathbb{F}_{p^r}$ be a finite extension of k_v. Fix $x \in \underline{\mathrm{Sh}}_{\mathrm{K}}(k)$. For each prime $\ell \neq p$, upon fixing an isomorphism

$$V_{\mathbb{Q}_\ell}^\vee \cong T_\ell(\mathcal{A}_x)^\vee \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \tag{5.2.5.1}$$

taking s_α to $s_{\alpha,\ell,x}$, which exists by [21, §3.4.2], the p^r-Frobenius on the right is given by an element $\gamma_\ell \in \mathbf{G}(\mathbb{Q}_\ell)$ well defined up to conjugation. In fact, [21, §3.4.2] shows that we may assume the isomorphism (5.2.5.1) arises from an isomorphism

$$V_{\mathbb{A}_f^p}^\vee \cong \widehat{V}(\mathcal{A}_x)^\vee$$

taking s_α to $s_{\alpha,\ell,x}$, and hence that $(\gamma_\ell)_{\ell\neq p}$ is an element of $\mathbf{G}(\mathbb{A}_f^p)$. We let $I_{\ell/k}$ denote the centralizer of γ_ℓ. For sufficiently divisible n, the centralizer of γ_ℓ^n stabilizes and we write I_ℓ for this centralizer. We also obtain $\delta \in G(W(k)[\frac{1}{p}])$ from x as explained in §5.2.4, and we define the $\mathbb{Q}_p$-group $I_{p/k}$ whose points in a $\mathbb{Q}_p$-algebra R is given by

$$I_{p/k}(R) := \{g \in G(W(k)[1/p] \otimes_{\mathbb{Q}_p} R) \mid g^{-1}\delta\sigma(g) = \delta\}.$$

Then $I_{p/k}$ is a subgroup of J_δ, and it grows if we keep δ fixed and let the finite field k grow. Thus when k'/k is a sufficiently large finite extension $I_{p/k'}$ stabilizes, and we denote it by I_p. We write γ_p for the norm $\delta\sigma(\delta)\ldots\sigma^{r-1}(\delta)$.

Finally we define the $\mathbb{Q}$-group whose points valued in a $\mathbb{Q}$-algebra R are given by

$$\mathrm{Aut}(\mathcal{A}_x \otimes_k \overline{\mathbb{F}}_p)(R) = (\mathrm{End}_{\mathbb{Q}}(\mathcal{A}_x \otimes_k \overline{\mathbb{F}}_p) \otimes_{\mathbb{Q}} R)^\times$$

and we let $I \subset \mathrm{Aut}(\mathcal{A}_x \otimes_k \overline{\mathbb{F}}_p)$ denote the subgroup which preserve the tensors $s_{\alpha,0,x}$ and $s_{\alpha,\ell,x}$ for all $\ell \neq p$. We have the following facts about these groups for points x in the basic locus.

Proposition 5.2.6. *Let $k = \mathbb{F}_{p^r}$ a finite extension of k_v and $x \in \underline{\mathrm{Sh}}_{\mathrm{K,bas}}(\mathbb{F}_{p^r})$.*

(1) *There exists $\gamma_0 \in \mathbf{G}(\mathbb{Q})$ which is elliptic in $\mathbf{G}(\mathbb{R})$ such that $(\gamma_0, (\gamma_\ell)_{\ell\neq p}, \delta)$ forms a Kottwitz triple of level r in the sense of* [22, §4.3.1]. *In particular, γ_0 is $\mathbf{G}(\overline{\mathbb{Q}}_\ell)$-conjugate to γ_ℓ for all ℓ (including $\ell = p$).*

(2) *For any prime ℓ (including $\ell = p$), the natural map $I \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to I_\ell$ is an isomorphism, and the group $(I/\mathbb{G}_m)(\mathbb{R})$ is compact. Here $\mathbb{G}_m \subset I$ arises from the image of the weight homomorphism of the Shimura datum $(\mathbf{G}, X)$.*

(3) *We write $I_0 \subset \mathbf{G}$ for the centralizer of γ_0^n for sufficiently divisible n such that the centralizers stabilize. Then there exists an inner twisting $I \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \xrightarrow{\sim} I_0 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ which makes I an inner form of I_0 and such that the diagram*

$$\begin{tikzcd} I_0 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell \arrow[r, "\sim"] \arrow[d, equal] & I_\ell \otimes_{\mathbb{Q}_l} \overline{\mathbb{Q}}_\ell \\ I_0 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell \arrow[r, "\sim"] & I \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell \arrow[u, "\sim"'] \end{tikzcd}$$

commutes up to inner automorphism for any prime ℓ.

Proof. (1) and (2) follow from the discussion in [44, §9.5]. (3) follows from the same argument as [22, Corollary 2.3.5] using [44, Theorem 9.4] in place of [22, Theorem 2.2.3]. ■

5.2.7. For $(\gamma_0, (\gamma_\ell)_{\ell\neq p}, \delta)$ a Kottwitz triple of level r, $(\gamma_0^m, (\gamma_\ell^m)_{\ell\neq p}, \delta)$ is a Kottwitz triple of level rm. We consider the smallest equivalence relation on the set of all Kottwitz triples of all levels under which $(\gamma_0, (\gamma_\ell)_{\ell\neq p}, \delta)$ is equivalent to $(\gamma_0^m, (\gamma_\ell^m)_{\ell\neq p}, \delta)$ for all $m \geq 1$. An equivalence class under this relation is called a *Kottwitz triple*. For $x \in \underline{\mathrm{Sh}}_{\mathrm{K,bas}}(\overline{\mathbb{F}}_p)$, we know that x is defined over some $k = \mathbb{F}_{p^r}$, and the associated Kottwitz triple $(\gamma_0, (\gamma_\ell)_{\ell\neq p}, \delta)$ of level r defines a Kottwitz triple which is independent of the choice of $\mathbb{F}_{p^r}$.

Recall the following notion of isogeny classes introduced in [44].

Definition 5.2.8. Let $x, x' \in \underline{\mathrm{Sh}}_{\mathrm{K}}(\overline{\mathbb{F}}_p)$. We say x and x' are *isogenous* if there exists a quasi-isogeny $\mathcal{A}_x \to \mathcal{A}_{x'}$ which takes $s_{\alpha,\ell,x}$ to $s_{\alpha,\ell,x'}$ and $s_{\alpha,0,x}$ to $s_{\alpha,0,x'}$. Clearly this defines an equivalence relation on $\underline{\mathrm{Sh}}_{\mathrm{K}}(\overline{\mathbb{F}}_p)$, and the equivalence classes will be called *isogeny classes*.

5.2.9. We define an equivalence relation $\sim$ on the set of all Kottwitz triples by setting $\mathfrak{t} \sim \mathfrak{t}'$ for Kottwitz triples $\mathfrak{t}, \mathfrak{t}'$ if there exist representatives $(\gamma_0, (\gamma_\ell)_{\ell\neq p}, \delta)$, $(\gamma_0', (\gamma_\ell')_{\ell\neq p}, \delta')$ of some level r for $\mathfrak{t}, \mathfrak{t}'$ respectively such that

(1) $(\gamma_\ell)_{\ell\neq p}$ and $(\gamma_\ell')_{\ell\neq p}$ are conjugate in $\mathbf{G}(\mathbb{A}_f^p)$

(2) δ and δ' are σ-conjugate in $G(K_0)$, where $K_0 = W(\mathbb{F}_{p^r})[\frac{1}{p}]$.

It is easy to see that if $\mathfrak{t}, \mathfrak{t}'$ are the Kottwitz triples associated to points $x, x' \in \underline{\mathrm{Sh}}_{\mathrm{K,bas}}(\overline{\mathbb{F}}_p)$ lying in the same isogeny class, then $\mathfrak{t} \sim \mathfrak{t}'$.

Proposition 5.2.10. *Let $x, x' \in \underline{\mathrm{Sh}}_{\mathrm{K,bas}}(\overline{\mathbb{F}}_p)$ and let $\mathfrak{t}$ (resp. $\mathfrak{t}'$) denote the Kottwitz triple associated to x (resp. x'). Then $\mathfrak{t} \sim \mathfrak{t}'$.*

Proof. We fix a sufficiently large finite field $k = \mathbb{F}_{p^r}$ such that x and x' are both defined over k and we fix representatives $(\gamma_0, (\gamma_\ell)_{\ell\neq p}, \delta)$ and $(\gamma_0', (\gamma_\ell')_{l\neq p}, \delta')$ of level r for $\mathfrak{t}$ and $\mathfrak{t}'$ respectively. Write I and I' for the $\mathbb{Q}$-groups associated to x and x' as above. We first claim that there exists $n \geq 1$ such that γ_0^n and $\gamma_0'^n$ are central. Indeed this follows verbatim from the argument in [43, Lemma 7.2.14] which works without the assumption that G is unramified. Therefore upon extending k we may assume γ_0 and γ_0' are both central.

Let Z° denote the connected component of the center of $\mathbf{G}$. Upon enlarging k, we may assume $t := \gamma_0^{-1}\gamma_0' \in Z^\circ(\mathbb{Q})$. We claim that the image of t in $Z^\circ(\mathbb{A}_f)$ lies in a compact open subgroup H. For $\ell \neq p$, we have $\gamma_0 = \gamma_\ell$, hence γ_0 lies in a compact subgroup of $Z^\circ(\mathbb{Q}_\ell)$ since γ_ℓ is the Frobenius automorphism of the ℓ-adic Tate module. The same argument works for γ_0' and hence t lies in a compact open subgroup of $\mathbf{G}(\mathbb{A}_f^p)$. For $\ell = p$, we have that γ_0 and γ_0' both have the same image in $\pi_1(G)_\Gamma$ since δ and δ' are both basic. Since the kernel of the map $X_*(Z^\circ)_{\Gamma_0}^\sigma \to \pi_1(G)_\Gamma$ is torsion, it follows that upon further extending k, we may assume that γ and γ_0' have the same image under the Kottwitz map

$\kappa : Z^\circ(\mathbb{Q}_p) \to X_*(Z^\circ)^\sigma_{\Gamma_0}$. Thus t lies in the kernel of κ which is a compact open subgroup of $Z^\circ(\mathbb{Q}_p)$.

Since $\mathbf{G}$ and I are inner forms (recall γ_0 is central), we may naturally consider Z° as a subgroup of I which contains the scalars $\mathbb{G}_m$. Then the compactness of $(I/\mathbb{G}_m)(\mathbb{R})$ implies $(Z^\circ/\mathbb{G}_m)(\mathbb{R})$ is compact. It follows that $H \cap Z^\circ(\mathbb{Q})$ is finite. Hence there exists m such that $\gamma_0^m = \gamma_0'^m$. Upon extending k, we may assume $\gamma_0 = \gamma_0'$. This implies $\gamma_\ell = \gamma_\ell'$.

Now since $x, x' \in \underline{\mathrm{Sh}}_{\mathrm{K,bas}}(k)$, there exists $g \in G(\breve{\mathbb{Q}}_p)$ such that $g^{-1}\delta\sigma(g) = \delta'$. Taking norms, we obtain

$$g^{-1}\gamma_0\sigma^r(g) = \gamma_0' = \gamma_0$$

and hence $g^{-1}\sigma^r(g) = 1$ since γ_0 is central. This implies $g \in G(\mathbb{Q}_{p^r})$ and hence δ and δ' are σ-conjugate in $G(\mathbb{Q}_{p^r})$. It follows that $\mathfrak{t} \sim \mathfrak{t}'$. ■

Proposition 5.2.6 and Proposition 5.2.10 together with the Hasse principle for adjoint groups imply the following corollary.

Corollary 5.2.11. *Let $x, x' \in \underline{\mathrm{Sh}}_{\mathrm{K,bas}}(\overline{\mathbb{F}}_p)$. Then the groups I and I' are isomorphic as inner forms of* $\mathbf{G}$. ■

Proposition 5.2.12. *Let $x, x' \in \underline{\mathrm{Sh}}_{\mathrm{K,bas}}(\overline{\mathbb{F}}_p)$. Then x and x' lie in the same isogeny class.*

Proof. Let $k = \mathbb{F}_{p^r}$ be a sufficiently large finite field such that x and x' are both defined over k. We let I and I' be the groups associated to x and x' respectively. We let $\mathrm{Isog}(\mathcal{A}_x, \mathcal{A}_{x'})$ be the scheme of quasi-isogenies between $\mathcal{A}_{x'}$ and $\mathcal{A}_{x'}$. We define

$$\mathcal{P}_{s_\alpha}(x, x') \subset \mathrm{Isog}(\mathcal{A}_x, \mathcal{A}_{x'})$$

to be the subscheme which takes $(s_{\alpha,\ell,x})_{l\neq p}$ (resp. $s_{\alpha,0,x}$) to $(s_{\alpha,\ell,x'})_{\ell\neq p}$ (resp. $s_{\alpha,0,x'}$). It suffices to show that $\mathcal{P}_{s_\alpha}(x, x')$ is a trivial I-torsor.

We first show $\mathcal{P}_{s_\alpha}(x, x')$ is an I-torsor. By Corollary 5.2.11, we may fix an isomorphism $I \cong I'$. Let $\mathbf{T} \subset I \cong I'$ be a maximal torus. The proof of [44, Theorem 9.4] shows that upon modifying x and x' in its isogeny class, we may assume that x and x' admit lifts $\tilde{x}$ and $\tilde{x}'$ to $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)(\overline{\mathbb{Q}})$ satisfying the conditions:

(1) $\mathbf{T} \subset \mathrm{Aut}(\mathcal{A}_x)$ and $\mathbf{T} \subset \mathrm{Aut}(\mathcal{A}_{x'})$ lift to $\mathbf{T} \subset \mathrm{Aut}(\mathcal{A}_{\tilde{x}})$ and $\mathbf{T} \subset \mathrm{Aut}(\mathcal{A}_{\tilde{x}'})$.
(2) The Hodge filtrations on $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_{\tilde{x}})$ and $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_{\tilde{x}'})$ are induced by the same $\mathbf{T}$-valued cocharacter $\mu^{\mathbf{T}}$.
(3) If $i, i' : \mathbf{T} \to \mathbf{G}$ are the inclusions obtained by regarding $\mathbf{T}$ as a subgroup of the Mumford–Tate groups of $\mathcal{A}_{\tilde{x}}$ and $\mathcal{A}_{\tilde{x}'}$ (these are well-defined up to $\mathbf{G}(\mathbb{Q})$-conjugacy), then $\tilde{x}$ and $\tilde{x}'$ are in the images of the maps

$$\begin{aligned} i &: \mathrm{Sh}(\mathbf{T}, h_{\mathbf{T}}) \to \mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)_{\mathbf{E}_{\mathbf{T}}} \\ i' &: \mathrm{Sh}(\mathbf{T}, h_{\mathbf{T}}) \to \mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)_{\mathbf{E}_{\mathbf{T}}} \end{aligned}$$

respectively. Here $\mathrm{Sh}(\mathbf{T}, h_{\mathbf{T}})$ is the Shimura variety for $(\mathbf{T}, h_{\mathbf{T}})$ and $\mathbf{E}_{\mathbf{T}}$ is its reflex field.

We let $\tilde{\mathcal{P}} \subset \mathrm{Isog}(\mathcal{A}_{\tilde{x}}, \mathcal{A}_{\tilde{x}'})$ be the scheme of isogenies which respect the Hodge cycles and the action of $\mathbf{T}$. We claim that $\tilde{\mathcal{P}}$ is a $\mathbf{T}$-torsor; for this it suffices to show that $\tilde{\mathcal{P}}$ is non-empty.

By Proposition 5.2.6, the map

$$i : \mathbf{T} \longrightarrow \mathbf{G} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \cong I \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$$

is conjugate to the natural inclusion, and a similar statement holds for the map

$$i' : \mathbf{T} \longrightarrow \mathbf{G} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \cong I' \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}.$$

It follows that there exists $g \in \mathbf{G}(\overline{\mathbb{Q}})$ such that $gig^{-1} = i'$. Since $i(\mathbf{T})$ is its own centralizer in $\mathbf{G}$, we have $c_\tau = g^{-1}\tau(g) \in i(\mathbf{T})(\overline{\mathbb{Q}})$ for any $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let K_∞ denote the centralizer of $i \circ h_T$. Then by the same argument as in [22, Proposition 4.4.13], the image of c in $\mathrm{H}^1(\mathbb{R}, \mathrm{K}_\infty)$ is trivial.

This defines a $\mathbf{T}$-torsor $\tilde{\mathcal{P}}'$ which is isomorphic to $\tilde{\mathcal{P}}$ by [22, Proposition 4.2.6]. Indeed the proposition in loc. cit. shows that $\mathcal{A}_{\tilde{x}'}$ is isomorphic to the twisted abelian variety $\mathcal{A}_{\tilde{x}}^{\tilde{\mathcal{P}}'}$ as in [22, §4.1] equipped with its collection of Hodge cycles and action of $\mathbf{T}$ induced from $\mathcal{A}_{\tilde{x}}$. It then follows by the construction of $\mathcal{A}_{\tilde{x}}^{\tilde{\mathcal{P}}'}$ that $\tilde{\mathcal{P}} \cong \tilde{\mathcal{P}}'$. It follows that $\mathcal{P}_{s_\alpha}$ is the I-torsor obtained by pushout from the $\mathbf{T}$-torsor $\tilde{\mathcal{P}}$.

By [22, Lemma 4.4.3], there is an isomorphism

$$\ker(\mathrm{H}^1(\mathbb{Q}, I) \to \mathrm{H}^1(\mathbb{R}, I)) \cong \ker(\mathrm{H}^1(\mathbb{Q}, \mathbf{G}) \to \mathrm{H}^1(\mathbb{R}, \mathbf{G})).$$

By [22, Lemma 4.4.5] applied to the inclusion $\mathbf{T}_{\mathbb{R}} \to K_\infty$, the image of c in $\mathrm{H}^1(\mathbb{R}, \mathbf{T})$ is trivial, and hence the image of c in $\mathrm{H}^1(\mathbb{Q}, I)$ lies in $\ker(\mathrm{H}^1(\mathbb{Q}, I) \to \mathrm{H}^1(\mathbb{R}, I))$. Since the image of c in $\mathrm{H}^1(\mathbb{Q}, \mathbf{G})$ is trivial, we have that c is trivial in $\mathrm{H}^1(\mathbb{Q}, I)$. It follows that the I-torsor $\mathcal{P}_{s_\alpha}(x, x')$ is trivial. ■

Proof of Proposition 5.2.2. Let $x \in \underline{\mathrm{Sh}}_{\mathrm{K,bas}}(\overline{\mathbb{F}}_p)$. We first define a natural map $X_\mu(\delta) \to \underline{\mathrm{Sh}}^{\mathrm{pfn}}_{\mathrm{K,bas}}$. The key input for this is the existence of such a map on $\overline{\mathbb{F}}_p$-points which was constructed in [44, Proposition 7.7]. We may then argue as in [43, Lemma 7.2.12]; we sketch the argument emphasizing the points which do not directly carry over to the ramified case.

As in [43, 7.2.6], we may construct an abelian variety $\mathcal{A}$ over $X_\mu(b)$ equipped with a p-power quasi isogeny $\mathcal{A} \to \mathcal{A}_x \times X_\mu(b)$. Moreover this quasi-isogeny equips $\mathcal{A}$ with tensors $\mathbf{s}'_{\alpha,0} \in \mathbb{D}(\mathcal{A}[p^\infty])^\otimes$, as well as a weak polarization and a prime-to-p level structure. Hence we obtain a map

$$\iota' : X_\mu(b) \longrightarrow \mathscr{A}_{g,K'}.$$

We claim ι' lifts to a unique map

$$\iota : X_\mu(b) \longrightarrow \underline{\mathrm{Sh}}^{\mathrm{pfn}}_{\mathrm{K}}$$

such that for each closed point $y \in X_\mu(b)$, we have $s_{\alpha,0,y} = \mathbf{s}'_{\alpha,0,y}$. The uniqueness follows from [44, Corollary 6.3] and the fact that two maps between perfect schemes coincide if and only if they coincide on the level of closed points. Thus it suffices to prove the lifting locally.

Let y be a closed point of $X_\mu(b)$ and $U \subset X_\mu(b)$ an affine open neighborhood containing y which is perfectly of finite presentation. We may assume U is the perfection of a reduced affine scheme $U_0 \cong \mathrm{Spec}\, R$ and that the quasi-isogeny $\mathcal{A}|_U \to \mathcal{A}_x \times U$ comes from pullback from a quasi-isogeny $\mathcal{A}_0 \to \mathcal{A}_x \times U_0$ over U_0. We thus obtain a map

$$\iota'_0 : U_0 \to \mathscr{S}_{\mathrm{K}'}(\mathbf{GSp}(V), S^\pm) \otimes_{\mathbb{Z}_{(p)}} \overline{\mathbb{F}}_p$$

and it suffices to show ι'_0 can be lifted to $\iota : U_0 \to \underline{\mathrm{Sh}}_{\mathrm{K}}$.

We form the pullback diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \underline{\mathrm{Sh}}_{\mathrm{K}} \arrow[d] \\ \mathrm{Spec}\, R \arrow[r] & \mathscr{S}_{\mathrm{K}'}(\mathbf{GSp}(V), S^\pm) \otimes_{\mathbb{Z}_{(p)}} \overline{\mathbb{F}}_p \end{tikzcd}.$$

Then Y is equipped with a polarized abelian variety $(\mathcal{A}_Y, \lambda_Y)$ and tensors

$$\mathbf{s}'_{\alpha,0,Y} \in \mathbb{D}(\mathcal{A}[p^\infty])[\frac{1}{p}]^\otimes, \quad \mathbf{s}_{\alpha,0,Y} \in \mathbb{D}(\mathcal{A}[p^\infty])[\frac{1}{p}]^\otimes,$$

where the $\mathbf{s}'_{\alpha,0,Y}$ are obtained from pullback of $\mathbf{s}'_{\alpha,0}$ along $Y \to \mathrm{Spec}\, R$, and the $\mathbf{s}_{\alpha,0,Y}$ are obtained from pullback of $\mathbf{s}_{\alpha,0}$ along $Y \to \underline{\mathrm{Sh}}_{\mathrm{K}}$. We let Y° denote the union of connected components which contain an $\overline{\mathbb{F}}_p$-point y such that $\mathbf{s}_{\alpha,0,y} = \mathbf{s}'_{\alpha,0,y}$. By [30, Lemma 5.10], $\mathbf{s}_{\alpha,0,Y^\circ} = \mathbf{s}'_{\alpha,0,Y^\circ}$. By [44, Proposition 6.5 (i)], the map $Y^\circ \to \mathrm{Spec}\, R$ is bijective on $\overline{\mathbb{F}}_p$-points and by [24, Proposition 4.2.2], the map $Y^\circ \to \mathrm{Spec}\, R$ is finite and is a closed immersion when completed at every point of the domain. In addition R is reduced; it follows that $Y^\circ \to \mathrm{Spec}\, R$ is an isomorphism.

The map ι induces a finite map

$$\iota_{\mathrm{isog}} : I(\mathbb{Q}) \backslash X_\mu(b) \times \mathbf{G}(\mathbb{A}_f^p)/\mathrm{K}^p \longrightarrow \underline{\mathrm{Sh}}_{\mathrm{K},b}^{\mathrm{pfn}}$$

which is bijective on closed points by [44, Proposition 9.1] and Proposition 5.2.12, and is a closed immersion when completed at every closed point of the domain. It follows that ι_{isog} is an isomorphism. ∎

Erratum for [45]

We take the opportunity to make the following corrections to the prequel paper [45]. In [45, Definition 5.2.7], the two appearances of $\mathbb{C}[Y^*]$ should be replaced by $\mathbb{C}$. In [45, Proposition 5.5.1], the identity is between two elements of $\mathbb{C}[\mathbf{q}^{-1}]$.

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References

[1] Bhatt, B., Scholze, P.: Projectivity of the Witt vector affine Grassmannian. Invent. Math. **209**, 329–423 (2017) MR 3674218

[2] Borel, A., Prasad, G.: Finiteness theorems for discrete subgroups of bounded covolume in semi-simple groups. Inst. Hautes Études Sci. Publ. Math. 119–171 (1989) MR 1019963

[3] Bruhat, F., Tits, J.: Groupes réductifs sur un corps local. Inst. Hautes Études Sci. Publ. Math. 5–251 (1972) MR 327923

[4] Bruhat, F., Tits, J.: Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée. Inst. Hautes Études Sci. Publ. Math. 197–376 (1984) MR 756316

[5] Chen, M., Kisin, M., Viehmann, E.: Connected components of affine Deligne-Lusztig varieties in mixed characteristic. Compos. Math. **151**, 1697–1762 (2015) MR 3406443

[6] Gashi, Q. R.: On a conjecture of Kottwitz and Rapoport. Ann. Sci. Éc. Norm. Supér. (4) **43**, 1017–1038 (2010) MR 2778454

[7] Görtz, U., Haines, T. J., Kottwitz, R. E., Reuman, D. C.: Dimensions of some affine Deligne-Lusztig varieties. Ann. Sci. École Norm. Sup. (4) **39**, 467–511 (2006) MR 2265676

[8] Görtz, U., He, X.: Dimensions of affine Deligne-Lusztig varieties in affine flag varieties. Doc. Math. **15**, 1009–1028 (2010) MR 2745691

[9] Greenberg, M. J.: Schemata over local rings. II. Ann. of Math. (2) **78**, 256–266 (1963) MR 0156855

[10] Haines, T., Rapoport, M.: On parahoric subgroups. Adv. Math. **219**, 118–198 (2008) MR 2435422

[11] Hamacher, P.: The dimension of affine Deligne-Lusztig varieties in the affine Grassmannian. Int. Math. Res. Not. IMRN 12804–12839 (2015) MR 3431637

[12] Hamacher, P., Viehmann, E.: Irreducible components of minuscule affine Deligne-Lusztig varieties. Algebra Number Theory **12**, 1611–1634 (2018) MR 3871504

[13] Hamacher, P., Viehmann, E.: Finiteness properties of affine Deligne-Lusztig varieties. Doc. Math. **25**, 899–910 (2020) MR 4148693

[14] He, X.: Geometric and homological properties of affine Deligne-Lusztig varieties. Ann. of Math. (2) **179**, 367–404 (2014) MR 3126571

[15] He, X.: Centers and cocenters of 0-Hecke algebras. In: Representations of reductive groups, Progr. Math. 312, Birkhäuser/Springer, Cham, 227–240 (2015) MR 3495798

[16] He, X.: Hecke algebras and p-adic groups. In: Current developments in mathematics 2015, Int. Press, Somerville, MA, 73–135 (2016) MR 3642544

[17] He, X., Nie, S.: Minimal length elements of extended affine Weyl groups. Compos. Math. **150**, 1903–1927 (2014) MR 3279261

[18] He, X., Rapoport, M.: Stratifications in the reduction of Shimura varieties. Manuscripta Math. **152**, 317–343 (2017) MR 3608295

[19] He, X., Yang, Z.: Elements with finite Coxeter part in an affine Weyl group. J. Algebra **372**, 204–210 (2012) MR 2990008

[20] Howard, B., Pappas, G.: Rapoport-Zink spaces for spinor groups. Compos. Math. **153**, 1050–1118 (2017) MR 3705249
[21] Kisin, M.: Integral models for Shimura varieties of abelian type. J. Amer. Math. Soc. **23**, 967–1012 (2010) MR 2669706
[22] Kisin, M.: mod p points on Shimura varieties of abelian type. J. Amer. Math. Soc. **30**, 819–914 (2017) MR 3630089
[23] Kisin, M., Madapusi Pera, K., Shin, S. W.: Honda-Tate theory for Shimura varieties. Duke Math. J. **171**, 1559–1614 (2022) MR 4484214
[24] Kisin, M., Pappas, G.: Integral models of Shimura varieties with parahoric level structure. Publ. Math. Inst. Hautes Études Sci. **128**, 121–218 (2018) MR 3905466
[25] Kottwitz, R. E.: Tamagawa numbers. Ann. of Math. (2) **127**, 629–646 (1988) MR 942522
[26] Kottwitz, R. E.: Isocrystals with additional structure. II. Compositio Math. **109**, 255–339 (1997) MR 1485921 (99e:20061)
[27] Li, W., Mantovan, E., Pries, R., Tang, Y.: Newton polygons arising from special families of cyclic covers of the projective line. Res. Number Theory **5**, Paper No. 12, 31 (2019) MR 3897613
[28] Liu, Y., Tian, Y.: Supersingular locus of Hilbert modular varieties, arithmetic level raising and Selmer groups. Algebra Number Theory **14**, 2059–2119 (2020) MR 4172702
[29] Lusztig, G.: Hecke algebras with unequal parameters. CRM Monograph Series 18, American Mathematical Society, Providence, RI (2003) MR 1974442
[30] Madapusi Pera, K.: Integral canonical models for spin Shimura varieties. Compos. Math. **152**, 769–824 (2016) MR 3484114
[31] Nie, S.: Semi-modules and irreducible components of affine Deligne-Lusztig varieties. ArXiv e-prints (2018)
[32] Nie, S.: Irreducible components of affine Deligne-Lusztig varieties. Camb. J. Math. **10**, 433–510 (2022) MR 4461835
[33] Pappas, G., Rapoport, M.: Twisted loop groups and their affine flag varieties. Adv. Math. **219**, 118–198 (2008) MR 2435422
[34] Rapoport, M.: A guide to the reduction modulo p of Shimura varieties. Astérisque 271–318 (2005) MR 2141705
[35] Rapoport, M., Richartz, M.: On the classification and specialization of F-isocrystals with additional structure. Compositio Math. **103**, 153–181 (1996) MR 1411570
[36] Richarz, T.: On the Iwahori Weyl group. Bull. Soc. Math. France **144**, 117–124 (2016) MR 3481263
[37] Rousseau, G.: Immeubles des groupes réducitifs sur les corps locaux. U.E.R. Mathématique, Université Paris XI, Orsay (1977) MR 0491992
[38] Springer, T. A.: Reductive groups. In: Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 3–27 (1979) MR 546587
[39] Springer, T. A.: Linear algebraic groups. second ed., Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA (2009) MR 2458469
[40] Tits, J.: Reductive groups over local fields. In: Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 29–69 (1979) MR 546588
[41] Viehmann, E.: The dimension of some affine Deligne-Lusztig varieties. Ann. Sci. École Norm. Sup. (4) **39**, 513–526 (2006) MR 2265677
[42] Vollaard, I., Wedhorn, T.: The supersingular locus of the Shimura variety of $\mathrm{GU}(1, n-1)$ II. Invent. Math. **184**, 591–627 (2011) MR 2800696

[43] Xiao, L., Zhu, X.: Cycles on Shimura varieties via geometric Satake. arXiv e-prints (2017)

[44] Zhou, R.: Mod p isogeny classes on Shimura varieties with parahoric level structure. Duke Math. J. **169**, 2937–3031 (2020) MR 4158671

[45] Zhou, R., Zhu, Y.: Twisted orbital integrals and irreducible components of affine Deligne-Lusztig varieties. Camb. J. Math. **8**, 149–241 (2020) MR 4085434

[46] Zhu, X.: The geometric Satake correspondence for ramified groups. Ann. Sci. Éc. Norm. Supér. (4) **48**, 409–451 (2015) MR 3346175

[47] Zhu, X.: Affine Grassmannians and the geometric Satake in mixed characteristic. Ann. of Math. (2) **185**, 403–492 (2017) MR 3612002